RADON-NIKODÝM INDEXES AND MEASURES OF WEAK NONCOMPACTNESS

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ABSTRACT. The aim of this paper is to present a quantitative version of the Radon-Nikodým Property and some of the main results related to it. This approach gives an extra insight to the classical results. We introduce two indexes: an index of representability of measures and an index of dentability. We review classic results in order to obtain relationships between these and other indexes.

1. INTRODUCTION

From its very beginning, Banach space theory has combined measure theory and point-set topological ideas with linear theory techniques to obtain powerful results and applications. In recent years a new trend in Banach spaces has started: the proposal is to deal with indexes related to topological or analytical properties (compactness, measurability, integrability, Dunford-Pettis property, etc.) and via inequalities between the indexes then offer a new and deeper look at the concepts under study. The advantages of this approach are that classical results can be sharpened, better understood and new results and applications can be found. Some recent publications along this line are [1, 2, 7, 11, 15, 16]

Throughout this paper $(E, \|\cdot\|)$ is a Banach space and (Ω, Σ, μ) is a complete probability space. By $\mathcal{L}^1(\mu, E)$ we denote the subspace of E^Ω which consists of Bochner integrable functions, and by $L^1(\mu, E)$ the Banach space of equivalence classes of Bochner integrable functions endowed with its usual norm. If $E = \mathbb{R}$ we simply write $\mathcal{L}^1(\mu)$ and $L^1(\mu)$ respectively. All our vector measures $m : \Sigma \to E$ are assumed to be countably additive, μ-continuous (*i.e.* $m(A) = 0$ whenever $A \in \Sigma$ and $\mu(A) = 0$) and of bounded variation.

The aim of this paper is to introduce and study indexes related to the so-called Radon-Nikodým property in Banach spaces in order to establish inequalities between them. In this way, we assign to a given $m : \Sigma \to E$ an *"index of representability"* $\mathcal{R}(m)$, see Definition 2.2, that can be characterized as the infimum of the constants δ for which there exists $g \in L^1(\mu, E)$ satisfying

$$\left\| m(A) - \int_A g \, d\mu \right\| \leq \delta \, \mu(A), \quad \text{for every } A \in \Sigma,$$

In proposition 3.3 we strengthen a result by Musial [17, theorem 11.1, p. 236] in order to prove that for every vector measure $m : \Sigma \to E^*$ there exists a *"weak"*

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Radon-Nikodým derivative $\psi : \Omega \to E^*$ in the sense that

$$\langle x, m(A)\rangle = \int_A \langle x, g(t)\rangle \, d\mu, \qquad \text{for every } A \in \Sigma \text{ and every } x \in E,$$

whose (strong) measurability is equivalent to the representability of m. The quantitative version of this last claim is established in proposition 3.5 as

$$(\diamondsuit) \qquad \mathcal{R}(m) = \mathrm{d}(\psi, \mathcal{M}(\mu, E^*))$$

where $\mathcal{M}(\mu, E^*)$ is the subset of $(E^*)^\Omega$ made up of all strongly (μ-) measurable functions. The *distances* considered here are the usual $\inf$ distances for sets in the subjacent space: in E^Ω we consider the $\sup$ distance, see section 3 for details. Our study for vector measures is completed with the corresponding study for *"representability of operators"*. We also give several examples.

In section 4 we introduce the index of dentability $\mathrm{Dent}(C)$ of a set $C \subseteq E$ as the infimum of those ε for which C has nonempty slices of radius small than ε. We study equivalent formulations of this index and its relationship with the index of representability of vector measures (see theorem 4.8).

Two noticeable properties for Dent are established. First we prove (see theorem 4.18) that every ω^*-compact convex set $C \subseteq E^*$ satisfies that

$$(\spadesuit) \qquad \begin{aligned} &\sup\{\mathrm{Dent}(D) : D \subseteq C\} \leq \sup\{\mathrm{Frag}(D) : D \subseteq C\} \leq \\ &\leq 2\ \sup\{\mathrm{Dent}(D) : D \subseteq C,\, D \text{ countable}\}, \end{aligned}$$

where $\mathrm{Frag}(D)$ is defined as the infimum of those ε for which D has nonempty ω^*-relatively open sets of radius small than ε.

Second, we show that

$$(\clubsuit) \qquad \mathrm{Dent}(C) \leq \gamma(C),$$

for any convex bounded subset $C \subseteq E$, see proposition 4.20: here γ is the measure of weak noncompactness defined by double limits

$$\gamma(H) := \sup\left\{\left|\lim_n \lim_m x_m^*(x_n) - \lim_m \lim_n x_m^*(x_n)\right| : (x_m^*)_m \subseteq B_{E^*}, (x_n)_n \subseteq H\right\}$$

where the supremum is taken over those sequences for which the previous limits exist (we refer to [16] for more information about γ).

We stress that inequalites ($\diamondsuit$), ($\spadesuit$) and ($\clubsuit$) and their proofs summarize, sharpen and offer new ways to the theory of Radon-Nikodým property as studied by many authors in our references. We believe that beyond the applications that we offer in this paper, the techniques developed here can be useful for other purposes in the theory of Banach spaces.

1.1. **Notation and terminology.** Our notation and terminology is standard and it is either explained when needed or can be found in our standard references for Banach spaces [8] and vector measures and integration [5].

The letters E and F are reserved to denote real Banach spaces with their norms $\|\cdot\|$. If $x \in E$ and $r > 0$ then $B[x, r]$ (resp. $B(x, r)$) is the closed (resp. open) ball of radius r centered at x; B_E and S_E are, respectively, the closed unit ball and the unit sphere of E. $L(E, F)$ is the space of bounded linear operators from E into F endowed with its norm

$$\|T\| := \sup\{\|T(x)\| : x \in B_E\}.$$

By $E^* := L(E, \mathbb{R})$ –respectively, $E^{**} := L(E^*, \mathbb{R})$– we denote the topological dual –respectively, bidual– of E. The weak topology in E is denoted by ω, and ω^* is the weak* topology in E^*. For $x \in E$ and $x^* \in E^*$ we write $\langle x, x^* \rangle = \langle x^*, x \rangle := x^*(x)$.

Given a set $D \subseteq E$ we denote by $\mathrm{Ext}(D)$ the set of extremal points of D and by $\mathrm{co}\,(D)$ its convex hull. Recall that the diameter of D is defined as

$$\mathrm{diam}(D) = \sup\{\|x - y\| : x, y \in D\}.$$

and its radius is given by

$$\mathrm{rad}(D) = \inf\{\delta > 0 : \text{ there exists } x \in E \text{ such that } D \subseteq B(x, \delta)\ \}.$$

Our agreement for the paper is that $\inf \emptyset = +\infty$. We will use $+\infty$ assuming, as usual, that $\lambda \cdot (+\infty) = +\infty$ for $\lambda > 0$ and that $\delta < +\infty$ for every $\delta \in \mathbb{R}$.

