# Measurability and semi-continuity of multifunctions

B. Cascales

*Departamento de Matemáticas, Universidad de Murcia,*
*30.100 Espinardo. Murcia, Spain*
*E-mail: beca@um.es*

The following pages contain details of a mini-course of three lectures given at the *V International Course of Mathematical Analysis of Andalucía (CIDAMA), Almeria, September 12-17, 2011.* When I was invited to give this mini-course and thought about possible topics for it, I decided to talk about multifunctions because they have always been present in my research on fields theoretically apart from each other as topology and integration theory. Therefore you will find here my biased views regarding part of the research that I have done over the years. The proofs for this material have been published elsewhere by me or by some other authors. This mini-survey is written attending to the invitation of the publishers of this book with the sole purpose of witnessing the given mini-course and with the aim of providing the reader with connections and ideas that usually are not written in research papers. I thank the organizers of CIDAMA V as well as the editors of the book for their kind invitation to give the lecture and write this mini-survey.

In these notes we shall deal with multifunctions (or set-valued maps). Multifunctions naturally appear in analysis and topology, for instance via inequalities, performing unions or intersections with sets indexed in another set, considering the set of points minimizing an expression, etc. First, we will present some results about semi-continuity of multifunctions, namely, lower semi-continuity and an application of Michael's selection theorem. Then we will deal with upper semi-continuity of multifunctions and an application to the generation of $K$-analytic structures with consequences in topology and functional analysis. We will finish by showing a few results about measurability for multifunctions related to the Kuratowski-Ryll-Narzesdky selection theorem and their implications to integrability of multifunctions for non separable Banach spaces.

*Keywords*: set-valued map, multifunction, lower semi-continuous, upper semi-continuous, measurable, compactness, metrizability, Lindelöf property, $K$-analytic space, Pettis integrability, Effros measurability

## 1. Settings, first definitions and introduction

Our notation and terminology is standard and it is either explained when needed or can be found in our references for Banach spaces,[1,2] topology[3,4] and vector measures and integration.[5]

By capital letters $D, E, S, X, Y, \Omega, \dots$ we denote sets. Sometimes these sets are endowed with a topology, *i.e.*, they are topological spaces. In particular by $(E, \|\cdot\|)$ we denote a real Banach space (or simply $E$ if $\|\cdot\|$ is tacitly assumed): $B_E$ stands for the closed unit ball in $E$, $S_E$ for the unit sphere, $E^*$ for the dual space of $E$ and $E^{**}$ for the bidual space of $E$; $w$ is the weak topology and $w^*$ is the weak$^*$ topology in the dual. Throughout this paper $(\Omega, \Sigma, \mu)$ is a complete finite measure space.

**Definition 1.1.** A multifunction (set-valued map) is a map $\psi$ from a set $X$ into the family of subsets $2^Y$ of another set $Y$, *i.e.*, for each $x \in X$ the image $\psi(x)$ is a subset of $Y$.

**Example 1.1.**

(1) The map $\log : \mathbb{C} \setminus \{0\} \to 2^{\mathbb{C}}$ that sends every $z \in \mathbb{C} \setminus \{0\}$ to the set $\log(z)$ of all logarithms of $z$ is a multifunction, see [6, p. 39].
(2) If $g, G : X \to \mathbb{R}$ are two given functions with $g(t) \le G(t)$ for every $t \in X$, then $\psi(t) := [g(t), G(t)]$ defines a multifunction $\psi : X \to 2^{\mathbb{R}}$, see figure 1.

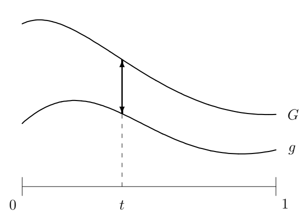


Fig. 1. Example of multifuncion

(3) If $f : Y \to X$ is an onto map, then $\psi(x) := f^{-1}(x)$, $x \in X$, defines multifunction $\psi : X \to 2^Y$.
(4) If $K$ is a Hausdorff compact space the map $\psi : C(K) \to 2^K$ given by

$$\psi(f) := \big\{x \in K : |f(x)| = \sup_{t \in K} |f(t)| =: \|f\|_\infty\big\}$$

is a multifunction defined in the Banach space of scalar-valued continuous functions $C(K)$.
(5) If $E$ is a Banach space the duality mapping $J : E \to 2^{B_{E^*}}$ given by

$$J(x) := \{x^* \in B_{E^*} : \|x\| = x^*(x)\}$$

is a multifunction, see [2, p. 343].

(6) If $E$ is a Banach space and $Y \subset E$ is a closed proximinal subspace, then the metric projection $P_Y : E \to 2^Y$ given by

$$P_Y(x) := \big\{ y \in Y : \|x - y\| = \inf_{z \in Y} \|x - z\| =: d(x, Y) \big\}$$

is a multifunction (recall that by definition $Y$ being proximinal means $P_Y(x) \neq \emptyset$ for every $x \in E$, see [7, §5]).

(7) If $E$ is a Fréchet space, see [8, §18.2], and $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$ is a basis of neighborhoods of 0 then $\psi : \mathbb{N}^{\mathbb{N}} \to 2^E$ given by

$$\psi(\alpha) := \bigcap_{k=1}^{\infty} n_k U_k, \text{ with } \alpha = (n_k)_k,$$

is a multifunction with $\psi(\mathbb{N}^{\mathbb{N}}) = E$, $\psi(\alpha) \subset \psi(\beta)$ if $\alpha \leq \beta$ (coordinatewise) in $\mathbb{N}^{\mathbb{N}}$ and $\{\psi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental family of bounded sets of $E$.

(8) If $E = \varinjlim E_n$ is an (LF) space, see [8, §19.5], and

$$U_1^m \supset U_2^m \supset \cdots \supset U_n^m \supset \cdots$$

is a basis of neighborhoods of 0 in $E_m$ then $\psi : \mathbb{N}^{\mathbb{N}} \to 2^{E'}$ given by

$$\psi(\alpha) := \overline{\operatorname{aco}\Big(\bigcup_{k=1}^{\infty} U_{n_k}^k\Big)}^{\circ}, \text{ with } \alpha = (n_k)_k,$$

is a multifunction with $\psi(\mathbb{N}^{\mathbb{N}}) = E'$, $\psi(\alpha) \subset \psi(\beta)$ if $\alpha \leq \beta$ (coordinatewise) in $\mathbb{N}^{\mathbb{N}}$ and $\{\psi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental family of equicontinuous subsets of $E'$ (polars $A^\circ$ are taken in the dual pair $\langle E, E' \rangle$, see [8, §20.8]).

(9) If $E$ is a Banach space, $f : \Omega \to E$ and $r : \Omega \to [0, \infty)$ are functions then $F : \Omega \to 2^E$ given by

$$F(\omega) := f(\omega) + r(\omega) B_E, \, \omega \in \Omega,$$

is a multifunction.

(10) If $\{f_i : \Omega \to E\}_{i \in I}$ is a family of functions we can consider the multifunction $F : \Omega \to 2^E$ defined by

$$F(\omega) := \overline{\operatorname{co}\{f_i(\omega) : \; i \in I\}}.$$

The three intimately connected notions below are the ones that we shall deal with in these notes.

**Definition 1.2.** Let $X$ and $Y$ be topological spaces and let $\psi : X \to 2^Y$ be a multifunction. We say that $\psi$ is lower semi-continuous (l.s.c.) if the set

$$\{x \in X : \psi(x) \cap O \neq \emptyset\}$$

is open for every open subset $O$ of $Y$, see [9, §43] and [10, Ch. 7].

**Definition 1.3.** Let $X$ and $Y$ be topological spaces and let $\psi : X \to 2^Y$ be a multifunction. We say that $\psi$ is upper semi-continuous (u.s.c.) if the set

$$\{x \in X : \psi(x) \cap F \neq \emptyset\}$$

is closed for every closed subset $F$ of $Y$, see [9, §43] and [10, Ch. 7].

It is easy to check that $\psi$ as above is u.s.c. if, and only if, for every $x_0 \in X$ and every open set $V \supset \psi(x_0)$ in $Y$, there is an open neighborhood $U \subset X$ of $x_0$ such that $\psi(x) \subset V$ for every $x \in U$, see figure 2.

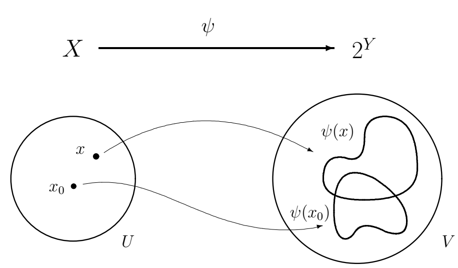


Fig. 2. Upper semi-continuity

**Definition 1.4.** Let $(\Omega, \Sigma)$ be a measurable space and let $E$ be a Banach space. A multifunction $F : \Omega \to 2^E$ is said to be Effros measurable if

$$\{t \in \Omega : \ F(t) \cap O \neq \emptyset\} \in \Sigma \text{ for each open set } O \subset E. \tag{E}$$

More general notions of measurability can be found in the literature: we remark that the notion above makes sense for any topological space in the range, see.[10–12]

Natural examples illustrating the above notions are easy to provide. Beyond those spread out in the literature we isolate, for the purposes of these notes, the following ones.

**Example 1.2.**

(1) *(A lower semi-continuous multifunction)* If $X$ is a topological space and we assume that in example 1.1.(2) $g : X \to \mathbb{R}$ is upper semi-continuous and $G : X \to \mathbb{R}$ is lower semi-continuos, then it is easily checked that $\psi(t) := [g(t), G(t)]$ defines a lower semi-continuous multifunction $\psi : X \to 2^{\mathbb{R}}$.

(2) *(An upper semi-continuous multifunction)* Let us consider $\mathbb{N}$ endowed with its discrete topology and $\mathbb{N}^{\mathbb{N}}$ with its product topology. The multifunction $\psi$ defined in example 1.1.(7) is upper semi-continuous whenever $E$ is Fréchet-Montel, see [8, §27.2] for the definition, and the basis $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$ of neighborhoods of 0 is made up of closed sets.

(3) *(A measurable multifunction)* Assume here that $E$ is a separable Banach space. When dealing with Effros measurability for Borel $\sigma$-algebras, the first examples that come to mind are l.s.c. multifunctions (and u.s.c. multifunctions if they take compact values, see [11, Cor. III.3]). A quite remarkable example regarding measurability of multifunctions is the one provided by example 1.1.(10) when $I = \mathbb{N}$ and each $f_n : \Omega \to E$ is measurable. A celebrated result by Castaing-Valadier says that all Effros measurable multifunctions $\psi : \Omega \to 2^E$ with closed values are of the form described in example 1.1.(10) with $I = \mathbb{N}$ and each $f_n$ measurable, see [11, Th. III.9].

**Definition 1.5.** Given a multifunction $\psi : X \to 2^Y$ a *selector* (*selection*) for $\psi$ is a single-valued function $f : X \to Y$ such that

$$f(x) \in \psi(x),$$

for every $x \in X$, see figure 3.

In our views the leading role of multifunctions in many aspects of mathematical analysis and topology is due to their proliferation and the strong consequences that can be obtained from their study. In the rest of these notes we shall present some results connected with our research that repeatedly go once and again to one of the ideas below:

(a) when dealing with multifunctions defined between topological spaces $\psi : X \to 2^Y$ semi-continuity properties of $\psi$ can be used many times:

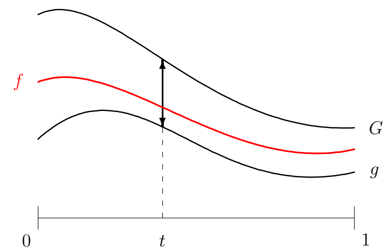


Fig. 3. Selection

- to transfer properties from $X$ to $Y$;
- to find "good" selectors for $\psi$ (from a topological point of view);

(b) when dealing with multifunctions $F : \Omega \to 2^E$ their measurability can be used many times:

- to find "good" selectors for $F$ (from a measurability point of view);
- to study properties of integrability for $F$.

For the study of questions as in (a) many names come to our minds, a few of which are: Argyros, Arkhangel'skiĭ, Jayne, Kuratowski, Mercourakis, Michael, Negrepontis, Talagrand, Rogers, etc. For the study of questions as in (b) authors like Aumann, Debreu, Hess, Kuratowsky, Ryll Nardzewski, etc. made very important contributions. Many other authors have made quite important contributions too to topics related to (a) and (b) above. Since it is imposible to name all of them we cut our list short without diminishing the importance of contributions of those that we cannot name. Let us stress though that very in particular, Debreu[13] and Aumann[14] established very important results in mathematics and in some models in economy when dealing with multifunctions (notice that Debreu and Aumann received the Nobel prize in economy, 1983 and 2055 respectively).

We finish this introduction collecting three superb selection results.

**Theorem 1.1 (Michael,[15]).** *Assume that $X$ is a paracompact space, that $E$ is a Banach space and that $\psi : X \to 2^E$ is a l.s.c. multifunction such that $\psi(x)$ is closed, convex and nonempty for every $x \in X$. Then $\psi$ has a continuos selector,* i.e., *there is a continuous function $f : X \to E$ such that $f(x) \in \psi(x)$ for every $x \in X$.*

Amongst the many applications of Michael's theorem we can mention Bartle-Graves' theorem (if $F \subset E$ is a closed subspace of $E$, the quotient map $\pi : E \to E/F$ has a positive homogeneous lifting, see [16, Prop. 1.19]) and Borsuk-Kakutani-Didunji's theorem (if $K$ is compact and $H \subset K$ is closed and metrizable, there is a simultaneous extension continuous operator $T : C(H) \to C(K)$ such that $\|T\| = 1$ and $T1 = 1$, see [16, Prop. 1.21]).

**Theorem 1.2 (Jayne-Rogers, [17, Th. 5.4]).** *Let $E$ be an Banach space. The following statements are equivalent:*

*(i)* *$E$ is Asplund,* i.e., *every separable subspace has separable dual;*
*(ii)* *the duality mapping $J : E \to 2^{B_{E^*}}$ has a Baire-1 selector,* i.e., *there is a sequence of norm-to-norm continuous maps $f_n : E \to E^*$ such that for every $x \in E$ there exists $\lim_n f_n(x) \in J(x)$.*

We should note that the implication $(i) \Rightarrow (ii)$ is based on the fact that the duality mapping $J$ is norm-to-$w^*$ upper semi-continuos and that whenever $E$ is an Asplund space then $(B_{E^*}, w^*)$ is norm-fragmented, see.[18,19] The above result can be found in [20, Th. 5.2, Rem. 5.11]. Such a remarkable selection result has played a fundamental role in renorming theory and in the study of boundaries in Banach spaces,[17,20–22]

**Theorem 1.3 (Kuratowski-Ryll Nardzewski,[23]).** *Let $(\Omega, \Sigma, \mu)$ be a complete probability space and $F : \Omega \to 2^E$ a multifunction with closed non empty values of $E$. If $E$ is separable and $F$ is Effros measurable, then $F$ admits a measurable selector $f$,* i.e., *there is a $f : \Omega \to E$ such that $f^{-1}(O) \in \Sigma$ for every open set $O \subset E$ and $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$.*

A proof for the above Kuratowski-Ryll Nardzewski's theorem can be found in [11, Th . III.6] and [10, Th. 14.2.1]. Over the years Kuratowski-Ryll Nardzewski's theorem has been the milestone result to build up several theories of multifunction integration that henceforth have been presented only for separable Banach spaces as range spaces.