For the probability space (Ω, Σ, μ) and $B \in \Sigma$, we write $\Sigma_B^+ := \{B' \in \Sigma : B' \subseteq B, \mu(B') > 0\}$ and $\Sigma^+ := \Sigma_\Omega^+$. Given $B \in \Sigma^+$, the average range of $m|_B$ is denoted by

$$\Gamma_B := \left\{ \frac{m(C)}{\mu(C)} : C \in \Sigma_B^+ \right\}. \tag{1}$$

Technically speaking Γ_B depends on m and μ but since most of the time we will only work with these two measures m and μ we avoid the tedious and selfexplanatory terminology $\Gamma_B^{\mu,m}$ unless it is strictly needed. Γ_Ω, the *average range* of m, is usually denoted by $\mathrm{AR}(m)$.

As previously said, throughout this paper m is assumed to be μ-continuous and of bounded variation. The variation of m is denoted by $|m|$.

By $L^\infty(\mu, E)$ we denote the space of equivalence classes of μ-essentially bounded strongly measurable functions $f : \Omega \to E$ endowed with its natural "ess sup"norm $\|\cdot\|_\infty$. $\mathcal{L}^\infty(\mu, E)$ stands for the subset of E^Ω of all functions belonging to some equivalence class of $L^\infty(\mu, E)$. If $E = \mathbb{R}$ we simply write $L^\infty(\mu)$ and $\mathcal{L}^\infty(\mu)$ respectively.

2. Representability of measures

The reader should not have any difficulty when proving the next well-known fact that follows. In case a reference is needed the following ones might well serve: [5, Corollary II.1.3], [4, Proposition 2.2] and [2].

Proposition 2.1. *Let $f : \Omega \to E$ be a function. The following conditions are equivalent:*

(i) *for every $\varepsilon > 0$ there is a countable partition $\{A_0, A_1, \dots\}$ of Ω in Σ such that $\mu(A_0) = 0$ and $\mathrm{diam}(f(A_n)) < \varepsilon$ for every $n \in \mathbb{N}$;*
(ii) *for every $\varepsilon > 0$ and every $A \in \Sigma^+$ there exists $B \in \Sigma_A^+$ such that $\mathrm{diam}\big(f(B)\big) \leq \varepsilon$;*
(iii) *f is strongly measurable.*

Next we define the index of *representability* of a vector measure.

Definition 2.2. *If $m : \Sigma \to E$ is a μ-continuous vector measure of bounded variation, its* index of representability *$\mathcal{R}(m)$ is defined as*

$$\mathcal{R}(m) := \inf\{\varepsilon > 0 : \textit{for every } A \in \Sigma^+ \textit{ there exists } B \in \Sigma_A^+ \textit{ such that } \mathrm{rad}(\Gamma_B) < \varepsilon\}$$

Recall that m as above is said to be representable if there exists $f \in L^1(\mu, E)$ such that $m(A) = \int_A f d\mu$ for every $A \in \Sigma$. The following proposition that offers a quantitative version of [5, Lemma 6, p. 135] states that the previous index gives us an estimate of how far is m from being representable.

Proposition 2.3. *Let $m : \Sigma \to E$ be a vector measure as in definition 2.2. If $\mathcal{R}(m) < \delta$ then there exists $g \in L^1(\mu, E)$ satisfying*

$$\left\| m(A) - \int_A g \, d\mu \right\| \leq \delta \, \mu(A), \textit{ for every } A \in \Sigma^+. \tag{2}$$

Conversely, if $\delta \geq 0$ satisfies (2) *for some Bochner integrable function f then $\mathcal{R}(m) \leq \delta$.*

Proof. If $\mathcal{R}(m) = +\infty$ we can take $g = 0$ and (2) is obviously true. If $\mathcal{R}(m) < \delta < +\infty$, by an exhaustion argument we can find a partition $\{B_n : n = 0, 1, 2, ...\}$ in Σ of Ω such that

$$\mu(B_0) = 0 \text{ and for every } n \in \mathbb{N}, \mu(B_n) > 0 \text{ and } \operatorname{rad}(\Gamma_{B_n}) < \delta.$$

Given $n \in \mathbb{N}$ take $x_n \in E$ verifying $\Gamma_{B_n} \subseteq B(x_n, \delta)$. For every $A \in \Sigma$ and $n \in \mathbb{N}$ we have that

$$\|m(A \cap B_n) - x_n \mu(A \cap B_n)\| \leq \delta \mu(A \cap B_n). \tag{3}$$

Indeed, if $\mu(A \cap B_n) = 0$ then $m(A \cap B_n) = 0$ and the inequality is clear. Otherwise $\mu(A \cap B_n) > 0$ and (3) follows from $\Gamma_{B_n} \subseteq B(x_n, \delta)$. On the one hand, taking $A = \Omega$ in the above inequality and with the help of the triangle inequality we have that

$$\sum_{n=1}^{\infty} \|x_n\| \mu(B_n) \leq |m|(\Omega) + \delta < +\infty,$$

and consequently $g := \sum_{n=1}^{\infty} x_n \chi_{B_n}$ is Bochner integrable. On the other hand, inequality (3) implies that

$$\left\| m(A) - \int_A g \, d\mu \right\| \leq \sum_{n=1}^{\infty} \|m(A \cap B_n) - x_n \mu(A \cap B_n)\| \leq \delta \mu(A).$$

Conversely, suppose that $\delta \geq 0$ verifies (2) for some $g \in L^1(\mu, E)$ and every $A \in \Sigma^+$. Since g is strongly measurable, for every $\varepsilon > 0$ there exists $B \in \Sigma_A^+$ such that $\operatorname{diam}(g(B)) < \varepsilon$. Fix $C \in \Sigma_B^+$. It follows from inequality (2) that

$$\left\| \frac{m(C)}{\mu(C)} - \frac{\int_C g \, d\mu}{\mu(C)} \right\| \leq \delta.$$

On the other hand, [5, Corollary II.2.8] implies that

$$\frac{\int_C g \, d\mu}{\mu(C)} \in \overline{\operatorname{co}}(g(C)) \subseteq \overline{\operatorname{co}}(g(B)).$$