## 2. Lower semi-continuity for multifunctions, an application

This section is the witness of how lower-semicontinuity and Michael selection theorem ignited the appearance of tools that allowed to rewrite most of the known results about pointwise and weak compactness in $C_p$-theory and functional analysis from a quantitative point of view.

A straightforward application of Michael's selection theorem 1.1 is the following result.

**Theorem 2.1 ( [16, Pro. 1.18]).** *Let $X$ be a paracompact space and let $f_1 \le f_2$ be two real functions on $X$ such that $f_1$ is upper semi-continuous and $f_2$ is lower semi-continuous. Then, there exists a continuous function $h \in C(X)$ such that $f_1(x) \le h(x) \le f_2(x)$ for all $x \in X$.*

**Proof.** It is easily proved that the multifunction $\psi : X \to 2^{\mathbb{R}}$ given by $\psi(x) := [f_1(x), f_2(x)]$, $x \in X$, is l.s.c. and therefore theorem 1.1 can be used to conclude the existence of the continuous selection $h$, see figure 4.

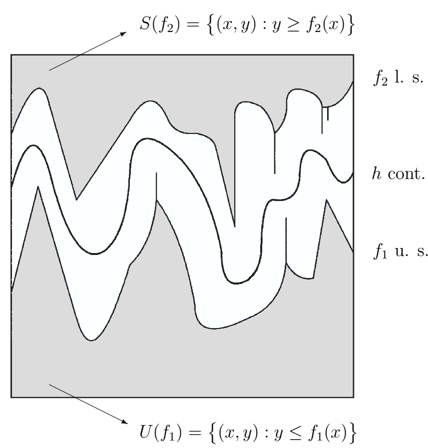


Fig. 4. A sandwich result

□

As a consequence of the above result we have.

**Theorem 2.2 ( [16, Pro. 1.19]).** *Let $X$ be a paracompact space. For a given bounded function $f \in \mathbb{R}^X$ the distance of $f$ to the subspace of bounded and continuous functions on $X$ is given by*

$$d(f, C_b(X)) = \frac{1}{2}\operatorname{osc}(f)$$

*where*

$$\operatorname{osc}(f) = \sup_{x\in X} \operatorname{osc}(f,x) = \sup_{x\in X} \inf\{\operatorname{diam} f(U) : U \subset X \text{ open}, x \in U\}.$$

**Proof.**

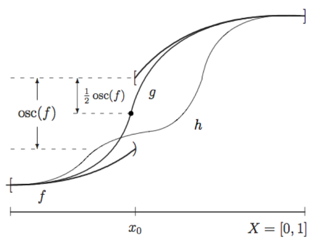


Fig. 5. Distance to continuous

As in [16, Pro. 1.18]. Put $\delta = \frac{1}{2}\operatorname{osc}(f)$. It is clear that the distance is at least $\delta$. To prove the other direction, define

$$f_1(x) := \inf_{U\in\mathcal{V}_x}\sup_{z\in U} f(z) - \delta$$

$$f_2(x) := \sup_{U\in\mathcal{V}_x}\inf_{z\in U} f(z) + \delta$$

Then $f_1 \le f_2$. It is easy to check that $f_1$ is upper semi-continuous and $f_2$ is lower semi-continuous. By theorem 2.1, there is a continuous function $h \in C(X)$ such that

$$f_1(x) \le h(x) \le f_2(x)$$

for every $x \in X$. On the other hand, for every $x \in X$ we have

$$f_2(x) - \delta \le f(x) \le f_1(x) + \delta$$

and therefore

$$h(x) - \delta \le f_2(x) - \delta \le f(x) \le f_1(x) + \delta \le h(x) + \delta.$$

So $d(f,h) \le \delta = \frac{1}{2}\operatorname{osc}(f)$ and this finishes the proof. □

When $X$ is only a normal space and the functions are not necessarily bounded a proof for the above result can be found in.[24]

Theorem 2.2 has been the key and inspiration to prove the four results that follow.

**Theorem 2.3 ([25,26]).** *Let $K$ be a compact space and let $H$ be a uniformly bounded subset of $C(K)$. We have*

$$\operatorname{ck}(H) \le \hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)) \le \gamma_K(H) \le 2\operatorname{ck}(H).$$

**Theorem 2.4 ([26]).** *Let $K$ be a compact topological space and let $H$ be a uniformly bounded subset of $\mathbb{R}^K$. Then*

$$\gamma_K(H) = \gamma_K(co(H))$$

*and as a consequence for $H \subset C(K)$ we obtain that*

$$\hat{d}(\overline{co(H)}^{\mathbb{R}^K}, C(K)) \le 2\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K))$$

*and if $H \subset \mathbb{R}^K$ is uniformly bounded then*

$$\hat{d}(\overline{co(H)}^{\mathbb{R}^K}, C(K)) \le 5\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)).$$

**Theorem 2.5 ([26]).** *Let $E$ be a Banach space and let $B_{E^*}$ be the closed unit ball in the dual $E^*$ endowed with the $w^*$-topology. Let $i : E \to E^{**}$ and $j : E^{**} \to \ell_\infty(B_{E^*})$ be the canonical embeddings. Then, for every $x^{**} \in E^{**}$ we have*

$$d(x^{**}, i(E)) = d(j(x^{**}), C(B_{E^*}))\,.$$

**Theorem 2.6 ([26,27]).** *Let $H$ be a bounded subset of a Banach space $E$. Then*

$$\operatorname{ck}(H) \le \operatorname{k}(H) \le \gamma(H) \le 2\operatorname{ck}(H) \le 2\operatorname{k}(H) \le 2\omega(H) \tag{1}$$

$$\gamma(H) = \gamma(\operatorname{co}(H)) \quad \textit{and} \quad \omega(H) = \omega(\operatorname{co}(H).$$

*For any $x^{**} \in \overline{H}^{w^*}$, there is a sequence $(x_n)_n$ in $H$ such that*

$$\|x^{**} - y^{**}\| \le \gamma(H)$$

*for any cluster point $y^{**}$ of $(x_n)_n$ in $E^{**}$. Furthermore, $H$ is relatively compact in $(E, w)$ if, and only if, it is zero one (equivalently all) of the numbers $\operatorname{ck}(H), \operatorname{k}(H), \gamma(H)$ and $\omega(H)$.*

The notation used is the following:

(1) The distance $d$ in $\mathbb{R}^K$ or $C(K)$ always refers to the supremum distance.
(2) If $T$ be a topological space and $A$ subset of of $T$, then $A^{\mathbb{N}}$ is considered as the set of all sequences in $A$. The set of all cluster points in $T$ of a sequence $\varphi \in A^{\mathbb{N}}$ is denoted by $\operatorname{clust}_T(\varphi)$.
(3) If $H$ be a subset $\mathbb{R}^K$ we define:

$$\operatorname{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\operatorname{clust}_{\mathbb{R}^K}(\varphi), C(K)),$$

$$\hat{d}(H, C(K)) := \sup_{g \in H} d(g, C(K)),$$

and

$$\gamma_K(H) := \sup\{\big|\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)\big| : (f_m) \subset H, (x_n) \subset K\},$$

assuming that the involved limits exist.
(4) If $E$ is a Banach space and $H \subset E$ is a bounded set, then $\overline{H}^{w^*}$ stands for its $w^*$-closure in $E^{**}$ and

$$k(H) = \hat{d}(\overline{H}^{w^*}, E) = \sup_{y \in \overline{H}^{w^*}} \inf_{x \in E} \|y - x\|,$$

$$\gamma(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m)_m \subset B_{E^*},\, (x_n)_n \subset H\},$$

assuming the involved limits exist,

$$\mathrm{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\mathrm{clust}_{E^{**},w^*}(\varphi), E)$$

and

$$\omega(H) := \inf\{\varepsilon > 0 : H \subset K_\varepsilon + \varepsilon B_E \text{ and } K_\varepsilon \subset X \text{ is } w\text{-compact}\}.$$

For obvious reasons the quantities that appear in theorem 2.6 are called measures of weak noncompactness, see.[28,29] Measures of noncompactness or weak noncompactness have been successfully applied to the study of compactness, operator theory, differential equations and integral equations, see for instance.[26,27,29–39] Theorem 2.6 tells us that *all classical approaches used so far to study weak compactness in Banach spaces (Tychonoff's theorem, Eberlein-Šmulian's theorem, Eberlein-Grothendieck double-limit criterion) are qualitatively and quantitatively equivalent.* Quantitative versions of James compactness theorem can be found in.[33] Surveys about these questions are.[24,40,41]

## 3. Upper semi-continuity for multifunctions, applications

This section explains how one can exploit the use of multifunctions $\psi : X \to 2^Y$ between topological spaces from two different but connected angles: (a) transferring properties of $X$ to properties of $Y$ when $\psi$ is upper semi-continuous; (b) ensuring how to automatically produce upper-semicontinuity from descriptive properties.

The two results that follow, theorems 3.1 and 3.2, have been during our years of research the most useful ones that we have ever found. The first one is related to property (a) above and the second one to property (b). The ideas behind them can be traced back to references.[42–45]

Recall that the *weight* $w(X)$ of a topological space $X$ is the minimal cardinality of a basis for the topology of $X$. By the *density* $d(X)$ we mean the minimal cardinality of a dense subset of $X$. The *Lindelöf number* $l(X)$ of $X$ is the smallest infinite cardinal number $\mathfrak{m}$ such that every open cover of $X$ has a subcover of cardinality $\leq \mathfrak{m}$.

**Theorem 3.1 ( [46, Pro. 2.1]).** *Let $X$ and $Y$ be topological spaces and let $\psi : X \to 2^Y$ be an upper semi-continuous compact-valued map such that the set $Y = \bigcup\{\psi(x) : x \in X\}$. Assume that $w(X)$ is infinite. Then,*

*(1) the Lindelöf number $l(Y^n) \leq w(X)$, for every $n = 1, 2, \ldots$;*

*(2) if $Y$ is moreover assumed to be metric then $d(Y) \le w(X)$.*

**Proof.** The proof below is the one that was published in [46, Pro. 2.1] and it is included in order that the reader can get the flavour of the techniques needed.

To prove (1) we observe first that for every $n = 1, 2, \dots$ the multi-valued map $\psi^n : X^n \to 2^{Y^n}$ given by

$$\psi^n(x_1, x_2, \dots, x_n) := \psi(x_1) \times \psi(x_2) \times \cdots \times \psi(x_n)$$

is compact-valued, upper semi-continuous and

$$Y^n = \bigcup \{\psi^n(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in X^n\}.$$

Since $w(X)$ is infinite we have that $w(X^n) = w(X)$ and therefore we only need to prove (1) for $n = 1$. Take $(G_i)_{i \in I}$ any open cover of $Y$. For each $x \in X$ the compact set $\psi(x)$ is covered by the family $(G_i)_{i \in I}$ and therefore we can choose a finite subset $I(x)$ of $I$ such that

$$\psi(x) \subset \bigcup_{i \in I(x)} G_i.$$

By upper semi-continuity, for each $x$ in $X$ we can take an open set $O_x$ of $X$ such that $x \in O_x$ and

$$\psi(O_x) \subset \bigcup_{i \in I(x)} G_i.$$

The family $(O_x)_{x \in X}$ is an open cover of $X$ and therefore there is a set $F \subset X$ such that $|F| \le w(X)$ and $X = \bigcup_{x \in F} O_x$, see [3, Theorem 1.1.14]. Then

$$Y = \psi(X) = \bigcup_{x \in F} \psi(O_x) = \bigcup_{x \in F} \bigcup_{i \in I(x)} G_i.$$

Hence $(G_i)_{i \in I}$ has a subcover of at most $w(X)$ elements.

For the proof of (2) we refer to [3, Theorem 4.1.15]. $\square$

**Theorem 3.2 ( [46, Th. 2.3],[45]).** *Let $X$ be a first-countable topological space, $Y$ a topological space in which the relatively countably compact subsets are relatively compact and let $\phi : X \to 2^Y$ be a multifunction satisfying the property*

$$\bigcup_{n \in \mathbb{N}} \phi(x_n) \text{ is relatively compact for each convergent sequence } (x_n)_n \text{ in } X. \tag{2}$$

*If for each $x$ in $X$ we define*

$$\psi(x) := \bigcap\{\overline{\phi(V)} : V \text{ neighborhood of } x \text{ in } X\}, \tag{3}$$

*then the multifunction so defined $\psi : X \to 2^Y$ is upper semi-continuous, compact-valued and satisfies $\phi(x) \subset \psi(x)$ for every $x$ in $X$.*

We recall that a topological space $Y$ is said to be $K$-*analytic* if there is a usco map $T : \mathbb{N}^{\mathbb{N}} \to 2^Y$ such that $T(\mathbb{N}^{\mathbb{N}}) := \bigcup\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\} = Y$,[47] Recall also that a regular topological space $T$ is *angelic* if every relatively countably compact subset $A$ of $T$ is relatively compact and its closure $\overline{A}$ is made up of the limits of sequences from $A$. In angelic spaces the different concepts of compactness and relative compactness coincide: the (relatively) countably compact, (relatively) compact and (relatively) sequentially compact subsets are the same, as seen in.[7] Examples of angelic spaces include metric spaces, spaces $C_p(K)$, when $K$ is a countably compact space, see[48,49] and all Banach spaces in their weak topologies.

**Corollary 3.1 ( [43, Corollary 1.1]).** *Let $Y$ be an angelic space. Assume that there is a family of subsets $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of $Y$ with the properties:*

($\alpha$) *$A_\alpha$ is compact for every $\alpha \in \mathbb{N}^{\mathbb{N}}$;*
($\beta$) *$A_\alpha \subset A_\beta$ if $\alpha \leq \beta$;*
($\gamma$) *$Y = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.*

*Then,*

*(1) $Y$ is $K$ analytic;*
*(2) if moreover $Y$ metrizable, then $Y$ is separable.*

**Proof.** Let us prove (1). To do so we will use theorem 3.2. We define $\phi(\alpha) := A_\alpha$, $\alpha \in \mathbb{N}^{\mathbb{N}}$. We check that $\phi$ satisfies the assumptions (2). Indeed, let $\pi_j : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be the $j$-th projection onto $\mathbb{N}$ and if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ we define, for every $j \in \mathbb{N}$,

$$m_j := \max\{\pi_j(\alpha_n) : n \in \mathbb{N}\},$$

and we write $\beta := (m_j)$. Note that $\alpha_n \leq \beta$ for every $n \in \mathbb{N}$ and then condition ($\beta$) ensures that $A_{\alpha_n} \subset A_\beta$ for every $n \in \mathbb{N}$. Thus

$$\bigcup_{n\in\mathbb{N}} \phi(\alpha_n) = \bigcup_{n\in\mathbb{N}} A_{\alpha_n} \subset A_\beta.$$

and since condition ($\alpha$) guarantees that $A_\beta$ is compact, we conclude that requirement (2) is fulfilled. Therefore we can use theorem 3.2 and produce the usco map $\psi : \mathbb{N}^{\mathbb{N}} \to 2^Y$ with the property $\phi(\alpha) \subset \psi(\alpha)$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. Now, condition $(\gamma)$ applies to conclude that $Y = \bigcup\{\psi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and therefore $Y$ is $K$-analytic.