Since $\operatorname{diam}(g(B)) = \operatorname{diam}(\overline{\operatorname{co}}(g(B))) < \varepsilon$, we deduce that

$$\left\| \frac{m(C)}{\mu(C)} - \frac{\int_B g \, d\mu}{\mu(B)} \right\| \leq \left\| \frac{m(C)}{\mu(C)} - \frac{\int_C g \, d\mu}{\mu(C)} \right\| + \left\| \frac{\int_C g \, d\mu}{\mu(C)} - \frac{\int_B g \, d\mu}{\mu(B)} \right\| < \delta + \varepsilon.$$

Thus $\operatorname{rad}(\Gamma_B) < \delta + \varepsilon$ for every $\varepsilon > 0$ and the proof is over. $\square$

It follows from the above proposition that $\mathcal{R}(m)$ is the infimum of those $\delta > 0$ satisfying (2), so $\mathcal{R}(m) = 0$ if m is representable. The converse is also true: suppose that $\mathcal{R}(m) = 0$ and use repeatedly inequality (2) for $\delta = 1/n$ to find a sequence of functions $(f_n)_n \subseteq L^1(\mu, E)$ such that

$$\left\| m(A) - \int_A f_n \, d\mu \right\| \leq \frac{1}{n}\, \mu(A), \text{ for every } A \in \Sigma.$$

Consequently

$$\left\| \int_A f_p \, d\mu - \int_A f_q \, d\mu \right\| \leq \left(\frac{1}{p} + \frac{1}{q} \right) \mu(A), \text{ for every } A \in \Sigma.$$

Using [5, p. 46, theorem 4 (iv)] the above inequality leads to

$$\|f_p - f_q\|_1 = \int_\Omega \|f_p - f_q\| d\mu \leq \left(\frac{1}{p} + \frac{1}{q} \right) \mu(\Omega) = \frac{1}{p} + \frac{1}{q},$$

which implies that $(f_n)_n$ is a Cauchy sequence in $L^1(\mu, E)$, so it converges to a function $f \in L^1(\mu, E)$. If $A \in \Sigma$ we can use the triangular inequality to deduce that

$$\left\| m(A) - \int_A f \, d\mu \right\| \leq \frac{1}{n} \mu(A) + \|f - f_n\|_1 \text{ for every } n \in \mathbb{N}$$

and therefore $m(A) = \int_A f \, d\mu$ for each $A \in \Sigma$.

Next proposition gathers the first properties of $\mathcal{R}(\cdot)$.

Proposition 2.4. *Let $m, m' : \Sigma \to E$ be μ-continuous vector measures of bounded variation and $\alpha \in \mathbb{R}$. The index of representability satisfies the following properties:*

(i) $\mathcal{R}(m + m') \leq \mathcal{R}(m) + \mathcal{R}(m')$.
(ii) *If $T : E \to F$ is a bounded linear operator then $\mathcal{R}(T \circ m) \leq \|T\| \mathcal{R}(m)$.*
(iii) $\mathcal{R}(\alpha m) = |\alpha| \mathcal{R}(m)$.

Proof. If $\mathcal{R}(m)$ or $\mathcal{R}(m')$ are infinite then property (i) is clear. Note that if we agree that $0 \cdot (+\infty) = 0$ and $\mathcal{R}(m) = +\infty$ then (ii) is also satisfied, so we will assume in both cases that the indexes are finite.

(i) Suppose that $\delta > \mathcal{R}(m)$, $\delta' > \mathcal{R}(m')$. Given $A \in \Sigma^+$ there exists $C \in \Sigma_A^+$ such that $\operatorname{rad}(\Gamma_C^m) < \delta$. But we can also find $D \in \Sigma_C^+$ with $\operatorname{rad}(\Gamma_D^{m'}) < \delta'$, so there exist $x, x' \in E$ with $\Gamma_D^m \subseteq B(x, \delta)$ and $\Gamma_D^{m'} \subseteq B(x', \delta')$. Using triangular inequality one deduces that $\Gamma_D^{m+m'} \subseteq B(x + x', \delta + \delta')$, which leads to the result.
(ii) Let $T : E \to F$ be a bounded linear operator, then $T \circ m$ is an F-valued vector measure. Now observe that if $\Gamma_C^m \subseteq B(x, \delta)$ then $\Gamma_C^{T \circ m} \subseteq B(T(x), \|T\|\delta)$, so $\mathcal{R}(T \circ m) \leq \|T\| \mathcal{R}(m)$.
(iii) This is a consequence of (ii). If $\alpha = 0$ then the relation is clear, while $\alpha \neq 0$ implies that $T : E \to E$ given by $T(x) = \alpha x$ verifies $\|T\| = 1/\|T^{-1}\| = \alpha$, so $\mathcal{R}(m) \leq \frac{1}{|\alpha|} \mathcal{R}(T \circ m) \leq \mathcal{R}(m)$.

□

In the examples that follow we compute the index of representability of some well-known examples of vector measures which are not representable.

Example 2.5. [3, Example 2.1.2],[5, Example III.1.2] *Consider a probability space (Ω, Σ, μ) where μ is non-atomic and let $m : \Sigma \to L^1(\mu)$ be the vector measure defined for each $A \in \Sigma$ by*

$$m(A) := \chi_A.$$

Then $\mathcal{R}(m) = 1$.

Proof. It is clear that m is a μ-continuous vector measure whose average range $\mathrm{AR}(m)$ is contained in B_E, so $\mathcal{R}(m) \leq 1$. Since μ is non-atomic, for every $A \in \Sigma^+$ we can find disjoint sets $B, B' \in \Sigma_A^+$. Hence

$$\left\| \frac{\chi_B}{\mu(B)} - \frac{\chi_{B'}}{\mu(B')} \right\|_1 = \int_B \frac{1}{\mu(B)}\, d\mu + \int_{B'} \frac{1}{\mu(B')}\, d\mu = 2$$

which implies that $\mathrm{rad}(\Gamma_A) \geq 1$ and consequently $\mathcal{R}(m) \geq 1$. $\square$

Example 2.6. *Let λ be the Lebesgue measure on $[0,1]$ and $\mathcal{M}$ the Lebesgue measurable sets. For $E = c_0$ the Banach space with the supremum norm $\|\cdot\|_\infty$, consider the vector measure $m : \mathcal{M} \to c_0$ given by*

$$m(A) = \left(\int_A r_n(t)\, d\lambda \right)_n$$

where $(r_n)_n$ are the Rademacher functions. Then $\mathcal{R}(m) = 1$.

Proof. Since $(r_n)_n$ is an orthonormal sequence in $L^2(\lambda)$ we conclude that $m(A) \in c_0$. On the other hand, m is clearly a finitely additive measure that satisfies

$$\|m(A)\|_\infty \leq \lambda(A) \text{ for every } A \in \mathcal{M}$$

From the above inequality it follows that m is countably additive, λ-continuous and of bounded variation. It also follows that $\mathcal{R}(m) \leq 1$. Let us suppose that there exists a function $f \in L^1(\lambda, E)$, $f = (f_1, f_2, ...)$ and a constant $0 \leq c < 1$ such that

$$\left\| m(A) - \int_A f\, d\lambda \right\|_\infty \leq c\, \lambda(A), \text{ for every } A \in \mathcal{M}.$$

In particular, each component verifies

$$\left| \int_A r_n(t)\, d\lambda - \int_A f_n\, d\lambda \right| \leq c\, \lambda(A), \text{ for every } A \in \mathcal{M}. \tag{4}$$

Thus

$$\int_A |r_n(t) - f_n(t)|\, d\lambda \leq c\, \lambda(A), \text{ for every } A \in \mathcal{M}. \tag{5}$$

We deduce from (5) that for every n there exists a λ-null set $A_n \in \mathcal{M}$ such that $t \in [0,1] \setminus A_n$ implies $|r_n(t) - f_n(t)| \leq c$, so $|f_n(t)| \geq 1 - c$. Hence, for each $t \in [0,1] \setminus \bigcup_n A_n$ we have that

$$|f_n(t)| > 1 - c, \text{ for every } n \in \mathbb{N},$$

which contradicts the fact that f takes its values in c_0 almost surely and allows us to conclude that $\mathcal{R}(m) \geq 1$. $\square$

2.1. **Representability of operators.** Let us recall that a bounded linear operator $T : L^1(\mu) \to E$ is said to be (Riesz) representable if there exists $g \in L^\infty(\mu, E)$ such that

$$T(f) = \int_\Omega fg \, d\mu,$$

see [5, Section 1. Chapter III]. In this case we have that $\|T\| = \|g\|_\infty$. The set of all representable operators is a closed subspace of $L(L^1(\mu), E)$ that we will denote by $L_{rep}(L^1(\mu), E)$.

If T is an operator as above, then $m : \Sigma \to E$ defined as $m(A) = T(\chi_A)$ is a μ-continuous vector measure whose average range is contained in $T(B_{L^1(\mu)})$. On the other hand, each μ-continuous vector measure with bounded average range extends to an operator $T \in L(L^1(\mu), E)$ verifying $m(A) = T(\chi_A)$ for every $A \in \Sigma$. The following proposition is a quantitative version of [5, lemma III.1.4].

Proposition 2.7. *Let $T : L^1(\mu) \to E$ be a bounded linear operator and let $m : \Sigma \to E$ be the vector measure defined by $m(A) = T(\chi_A)$ for every $A \in \Sigma$. Then*

$$\mathcal{R}(m) = \mathrm{d}(T, L_{rep}(L^1(\mu), E))$$

where the distance is in the norm-operator.

Proof. Since $\mathrm{AR}(m)$ is bounded we have that $\mathcal{R}(m)$ is finite. Given $\delta > \mathcal{R}(m)$ and according to Proposition 2.3 there exists $g \in L^1(\mu, E)$ such that

$$\left\| m(A) - \int_A g \, d\mu \right\| \le \delta\mu(A), \text{ for every } A \in \Sigma. \tag{6}$$

Using $\|m(A)\| \le \|T\|\mu(A)$, inequality (6) tells us that

$$\left\| \int_A g \, d\mu \right\| \le \big(\delta + \|T\|\big)\mu(A), \text{ for every } A \in \Sigma$$

By [5, p. 46, theorem 4 (iv)] this last inequality leads to

$$\int_A \|g\| \, d\mu \le \big(\delta + \|T\|\big)\mu(A), \text{ for every } A \in \Sigma,$$

from where it follows that $\|g\| \le \delta + \|T\|$ almost everywhere, and therefore g belongs to $L^\infty(\mu, E)$.

If $S : L^1(\mu) \to E$ is the representable operator given by $S(f) = \int_A fg \, d\mu$, $f \in L^1(\mu)$, then inequality (6) can be read as

$$\|T(\chi_A) - S(\chi_A)\| \le \delta \, \mu(A), \text{ for every } A \in \Sigma.$$

From here we obtain that

$$\|T(f) - S(f)\| \le \delta\|f\|_1, \text{ for every } f \in L^1(\mu),$$

which says that $\mathrm{d}(T, L_{rep}(L^1(\mu), E)) \le \delta$; since $\delta > \mathcal{R}(m)$ is arbitrary we conclude that $\mathrm{d}(T, L_{rep}(L^1(\mu), E)) \le \mathcal{R}(m)$.

The converse inequality $\mathrm{d}(T, L_{rep}(L^1(\mu), E)) \ge \mathcal{R}(m)$ straightforwardly follows from Proposition 2.3 and is left to the reader. $\square$

3. GELFAND DERIVATIVE

In this section we will use the tools introduced in the previous one to study properties of strong measurability of ω^*-densities of vector measures and operators.

The following concepts are needed here: a function $\psi : \Omega \to E^*$ is said to be ω^*-*scalarly measurable* (resp. μ-*Gelfand integrable*) if for every $x \in E$ the function $\langle x, \psi\rangle : \Omega \to \mathbb{R}$ given by $t \mapsto \langle x, \psi(t)\rangle$ is measurable (resp. μ-integrable). If ψ is Gelfand integrable, then for each $A \in \Sigma$ there exists a vector $\int_A \psi\, d\mu \in E^*$ (called the *Gelfand integral* of ψ over A) satisfying

$$\langle x, \int_A \psi\, d\mu\rangle = \int_A \langle x, \psi\rangle\, d\mu \text{ for every } x \in E.$$

For basic information about the Gelfand integral see [5, p. 53-54].

With the help of the lifting theorem (see below) it can be proved that given a μ-continuous (norm) vector measure of bounded variation $m : \Sigma \to E^*$ there exists a Gelfand integrable function $\psi : \Omega \to E^*$ satisfying

$$\text{(I)}\ \langle x, m(A)\rangle = \int_A \langle x, \psi(t)\rangle d\mu,$$

for every $x \in E$ and $A \in \Sigma$, see [17, theorem 11.1, p. 236]. If we think in terms of continuous linear operators $T : L^1(\mu) \to E^*$, the application of the previous result to the vector measure $m(A) = T(\chi_A)$ provides a ω^*-measurable function $\psi : \Omega \to E^*$ with the property

$$\text{(I')}\ \langle x, T(f)\rangle = \int_\Omega \langle x, \psi(t)\rangle f(t) d\mu$$

for every $x \in E$ and for every $f \in L^1(\mu)$. It is pointed out in [5, p. 84], that the existence of such a ω^*-measurable function $\psi : \Omega \to E^*$ satisfying (I') is *"mostly an esthetic generalization* of the representation results that deal with Bochner integrabiliy*, because the measurability properties of the kernel ψ are not, in general, strong enough to exhibit structural properties of the operator under representation"*. In terms of vector measures the following example, [17, example 3.1, p. 186], illustrate the comment above.