Statement (2) straightforwardly follows from statement (1) in combination with (2) in theorem 3.1, if we bear in mind that $\mathbb{N}^{\mathbb{N}}$ is second countable, *i.e.*, the weight $w(\mathbb{N}^{\mathbb{N}})$ is countable. □

**Theorem 3.3 ( [40, Theorem 2.6]).** *Let $K$ be a compact space and let $\Delta$ be the diagonal of $K \times K$. The following statements are equivalent:*

*(1) $K$ is metrizable;*
*(2) $(C(K), \|\cdot\|_\infty)$ is separable;*
*(3) $\Delta$ is a $G_\delta$ subset of $K \times K$;*
*(4) $\Delta = \bigcap_n G_n$ with each $G_n$ open in $K \times K$ and $\{G_n : n \in \mathbb{N}\}$ being a basis of open neighbourhoods of $\Delta$;*
*(5) $(K \times K) \setminus \Delta = \bigcup_n F_n$, with $\{F_n : n \in \mathbb{N}\}$ an increasing family of compact subsets in $(K \times K) \setminus \Delta$;*
*(6) $(K \times K) \setminus \Delta = \bigcup_n F_n$, with $\{F_n : n \in \mathbb{N}\}$ an increasing family of compact sets that swallows all the compact subsets in $(K \times K) \setminus \Delta$;*
*(7) $(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ a family of compact sets that swallows all the compact subsets in $(K \times K) \setminus \Delta$ such that $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;*
*(8) $(K \times K) \setminus \Delta$ is Lindelöf.*

**Proof.** We refer to the proof of this theorem to [40, Theorem 2.6]. We reproduce here only the implication (7) $\Rightarrow$ (2). Assume that (7) holds and let us define $O_\alpha := (K \times K) \setminus A_\alpha$, $\alpha \in \mathbb{N}$. The family $\mathcal{O} := \{O_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of open neighbourhoods of $\Delta$ that satisfies the decreasing condition

$$O_\beta \subset O_\alpha, \text{ if } \alpha \leq \beta \text{ in } \mathbb{N}^{\mathbb{N}}. \tag{4}$$

Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and any $m \in \mathbb{N}$ we write $\alpha|^m := (n_m, n_{m+1}, n_{m+2}, \dots)$ and define

$$B_\alpha := \Big\{ f \in n_1 B_{C(K)} : (m \in \mathbb{N}, \text{ and } (x,y) \in O_{\alpha|^m}) \Rightarrow |f(x) - f(y)| \leq \frac{1}{m} \Big\}.$$

Note that each $B_\alpha$ is $\|\cdot\|_\infty$-bounded, closed and equicontinuous as a family of functions defined on $K$. Therefore, Ascoli's theorem, see [4, p. 234], implies that $B_\alpha$ is compact in $(C(K), \|\cdot\|_\infty)$. The decreasing property (4) implies that $B_\alpha \subset B_\beta$ if $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$. We claim that $C(K) = \bigcup\{B_\alpha : \alpha \in \mathbb{N}\}$. To see this, given $f \in C(K)$ take $M > 0$ such that $\|f\|_\infty \leq M$. On the other hand since $\mathcal{O}$ is a basis of neighborhoods of $\Delta$, there exists a sequence $\big(\alpha_m = (n_k^m)\big)$ in $\mathbb{N}^{\mathbb{N}}$ such that

$$|f(x) - f(y)| \le \frac{1}{m} \text{ for every } (x,y) \in O_{\alpha_m}.$$

If we define now $n_1 := \max\{n_1^1, M\}$ and $n_k := \max\{n_k^1, n_{k-1}^2, \ldots, n_1^k\}$, $k = 2, 3, \ldots$, then for the sequence $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ we have that $f \in B_\alpha$. The family $\{B_\alpha : \alpha \in \mathbb{N}\}$ of subsets of $(C(K), \|\cdot\|_\infty)$ satisfies the hypothesis of corollary 3.1 and we conclude that $(C(K), \|\cdot\|_\infty)$ is separable. This finishes the proof of (7) $\Rightarrow$ (2) . $\square$

Do not be misled by the purely topological aspect of the above theorem. Our contribution there, that is, implication (7) $\Rightarrow$ (2), was first stated also in a topological setting (apparently different) in [44, Theorem 1] as kind of lemma to establish metrizability results for compact sets in locally convex spaces. On the light of this result we introduced the class $\mathfrak{G}$ of locally convex spaces:

**Definition 3.1 ([44]).** *A locally convex space $E$ belongs to the class $\mathfrak{G}$ if there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of $E'$ satisfying the properties:*

*(a) for any $\alpha \in \mathbb{N}^{\mathbb{N}}$ the countable subsets of $A_\alpha$ are equicontinuous;*
*(b) $A_\alpha \subset A_\beta$ if $\alpha \le \beta$;*
*(c) $X = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.*

The class $\mathfrak{G}$ is a very wide class of locally convex spaces and it is stable under the usual operations in functional analysis of countable type (completions, closed subspaces, quotients, direct sums, products, etc.) that contains metrizable locally convex spaces and their duals and for which (7) $\Rightarrow$ (2) collected in theorem 3.3 implies:

**Theorem 3.4 ([44]).** *If $E$ is a locally convex space in class $\mathfrak{G}$, then its compact (even its precompact) subsets are metrizable.*

**Proof.** See [44, Theorem 2]. $\square$

We should mention that theorem 3.4 solved a number of open question in those times, for instance one posed by Floret in,[50] in which he asked about the sequential behaviour of compact subsets of (LM)-spaces, *i.e.*, inductive limits of metrizable locally convex spaces. Note that since $\mathfrak{G}$ contains metric spaces and their duals theorem 3.4 provides metrizability of compact subsets for (LM)-spaces as well as many other classes of spaces for which these properties were unknown by then. Since the appearance of[44] a number of authors coming from topology and functional analysis have been working on topics connected with those developed there: we refer to the recent book[51] for more references, applications and consequences of these ideas. Another possible survey reference is.[40]

Next we isolate the result below to show the simplicity, beauty and power of the techniques involving $K$-analytic structures.

**Theorem 3.5 (Dieudonné, Theorem §.2.(5)[8]).** *Every Fréchet-Montel space $E$ is separable (in particular, for any open set $\Omega \subset \mathbb{C}$ the space of holomorphic functions $(\mathcal{H}(\Omega), \tau_k)$ with its compact-open topology is separable).*

**Proof.** Fix $U_1 \supset U_2 \supset \cdots \supset U_n \dots$ a basis of absolutely convex closed neighborhoods of 0. Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, let us define

$$A_\alpha := \bigcap_{k=1}^{\infty} n_k U_k.$$

The family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is made up of closed bounded sets, covers $E$ and satisfies $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$. Since $E$ is Montel, each $A_\alpha$ is compact and since $E$ is Fréchet it is metrizable and therefore corollary 3.1 applies to say that $E$ is separable. $\square$

We finish pointing out some some recent developments that got started with[52,53] where the following definition can be found.