Example 3.1. *Let μ be the Lebesgue measure on $[0, 1]$, Σ the Lebesgue measurable sets and $m : \Sigma \to \ell_2([0, 1])$ the null vector measure i.e. $m(A) = 0$ for every $A \in \Sigma$. Clearly the null function is a (strongly measurable) Gelfand derivative (in fact the Radon-Nikodým derivative of m). On the other hand, let $\{e_t : t \in [0, 1]\}$ be the canonical base of the Hilbert space $\ell_2([0, 1])$. The function $f : [0, 1] \to \ell_2([0, 1])$ defined by $f(t) = e_t$, $t \in [0, 1]$ satisfies that $\langle x, f(t)\rangle = 0$, μ-almost everywhere, for every $x \in \ell_2([0, 1])$. So f is ω^*-meaurable and satisfies (I) but it is not strongly measurable.*

Our aim now is to prove that for vector measures m as those considered here a ω^*-measurable function ψ verifying (I) can be obtained with an additional property that avoids the commented pathologies.

In what follows we fix a lifting $\rho : \Sigma \to \Sigma$ on (Ω, Σ, μ), see [13, p. 46, Theorem 3], [9, 341K] and [21, Appendix G]. Recall that ρ satisfies the following properties:

(1) If $A, B \in \Sigma$ and $\mu(A\Delta B) = 0$ then $\rho(A) = \rho(B)$;
(2) $\mu(\rho(A)\Delta A) = 0$ for every $A \in \Sigma$;

(3) $\rho(A \cap B) = \rho(A) \cap \rho(B)$ for every $A, B \in \Sigma$;
(4) $\rho(\emptyset) = \emptyset$, $\rho(\Omega) = \Omega$;
(5) $\rho(\Omega \backslash A) = \Omega \backslash \rho(A)$ for every $A \in \Sigma$;
(6) $\rho(A \cup B) = \rho(A) \cup \rho(B)$ for every $A, B \in \Sigma$.

We note that the lifting ρ can be extended to a map $\rho : L^\infty(\mu) \to \mathcal{L}^\infty(\mu)$ that assigns to each equivalence class $f \in L^\infty(\mu)$ a function $\rho(f) \in f$.

The following lemma is needed later. The definition of Γ_B below is given in equation (1).

Lemma 3.2. *If $B \in \Sigma^+$, then $\Gamma_B = \Gamma_{\rho(B)}$.*

Proof. Observe that for any $B' \in \Sigma_B^+$ we have that $\rho(B') \subseteq \rho(B)$, $\mu(\rho(B')) = \mu(B')$ and $m(\rho(B')) = m(B')$ by μ-continuity. Therefore

$$\frac{m(B')}{\mu(B')} = \frac{m(\rho(B'))}{\mu(\rho(B'))} \in \Gamma_{\rho(B)},$$

which means that $\Gamma_B \subseteq \Gamma_{\rho(B)}$.

Conversely, if $B' \subseteq \rho(B)$ then $B' = B' \cap \big(B \cup (\rho(B) \backslash B)\big)$ and $\mu(\rho(B) \backslash B) = 0$ implies that $\mu(B' \cap B) = \mu(B')$ and $m(B' \cap B) = m(B')$. Consequently

$$\frac{m(B')}{\mu(B')} = \frac{m(B' \cap B)}{\mu(B' \cap B)} \in \Gamma_B,$$

and the proof is over. □

Proposition 3.3. *If $m : \Sigma \to E^*$ is a μ-continuous (norm) vector measure of bounded variation, then there is a ω^*-measurable function $\psi : \Omega \to E^*$ with the following properties:*

(I) $\langle x, m(A) \rangle = \int_A \langle x, \psi(t) \rangle d\mu$ *for every $x \in E$ and $A \in \Sigma$;*
(II) *if $A \in \Sigma$, then $\psi(t) \in \overline{\Gamma_A}^{\omega^*}$ for μ-almost every $t \in A$.*

Proof. The existence of a ω^*-measurable function $\psi : \Omega \to E^*$ verifying condition (I) is proved by Musial in [17, theorem 11.1, p. 236]. Here we refine his construction to obtain a function that also satisfies the second condition.

We assume first that there exists $M > 0$ such that $\|m(A)\| \leq M\mu(A)$ for each $A \in \Sigma$. Following Musial's proof, for every $x \in B_E$ we consider a μ-Radon-Nikodým derivative $g_x \in L^\infty(\mu)$ of the scalar measure $\langle x, m \rangle$. The function $\psi : \Omega \to E^*$ given by $\psi(t)(x) = \rho(g_x)(t)$ is well-defined, ω^*-measurable and satisfies (I).

We show now that ψ also verifies (II). For every finite partition π of Ω into Σ-sets we write

$$s_\pi := \sum_{A \in \pi} \frac{m(A)}{\mu(A)} \chi_A$$

Fix $x \in B_E$. By [5, lemma 1, p. 67] the net

$$\langle x, s_\pi \rangle = \sum_{A \in \pi} \frac{\langle x, m \rangle (A)}{\mu(A)} \chi_A$$

converges in $L^\infty(\mu)$ to g_x when considering partitions ordered by refinement and with the agreement that $0/0 = 0$. Hence

$$\rho(\langle x, s_\pi \rangle) = \sum_{A \in \pi} \frac{\langle x, m \rangle (A)}{\mu(A)} \chi_{\rho(A)}$$

converges to $\rho(g_x)$ uniformly in Ω because of the continuity property of ρ, see [13, p. 35].

With all the above, if we define

$$\rho(s_\pi) := \sum_{A\in\pi} \frac{m(A)}{\mu(A)} \chi_{\rho(A)},$$

then the net $\rho(s_\pi)(t)$ ω^*-converges to $\psi(t)$ for every $t \in \Omega$. For the given A, if $t \in A \cap \rho(A)$ then $\psi(t)$ is also the ω^*-limit of $\rho(s_\pi)(t)$ when taking only partitions π finer than $\{\rho(A), \Omega \setminus \rho(A)\}$, so $\psi(t) \in \overline{\Gamma_{\rho(A)}}^{\omega^*}$. Bearing in mind now lemma 3.2 we conclude that $\psi(t) \in \overline{\Gamma_A}^{\omega^*}$ for μ-almost every $t \in A$, and the validity of condition (II) is stated.