**Definition 3.2 ([52,53]).** *Given topological spaces $M$ and $Y$, an $M$-ordered compact cover of a space $Y$ is a family $\mathcal{F} = \{F_K : K \in \mathcal{K}(M)\} \subset \mathcal{K}(Y)$ such that*

$$\bigcup \mathcal{F} = Y \text{ and } K \subset L \text{ implies } F_K \subset F_L \text{ for any } K, L \in \mathcal{K}(M).$$

*$Y$ is said to be* dominated *(resp.* strongly dominated*) by the space $M$ if there exists an $M$-ordered compact cover $\mathcal{F}$ (resp. that moreover swallows all compact subsets of $Y$, in the sense that for any compact $C \subset Y$ there is $F \in \mathcal{F}$ such that $C \subset F$) of the space $Y$.*

It can be proved,[53] that condition (7) in theorem 3.3 is equivalent to $(K \times K) \setminus \Delta$ to be strongly dominated by a Polish space (Polish space means topological space that is metrizable, separable and complete for some metric given the topology).

**Proposition 3.1 ([54]).** *Let $K$ be a compact space and $\mathfrak{m}$ a cardinal number. The following statements are equivalent:*

*(1)* $w(K) \leq \mathfrak{m}$;
*(2) There exists a metric space $M$ with $w(M) \leq \mathfrak{m}$ and a family $\mathcal{O} = \{O_L : L \in \mathcal{K}(M)\}$ of open subsets in $K \times K$ that is basis of the neighborhoods of $\Delta$ such that $O_{L_1} \subset O_{L_2}$ whenever $L_2 \subset L_1$ in $\mathcal{K}(M)$;*
*(3)* $(K \times K) \setminus \Delta$ *is strongly dominated by a metric space $M$ with $w(M) \leq \mathfrak{m}$.*

The following questions is to the best of our knowledge still unanswered in full generality.

**Question 3.1.** if $K$ is a compact space such that $(K \times K) \setminus \Delta$ is dominated by a Polish space, is $K$ metrizable?

In the presence of some extra set theoretical axiom the answer is positive.

**Theorem 3.6 ([53]).** *Under $MA(\omega_1)$, if $K$ is a compact space such that $(K \times K) \setminus \Delta$ is dominated by a Polish space then $K$ is metrizable.*

## 4. Measurability for multifunctions, an application

In this section we will briefly present why we have been interested about measurable selectors for multifunctions and how we came across what we called property (P) that has shown to be useful for the existence of these measurable selectors.

Our interest about measurable selectors for measurable multifunctions goes back to our interest about integration for multifunctions. As said already, integration for multifunctions has its origin in the papers by Aumann[14] and Debreu:[13] good references on measurable selections and integration of multifunctions are the monographs[10,11] and the survey;[12] a common thing in all these studies that deal with multifunctions whose values are subsets of a Banach space $E$ is that $E$ was always assumed to be *separable*. The main reason for this limitation on $E$ relies on the fact that an integrable multifunction should have integrable (measurable) selectors and the tool to find these measurable selectors has always been the well-known selection theorem of Kuratowski and Ryll-Nardzewski[23] that only works when the range space is separable. Therefore if one wishes to find measurable selectors outside the universe of the $E$'s being separable a different approach should be done. With this in mind the following definition was introduced.

**Definition 4.1 ( [55, Definition 2.1]).** *We say that a multifunction $F : \Omega \to 2^E$ satisfies property (P) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma_A^+$ and $D \subset E$ with $\mathrm{diam}(D) < \varepsilon$ such that*

$$F(t) \cap D \neq \emptyset \text{ for every } t \in B,$$

*see figure 6.*

Here the notation that we use is the following: starting with our complete probability space $(\Omega, \Sigma, \mu)$ we write $\Sigma^+$ to denote the family of all $A \in \Sigma$ with $\mu(A) > 0$; given $A \in \Sigma^+$, the collection of all subsets of $A$ belonging to $\Sigma^+$ is denoted by $\Sigma_A^+$. We should mention here that our property (P) above

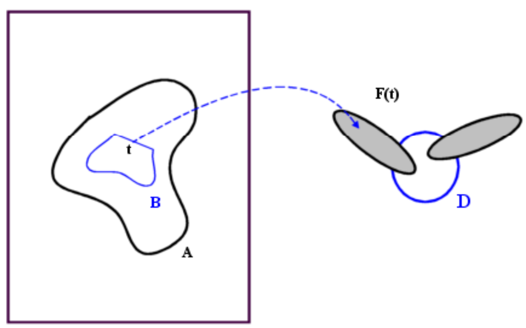


Fig. 6. Property (P)

is inspired in the topological notion of *fragmentability* for multifunctions that can be found in.[56]

It can be proved, see,[55] that for a multifunction $F : \Omega \to 2^E$ we have the following properties:

*(i) If there exists a multifunction $G : \Omega \to 2^E$ satisfying property (P) such that $G(t) \subset F(t)$ for $\mu$-a.e. $t \in \Omega$, then $F$ satisfies property (P) as well.*
*(ii) If there exists a strongly measurable function $f : \Omega \to E$ such that $F(t) = \{f(t)\}$ for $\mu$-a.e. $t \in \Omega$, then $F$ satisfies property (P).*
*(iii) If $F$ admits strongly measurable selectors, then $F$ satisfies property (P).*

Recall that $f : \Omega \to E$ is said to be strongly measurable if it is the $\mu$-a. e. limit os a sequence of measurable simple functions. It is easy to observe that property (P) for a multifunction helps to isolate the ideas behind the classical Kuratowski and Ryll-Nardzewski theorem that we present below:

proposition 4.1, lemma 4.1 and the proof of theorem 4.2 are co-authored with V. Kadets and J. Rodríguez. These results were written in some preliminary version of[55] but we finally took them out from the version that was sent off for publication.

**Proposition 4.1 (Cascales, Kadets and Rodríguez).** *Suppose $E$ is separable. Let $F : \Omega \to 2^E$ be a Effros measurable multifunction. Then $F$ satisfies property (P).*

**Proof.** Fix $\varepsilon > 0$ and $A \in \Sigma^+$. Since $E$ is separable, we can write $E = \bigcup_{n\in\mathbb{N}} C_n$, where each $C_n$ is an open ball with $\operatorname{diam}(C_n) \leq \varepsilon$. By hypothesis, all the sets $B_n := \{t \in \Omega : F(t) \cap C_n \neq \emptyset\}$ belong to $\Sigma$ and, moreover, $\Omega = \bigcup_{n\in\mathbb{N}} B_n$. Since $\mu(A) > 0$, there is $n \in \mathbb{N}$ such that $B := A \cap B_n \in \Sigma_A^+$. Now, the set $D := C_n$ intersects $F(t)$ for all $t \in B$. □

**Lemma 4.1 (Cascales, Kadets and Rodríguez).** *Let $F : \Omega \to 2^E$ be a multifunction satisfying property (P). Then for each $\varepsilon > 0$ there exists a strongly measurable countably-valued function $f : \Omega \to E$ such that $F(t) \cap \overline{B(f(t), \varepsilon)} \neq \emptyset$ for $\mu$-a.e. $t \in \Omega$.*

**Proof.** Property (P) and a standard exhaustion argument allow us to find a sequence $(A_n)$ of pairwise disjoint measurable subsets of $\Omega$ with $\mu(\Omega \setminus \bigcup_{n\in\mathbb{N}} A_n) = 0$ and a sequence $(D_n)$ of subsets of $X$ with diameter less than or equal to $\varepsilon$ such that, for each $n \in \mathbb{N}$, we have $F(t) \cap D_n \neq \emptyset$ for every $t \in A_n$. Take $x_n \in D_n$ for all $n \in \mathbb{N}$ and define $f : \Omega \to E$ by $f := \sum_{n\in\mathbb{N}} x_n \chi_{A_n}$. This function satisfies the desired property: for each $n \in \mathbb{N}$ and each $t \in A_n$ there is some $y \in F(t) \cap D_n$ and, bearing in mind that $\operatorname{diam}(D_n) \leq \varepsilon$, we get $y \in F(t) \cap \overline{B(f(t), \varepsilon)}$. The proof is over. □

**Proposition 4.2 (Kuratowski and Ryll-Nardzewski).** *Suppose $E$ is separable. Let $F : \Omega \to 2^E$ be an Effros measurable multifunction having norm closed values. Then $F$ admits strongly measurable selectors.*

**Proof.** [by Cascales, Kadets and Rodríguez]. Let $(\varepsilon_m)$ be a decreasing sequence of positive real numbers converging to 0.

By Proposition 4.1, $F$ satisfies property (P) and therefore Lemma 4.1 ensures the existence of a strongly measurable countably-valued function $f_1 : \Omega \to E$ such that $F(t) \cap \overline{B(f_1(t), \varepsilon_1)} \neq \emptyset$ for all $t \in A_1$, where $A_1 \in \Sigma$ and $\mu(\Omega \setminus A_1) = 0$.

Define $F_1(t) := F(t) \cap \overline{B(f_1(t), \varepsilon)}$ if $t \in A_1$, $F_1(t) = \{0\}$ otherwise. It is easily checked that the multifunction $F_1$ is Effros measurable. Again, proposition 4.1 and lemma 4.1 allow us to find a strongly measurable countably-valued function $f_2 : \Omega \to E$ such that $F_1(t) \cap \overline{B(f_2(t), \varepsilon_2)} \neq \emptyset$ for all $t \in A_2$, where $A_2 \in \Sigma$ and $\mu(\Omega \setminus A_2) = 0$. In this way, we can construct a sequence $f_m : \Omega \to E$ of strongly measurable countably-valued functions and a sequence $(A_m)$ in $\Sigma$ with $\mu(\Omega \setminus A_m) = 0$ such that

$$F(t) \cap \left( \bigcap_{m=1}^{p} \overline{B(f_m(t), \varepsilon_m)} \right) \neq \emptyset \quad \text{for all } p \in \mathbb{N} \tag{5}$$

whenever $t \in A := \bigcap_{m \in \mathbb{N}} A_m$.

Fix $t \in A$. We claim that the sequence $(f_m(t))$ converges in norm to some point in $F(t)$. Indeed, given $j \geq i$ we can use (5) to find

$$x_j \in F(t) \cap \left( \bigcap_{m=1}^{j} \overline{B(f_m(t), \varepsilon_m)} \right),$$

so that $\|f_i(t) - x_j\| \leq \varepsilon_i$ and $\|f_j(t) - x_j\| \leq \varepsilon_j$, hence $\|f_i(t) - f_j(t)\| \leq \varepsilon_i + \varepsilon_j \leq 2\varepsilon_i$. This shows that $(f_m(t))$ is Cauchy and so it converges in norm. Since $x_j$ belongs to $F(t)$ and $\|f_j(t) - x_j\| \leq \varepsilon_j$ for all $j \in \mathbb{N}$, the limit of $(f_m(t))$ also belongs to $F(t)$, as claimed.

Let $f : \Omega \to E$ be a function such that $f(t) = \lim_{m \to \infty} f_m(t)$ whenever $t \in A$ and an arbitrary $f(t) \in F(t)$ whenever $t \in \Omega \setminus A$. Clearly, $f$ is a selector of $F$. Since each $f_m$ is strongly measurable and $\mu(\Omega \setminus A) = 0$, it follows that $f$ is strongly measurable and the proof is finished. □

The good thing regarding property (P) above is that beyond giving a new insight for the classical proof of Kuratowski and Ryll-Nardzewski's theorem it allowed us to characterize when a given multifunction does have measurable selectors.

In what follows the symbol $wk(E)$ (resp. $cwk(E)$) stands for the collection of all weakly compact (resp. convex weakly compact) non-empty subsets of the Banach space $E$.

**Theorem 4.1 ( [55, Theorem 2.5]).** *For a multifunction $F : \Omega \to wk(E)$ the following statements are equivalent:*

*(i) $F$ admits a strongly measurable selector.*
*(ii) $F$ satisfies property (P).*

*(iii)* *There exist a set of measure zero $\Omega_0 \in \Sigma$, a separable subspace $Y \subset E$ and a multifunction $G : \Omega \setminus \Omega_0 \to wk(Y)$ that is Effros measurable and such that $G(t) \subset F(t)$ for every $t \in \Omega \setminus \Omega_0$.*

We write $\delta^*(x^*, C) := \sup\{x^*(x) : x \in C\}$ for any set $C \subset E$ and any $x^* \in E^*$. A multifunction $F : \Omega \to 2^E$ is said to be scalarly measurable if for each $x^* \in E^*$ the function $t \mapsto \delta^*(x^*, F(t))$ is measurable. In particular a single valued function $f : \Omega \to E$ is scalarly measurable if the composition $x^* \circ f$ is measurable for every $x^* \in E^*$. Note that every Effros measurable multifunction $F$ is scalarly measurable.

Here is second result about scalar measurability for multifunctions that seems that has had some impact in integration for multifunctions.

**Theorem 4.2 ( [55, Theorem 3.8]).** *Every scalarly measurable multifunction $F : \Omega \to wk(E)$ admits a scalarly measurable selector.*

To finish let us mention the impact of measurable selections on multifunction integration. A multifunction $F : \Omega \to cwk(E)$ is said to be *Pettis integrable* if

- ► $\delta^*(x^*, F)$ is integrable for each $x^* \in E^*$;
- ► for each $A \in \Sigma$, there is $\int_A F\, d\mu \in cwk(E)$ such that

$$\delta^*\Big(x^*, \int_A F\, d\mu\Big) = \int_A \delta^*(x^*, F)\, d\mu \quad \text{for every } x^* \in E^*.$$

For the notion of Pettis ingegrability for single valued functions we refer to.[5]

**Theorem 4.3 ( [57, Theorem 2.5]).** *Let $F : \Omega \to cwk(E)$ be a Pettis integrable multifunction. Then $F$ admits a Pettis integrable selector.*

**Theorem 4.4 ( [57, Theorem 2.6]).** *Let $F : \Omega \to cwk(E)$ be a Pettis integrable multifunction. Then $F$ admits a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(E^*,w^*)}$ of Pettis integrable selectors such that*

$$F(\omega) = \overline{\{f_\alpha(\omega) :\ \alpha < \mathrm{dens}(E^*, w^*)\}} \quad \textit{for every } \omega \in \Omega.$$

*Moreover,*

$$\int_A F\, d\mu = \overline{\Big\{\int_A f\, d\mu :\ f \textit{ is a Pettis integrable selector of } F\Big\}}$$