Now we deal with the general case. Fix a partition $\{\Omega_n : n \in \mathbb{N}\}$ of Ω, such that for every $n \in \mathbb{N}$ there exists $M_n > 0$ verifying

$$\|m(A \cap \Omega_n)\| \leq M_n \mu(A \cap \Omega_n),$$

for each $A \in \Sigma$, see [5, p. 63, theorem 5]. Defining $A_n := \rho(\Omega_n)$, we obtain a sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets with $\mu(\Omega \setminus \bigcup_n A_n) = 0$ and satisfying

$$\|m(A \cap A_n)\| \leq M_n \mu(A \cap A_n),$$

for every $A \in \Sigma$. Note that the restriction of ρ to Σ_{A_n} is a lifting on (A_n, Σ_{A_n}, μ) so we can apply the first part in order to deduce the existence of a ω^*-measurable function $\psi_n : A_n \to E^*$ satisfying the conditions (I) and (II). It is clear that $\psi : \Omega \to E^*$ defined as $\psi = \sum_{n=1}^{\infty} \psi_n \chi_{A_n}$ is ω^*-measurable. If $x \in E$ then

$$\int_{\cup_{i=1}^{n} A_n} |\langle x, \psi\rangle| \, d\mu = \sum_{i=1}^{n} \int_{A_i} |\langle x, \psi\rangle| \, d\mu =$$
$$= \sum_{i=1}^{n} |\langle x, m\rangle| \,(A_i) \leq \|x\| |m|(\Omega) < \infty.$$

so $\langle x, \psi\rangle \in L^1(\mu)$ by the monotone convergence theorem. Moreover, the dominated convergence theorem implies that

$$\langle x, m(A)\rangle = \sum_{n=1}^{\infty} \langle x, m(A \cap A_n)\rangle = \sum_{n=1}^{\infty} \int_{A\cap A_n} \langle x, \psi_n\rangle \, d\mu = \int_A \langle x, \psi\rangle \, d\mu$$

for every $A \in \Sigma$.

Our function ψ also satisfies (II). Indeed, if $A \in \Sigma$ then for μ-almost every $t \in \bigcup_n (A \cap A_n)$ the element $\psi(t) \in \bigcup_n \overline{\Gamma_{A\cap A_n}}^{\omega^*} \subseteq \overline{\Gamma_A}^{\omega^*}$. □

Definition 3.4. *If $m : \Sigma \to E^*$ is a μ-continuous (norm) vector measure of bounded variation, we say that a ω^*-measurable function $\psi : \Omega \to E^*$ is a* Gelfand derivative *of m if it satisfies (I) and (II) of proposition 3.3.*

If $f, g \in (E^*)^\Omega$ the distance between f and g is

$$\mathrm{d}(f, g) := \sup\{\|f(t) - g(t)\| : t \in \Omega\} \in [0, +\infty]$$

For $F \subseteq E^*$ we write $\mathcal{M}(\mu, F, E^*)$ (resp. $\mathcal{L}^1(\mu, F, E^*)$) to denote the subset of all functions in $(E^*)^\Omega$ which are strongly measurable (resp. Bochner integrable) and μ-essentially F-valued.

Proposition 3.5. *Let $m : \Sigma \to F$ be a μ-continuous measure of bounded variation with values in a Banach space F. If E^* is a dual Banach space which contains isometrically F ($i : F \hookrightarrow E^*$) then every Gelfand derivative $\psi : \Omega \to E^*$ of $i \circ m$ verifies that*

$$\mathcal{R}(m) = \mathrm{d}(\psi, \mathcal{M}(\mu, F, E^*)) = \mathrm{d}(\psi, \mathcal{L}^1(\mu, F, E^*))$$

In particular, if $\mathcal{R}(m) = 0$ then ψ is a Radon-Nikodým derivative of m.

Proof. The proof is similar to the one of [2, theorem 2.3]. We identify the elements of F with their image in E^* so that we can write $\Gamma_B = \Gamma_B^{\mu,m} = \Gamma_B^{\mu,i\circ m}$.

Suppose that $\mathrm{d}(\psi, \mathcal{M}(\mu, F, E^*)) < \varepsilon$ so we can find $g \in \mathcal{M}(\mu, F, E^*)$ satisfying $\|\psi(t) - g(t)\| < \varepsilon$ for every $t \in \Omega$. If $(s_n)_{n\in\mathbb{N}}$ is a sequence of simple F-valued functions that converges μ-a.e. to g then, by Egorov's theorem [6, p. 94, theorem 1], there exists a set D verifying $\mu(\Omega \setminus D) < \mu(A)$ and $n \in \mathbb{N}$ with $\|\psi(t) - s_n(t)\| < \varepsilon$ for every $t \in D$. Clearly $A \cap D \in \Sigma^+$, and using that s_n is simple there are $y \in F$ and $B \in \Sigma^+_{A\cap D}$ with $\|\psi(t) - y\| < \varepsilon$ for all $t \in B$. Hence, if $B' \in \Sigma^+_B$ and $x \in B_E$ we have that

$$\left|\langle x, m(B')\rangle - \mu(B')\langle x, y\rangle\right| = \left|\int_{B'} \langle x, \psi\rangle \, d\mu - \int_{B'} \langle x, y\rangle \, d\mu\right| \le \varepsilon \, \mu(B')$$

Taking the supremum on $x \in B_E$ we deduce $\|m(B') - \mu(B')y\| \le \varepsilon\mu(B')$. Therefore $\Gamma_B \subseteq B(y, \varepsilon)$ and $\mathrm{rad}(\Gamma_B) < \varepsilon$. This shows that $\mathcal{R}(m) \le \mathrm{d}(\psi, \mathcal{M}(\mu, F, E^*))$.

Fix $\varepsilon > \mathcal{R}(m)$. Given $A \in \Sigma^+$ there exists $\widetilde{B} \in \Sigma^+_A$ and $y \in F$ such that $\overline{\Gamma_{\widetilde{B}}}^{\omega^*}$ is contained in the ball $B_{E^*}(y, \varepsilon)$. If B is the set of all $t \in \widetilde{B}$ verifying that $\psi(t)$ belongs to this ball, then $B \in \Sigma^+_{\widetilde{B}}$ by property (II) and the fact that μ is complete. In other words, for every $A \in \Sigma^+$ there exists $B \in \Sigma^+_A$ and $y \in E^*$ with $\|\psi(t) - y\| < \varepsilon$ for each $t \in B$. By an exhaustion argument we can construct a countable family of disjoint sets $\{B_n : n \in \mathbb{N}\}$ with $\mu(\Omega \setminus \cup_n B_n) = 0$ and a sequence of elements $(y_n)_{n\in\mathbb{N}}$ in F with the previous property so that $g = \sum_n y_n \chi_{B_n} + f\chi_{\Omega\setminus\cup B_n}$ is an integrable function satisfying $\|\psi(t) - g(t)\| < \varepsilon$ for every $t \in B$. The fact that g is integrable is consequence of the monotone convergence theorem since

$$\int_{\cup_{i=n}^m B_n} \|g\| \, d\mu = \sum_{n=1}^m \|y_n\|\mu(B_n) \le \sum_{n=1}^m \left(\varepsilon + \frac{\|m(B_n)\|}{\mu(B_n)}\right)\mu(B_n) \le \varepsilon + |m|(\Omega)$$

for every $m \in \mathbb{N}$. This shows that $\mathrm{d}(\psi, \mathcal{L}^1(\mu, F, E^*)) \le \mathcal{R}(m)$.

On the other hand, the inequality $\mathrm{d}(\psi, \mathcal{M}(\mu, F, E^*)) \le \mathrm{d}(\psi, \mathcal{L}^1(\mu, F, E^*))$ is obvious.

Observe that the proof shows that $\mathcal{R}(m)$ is finite if and only if any of the distances $\mathrm{d}(\psi, \mathcal{M}(\mu, F, E^*))$ or $\mathrm{d}(\psi, \mathcal{L}^1(\mu, F, E^*))$ is finite, so the equality remains true if one of these indexes is infinity.

Finally, if $\mathcal{R}(m) = 0$ then $\mathrm{d}(\psi, \mathcal{L}^1(\mu, F, E^*)) = 0$ so ψ is a Bochner integrable function with values in F almost everywhere that satisfies

$$\langle x, m(A)\rangle = \int_A \langle x, \psi(t)\rangle d\mu = \langle x, \int_A \psi \, d\mu\rangle \text{ for every } x \in E \text{ and } A \in \Sigma,$$

Therefore

$$m(A) = \int_A \psi \, d\mu \text{ for every } A \in \Sigma.$$

□

Remark 3.6.

(i) *Notice that although a Gelfand derivative* $\psi : \Omega \to E^*$ *may take values out of* F *we can control* $\mathrm{d}(\psi(t), F) \leq \mathcal{R}(m)$ *for* μ*-almost every* $t \in \Omega$.

(ii) *If we replace in (II) the condition*

$$\psi(t) \in \overline{\Gamma_A}^{\omega^*}$$

by the condition (II')

$$\psi(t) \in \overline{\mathrm{co}}^{\omega^*}\Gamma_A,$$

then proposition 3.5 is also valid. As of now we do not know if this fact can be useful to some further study.

Immediate consequences of the previous propositions are the following corollaries.

Corollary 3.7. *Let* $m : \Sigma \to F$ *be a* μ*-continuous measure of bounded variation with values in a Banach space* F. *If* $i : F \hookrightarrow F^{**}$ *is the canonical inclusion in the bidual then every Gelfand derivative* $\psi : \Omega \to F^{**}$ *of* $i \circ m$ *verifies that*

$$\mathcal{R}(m) = \mathrm{d}(\psi, \mathcal{M}(\mu, F, F^{**})) = \mathrm{d}(\psi, \mathcal{L}^1(\mu, F, F^{**}))$$

In particular, if $\mathcal{R}(m) = 0$ *then* ψ *is a Radon-Nikodým derivative of* m.

Corollary 3.8. *Let* $T : L^1(\mu) \to F$ *be a continuous linear operator. For every dual Banach space* E^* *containing* F *isometrically (*$i : F \hookrightarrow E^*$*) there is a* ω^**-measurable function* $\psi : \Omega \to E^*$ *with the following properties:*

(A) $\langle x, i \circ T(f)\rangle = \int_\Omega \langle x, \psi(t)\rangle f(t) d\mu$ *for every* $x \in E$ *and* $f \in L^1(\mu)$;

(B) *if* $A \in \Sigma$ *then* $\psi(t) \in \overline{\left\{\frac{i \circ T(\chi_B)}{\mu(B)} : B \in \Sigma_A^+\right\}}^{\omega^*}$ *for* μ*-almost every* $t \in A$.

For every function $\psi : \Omega \to E^*$ *satisfying (A) and (B) we have that*

$$\mathrm{d}(T, L_{rep}(L^1(\mu), F)) = \mathrm{d}(\psi, \mathcal{M}(\mu, F, E^*)) = \mathrm{d}(\psi, \mathcal{L}^1(\mu, F, E^*)) \in [0, \infty]$$

In addition, if T *is representable then* ψ *is a density of* T.

Proof. Combine propositions 2.7 and 3.5. □

Example 3.9. *Let* $([0,1], \mathcal{M}, \lambda)$ *be the Lebesgue measure and* $m : \mathcal{M} \to L^1(\lambda)$ *the vector measure given by* $m(A) = \chi_A$. *For every lifting* ρ *on the previous probability space, the function* $\psi : [0,1] \to L^1(\lambda)^{**}$ *that associates to each* $t \in \Omega$ *the finitely additive measure* $\nu_t : \mathcal{M} \to \mathbb{R}$ *defined as*

$$\nu_t(A) = \begin{cases} 0 & \text{if } t \notin \rho(A) \\ 1 & \text{if } t \in \rho(A) \end{cases}$$

is a Gelfand derivative of m. *Moreover* $\mathcal{R}(m) = 1$ *and* $\mathrm{d}(\psi(t), L^1(\lambda)) = 1$ *for* λ*-almost every* $t \in [0,1]$.

Proof. We consider $L^1(\lambda)$ as a subspace of $L^1(\lambda)^{**}$ which consists of all finitely additive λ-continuous (scalar) measures on $\mathcal{M}$ endowed with the norm $|\nu|(\Omega)$ (total variation). The space $L^1(\lambda)$ is identified with its countably additive elements (see [23, theorem 2.3, p. 53]). The function ψ is well-defined and verifies for every $A, B \in \Sigma$

$$\int_A \langle \chi_B, \nu_t\rangle \, d\lambda = \int_A \int_{[0,1]} \chi_B \, d\nu_t \, d\lambda = \int_A \chi_{\rho(B)}(t) \, d\lambda = \langle \chi_B, m(A)\rangle.$$

Extending this expression by linearity for simple functions and by density to all the elements of $L^\infty(\lambda)$ we deduce that ψ verifies (I).