*for every $A \in \Sigma$.*

## References


1. J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics, Vol. 96 (Springer-Verlag, New York, 1985).
2. M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach space theory*CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC (Springer, New York, 2011). The basis for linear and nonlinear analysis.
3. R. Engelking, *General topology* (PWN—Polish Scientific Publishers, Warsaw, 1977). Translated from the Polish by the author, Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60].
4. J. L. Kelley, *General topology* (Springer-Verlag, New York, 1975). Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27.
5. J. Diestel and J. J. Uhl, Jr., *Vector measures* (American Mathematical Society, Providence, R.I., 1977). With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.
6. J. B. Conway, *Functions of one complex variable*, Graduate Texts in Mathematics, Vol. 11, second edn. (Springer-Verlag, New York, 1978).
7. K. Floret, *Weakly compact sets*, Lecture Notes in Mathematics, Vol. 801 (Springer, Berlin, 1980). Lectures held at S.U.N.Y., Buffalo, in Spring 1978.
8. G. Köthe, *Topological vector spaces. I*Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159, Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159 (Springer-Verlag New York Inc., New York, 1969).
9. K. Kuratowski, *Topology. Vol. II*New edition, revised and augmented. Translated from the French by A. Kirkor, New edition, revised and augmented. Translated from the French by A. Kirkor (Academic Press, New York-London; Państwowe Wydawnictwo Naukowe Polish Scientific Publishers, Warsaw, 1968).
10. E. Klein and A. C. Thompson, *Theory of correspondences*Canadian Mathematical Society Series of Monographs and Advanced Texts, Canadian Mathematical Society Series of Monographs and Advanced Texts (John Wiley & Sons Inc., New York, 1984). Including applications to mathematical economics, A Wiley-Interscience Publication.
11. C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions* (Springer-Verlag, Berlin, 1977). Lecture Notes in Mathematics, Vol. 580.
12. C. Hess, Set-valued integration and set-valued probability theory: an overview, in *Handbook of measure theory, Vol. I, II*, (North-Holland, Amsterdam, 2002) pp. 617–673.
13. G. Debreu, Integration of correspondences, in *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1*, (Univ. California Press, Berkeley, Calif., 1967) pp. 351–372.
14. R. J. Aumann, *J. Math. Anal. Appl.* **12**, 1 (1965).
15. E. Michael, *Ann. of Math. (2)* **63**, 361 (1956).

16. Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, Vol. 48 (American Mathematical Society, Providence, RI, 2000).
17. J. E. Jayne and C. A. Rogers, *Selectors* (Princeton University Press, Princeton, NJ, 2002).
18. I. Namioka and R. R. Phelps, *Duke Math. J.* **42**, 735 (1975).
19. C. Stegall, *Israel J. Math.* **29**, 408 (1978).
20. R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 64 (Longman Scientific & Technical, Harlow, 1993).
21. B. Cascales, V. P. Fonf, J. Orihuela and S. Troyanski, *J. Funct. Anal.* **259**, 1346 (2010).
22. A. Moltó, J. Orihuela, S. Troyanski and M. Valdivia, *A nonlinear transfer technique for renorming*, Lecture Notes in Mathematics, Vol. 1951 (Springer-Verlag, Berlin, 2009).
23. K. Kuratowski and C. Ryll-Nardzewski, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13**, 397 (1965).
24. C. Angosto and B. Cascales, A new look at compactness via distances to function spaces, in *Advanced courses of mathematical analysis III*, (World Sci. Publ., Hackensack, NJ, 2008) pp. 49–66.
25. C. Angosto and B. Cascales, *J. Math. Anal. Appl.* **343**, 479 (2008).
26. B. Cascales, W. Marciszesky and M. Raja, *Topology Appl.* **153**, 2303 (2006).
27. C. Angosto and B. Cascales, *Topology Appl.* **156**, 1412 (2009).
28. J. Banaś and A. Martinón, *Portugal. Math.* **52**, 131 (1995).
29. A. Kryczka, S. Prus and M. Szczepanik, *Bull. Austral. Math. Soc.* **62**, 389 (2000).
30. C. Angosto, Distance to function spaces, PhD thesis, Universidad de Murcia2007.
31. K. Astala and H.-O. Tylli, *Math. Proc. Cambridge Philos. Soc.* **107**, 367 (1990).
32. F. S. D. Blasi, *Colloq. Math.* **65**, 333 (1992).
33. B. Cascales, O. F. K. Kalenda and J. Spurný, *Proc. Edinb. Math. Soc. (2)* **55**, 369 (2012).
34. M. Fabian, P. Hájek, V. Montesinos and V. Zizler, *Rev. Mat. Iberoamericana* **21**, 237 (2005).
35. A. S. Granero, P. Hájek and V. M. Santalucía, *Math. Ann.* **328**, 625 (2004).
36. A. S. Granero, *Rev. Mat. Iberoam.* **22**, 93 (2006).
37. A. S. Granero and M. Sánchez, Convexity, compactness and distances, in *Methods in Banach space theory*, , London Math. Soc. Lecture Note Ser. Vol. 337 (Cambridge Univ. Press, Cambridge, 2006) pp. 215–237.
38. A. Kryczka, *Bull. Austral. Math. Soc.* **69**, 49 (2004).
39. A. Kryczka and S. Prus, *Studia Math.* **147**, 89 (2001).
40. B. Cascales and J. Orihuela, A biased view of topology as a tool in functional analysis, in *Recent Progress in Topology III*, (Atlantis Press, 2013) pp. 93–165.
41. B. Cascales, J. Orihuela and M. R. Galán, Compactness, optimality and

risk, in *Computational and Analytical Mathematics*, eds. D. H. Bailey, H. H. Bauschke, P. Borwein, F. Garvan, M. Théra, J. D. Vanderwerff and H. Wolkowicz, Springer Proceedings in Mathematics & Statistics, Vol. 50 (Springer, New York, Amsterdam, 2013) pp. 153–207. In honor of Jonathan Borwein's 60th birthday, Papers from the workshop (JonFest) held at Simon Fraser University, Burnaby, BC, May 16–20, 2011.

42. B. Cascales and J. Orihuela, *Proc. Roy. Soc. Edinburgh Sect. A* **103**, 293 (1986).
43. B. Cascales, *Arch. Math. (Basel)* **49**, 232 (1987).
44. B. Cascales and J. Orihuela, *Math. Z.* **195**, 365 (1987).
45. B. Cascales and J. Orihuela, *J. Math. Anal. Appl.* **156**, 86 (1991).
46. B. Cascales, J. Ką̧kol and S. A. Saxon, *J. Math. Anal. Appl.* **269**, 500 (2002).
47. M. Talagrand, *Ann. of Math. (2)* **110**, 407 (1979).
48. A. Grothendieck, *Amer. J. Math.* **74**, 168 (1952).
49. J. L. Kelley and I. Namioka, *Linear topological spaces*, Graduate Texts in Mathematics, Vol. 36 (Springer-Verlag, New York, 1976). With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, and Kennan T. Smith, Second corrected printing.
50. K. Floret, Some aspects of the theory of locally convex inductive limits, in *Functional analysis: surveys and recent results, II (Proc. Second Conf. Functional Anal., Univ. Paderborn, Paderborn, 1979)*, , North-Holland Math. Stud. Vol. 38 (North-Holland, Amsterdam, 1980) pp. 205–237.
51. J. Ką̧kol, W. Kubiś and M. López-Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis* (Developments in Mathematics 24, Springer Verlag, New York, 2011).
52. V. V. Tkachuk, *Acta Math. Hungar.* **107**, 253 (2005).
53. B. Cascales, J. Orihuela and V. Tkachuk, *Topology Appl.* **158**, 204 (2011).
54. B. Cascales, M. Muñoz and J. Orihuela, *RACSAM* **106**, 341 (2012).
55. B. Cascales, V. Kadets and J. Rodríguez, *J. Convex Anal.* **17**, 229 (2010).
56. J. E. Jayne, J. Orihuela, A. J. Pallarés and G. Vera, *J. Funct. Anal.* **117**, 243 (1993).
57. B. Cascales, V. Kadets and J. Rodríguez, *J. Funct. Anal.* **256**, 673 (2009).