We have now to check (II). Given $A \in \mathcal{M}$ we will see that $\psi(s) \in \overline{\Gamma_A}^{\omega^*}$ for every $s \in A \cap \rho(A)$. Fix a ω^*-open neighbourhood of $\psi(s)$ of the form

$$W = \left\{ y^{**} \in L^1(\lambda)^{**} : |\langle y^{**} - \psi(s), h_i \rangle| < \varepsilon \text{ for every } i = 1, ..., m \right\}$$

where $h_i \in L^\infty(\lambda)$. We are going to show that it intersects Γ_A. Since simple functions are dense in $L^\infty(\lambda)$, we can assume that each function h_i can be written as $h_i = \sum_{j=1}^{k_i} a_j^i \chi_{\rho(A_j^i)}$ where the family $\{\rho(A_j^i) : j = 1, ..., k_i\}$ is a finite partition of Ω for every $i = 1, ..., m$. Denoting by C the intersection of $\rho(A) \cap A$ with the sets $\rho(A_j^i)$ containing s among its elements, we deduce that $C \in \mathcal{M}_A^+$. Moreover for every i we have that

$$\left| \left\langle \frac{\chi_C}{\mu(C)} - \psi(s), h_i \right\rangle \right| = \left| \sum_{j=1}^{k_i} a_j^i \left(\frac{\mu(C \cap \rho(A_j^i))}{\mu(C)} - \nu_s(\rho(A_j^i)) \right) \right| = \left| \sum_{j=1}^{k_i} a_j^i 0 \right| = 0.$$

Therefore ψ is a Gelfand derivative of m.

We know that $\mathcal{R}(m) = 1$ by example 2.5, but we also have that $\mathrm{d}(\psi(t), L^1(\mu)) = 1$ for almost every $t \in [0,1]$. To see this last assertion note that for almost every $t \in [0,1]$ we can find a decreasing sequence of sets $A_n = \rho(A_n)$ containing t and with $\lim_n \lambda(A_n) = 0$. Therefore, given a λ-continuous countably additive measure $\eta : \mathcal{M} \to \mathbb{R}$ we have that

$$\|\nu_t - \eta\| \geq |1 - \eta(A_n)| \text{ for every } n \in \mathbb{N}\,.$$

Using that $\lim_n \eta(A_n) = 0$ we conclude the result. □

Example 3.10. *Let* $([0,1], \mathcal{M}, \lambda)$ *be as above and consider the vector measure* $m : \mathcal{M} \to c_0$ *from example 2.6 given by*

$$m(A) = \left(\int_A r_n(t)\, d\lambda \right)_n$$

The function $\psi : [0,1] \to \ell^\infty$ *defined as* $\psi(t) = (r_n(t))_n$ *is a Gelfand derivative of* m. *Moreover* $\mathcal{R}(m) = 1$ *and* $\mathrm{d}(\psi(t), c_0) = 1$ *for* λ*-almost every* $t \in [0,1]$.

Proof. Fix an arbitrary $A \in \Sigma^+$. For every $(x_n)_n \in \ell^1 = c_0^*$ we have that

$$\int_A \sum_{n=1}^\infty x_n r_n(t)\, d\lambda = \sum_{n=1}^\infty \int_A r_n(t) x_n\, d\lambda$$

by the dominated convergence theorem, which shows (I).

To see (II) take the subset A' of A which consists of all elements of A that remain when removing those $t \in A$ which are dyadic numbers, (i.e. those of the form $t = k/2^n$ for some $k, n \in \mathbb{N}$) as well as those which belong to a μ-null set of the form

$$\{t \in A : (r_n(t))_{n=1}^m = (a_n)_{n=1}^m\}$$

for some $(a_n)_{n=1}^m \in \{-1,1\}^m$, $m \in \mathbb{N}$. Note that we have removed a countable number of μ-null sets, so $A' \subseteq A$ verifies $\mu(A) = \mu(A')$.

We are going to prove that $(r_n(s))_n \in \overline{\Gamma_A}^{\omega^*}$ for each $s \in A'$, which will show (II). In order to simplify the notation we will prove that a ω^*-open neighbourhood of $(r_n(s))_n$ of the form

$$V = \left\{ (y_n)_n \in \ell_\infty : \left| \sum_{n=1}^{\infty} x_n (y_n - r_n(s)) \right| < \varepsilon \right\} \text{ (for some } (x_n)_n \in \ell^1 \text{)}$$

intersects Γ_A. The general case of an arbitrary ω^*-neighbourhood is analogous.

We start by fixing $m_0 \in \mathbb{N}$ such that $\sum_{n>m_0} |x_n| < \varepsilon/2$. If $s \in A'$ then $B := \{t \in A' : (r_n(t))_{n=1}^{m_0} = (r_n(s))_{n=1}^{m_0}\}$ is an element of Σ with positive measure (because of the construction of A'), so

$$\frac{\int_B r_n(t)\, d\mu}{\mu(B)} = \frac{r_n(s)\mu(B)}{\mu(B)} = r_n(s) \qquad \text{for } n = 1, ..., m_0.$$

Therefore $m(B)/\mu(B) \in V$ since

$$\left| \sum_{n=1}^{\infty} x_n \left(\frac{\int_B r_n(t)\, d\mu}{\mu(B)} - r_n(s) \right) \right| \leq 2 \sum_{n>m_0} |x_n| < \varepsilon.$$

We know that $\mathcal{R}(m) = 1$ by example 2.6. Finally, note that the sequence $\psi(t)$ has range in $\{-1, 1\}^{\mathbb{N}}$ for every t which is not dyadic, so $\mathrm{d}(\psi(t), c_0) = 1$ for λ-almost everywhere. □

4. Index of dentability

The concept of dentability and its relationship with the Radon-Nikodým property was originally studied by Rieffel [20], who showed that a Banach space has the Radon-Nikodým Property whenever every bounded subset of it is dentable. Aftwerwards, Huff [12] proved the converse.

If $C \subseteq E$, a slice of C is a nonempty set S of the form $S = C \cap H$ where H is a ω-open half-space, i.e.

$$H = \{x \in E : \langle x, x^* \rangle < \alpha\}, \text{for some } x^* \in E, \alpha \in \mathbb{R}$$

Definition 4.1. *For each subset C of E define:*

$$\mathrm{Dent}(C) = \inf\{\varepsilon > 0 : \textit{there is a slice } S \textit{ of } C \textit{ with } \mathrm{rad}(S) < \varepsilon\}$$

Note that $\mathrm{Dent}(C) \leq \mathrm{rad}(C)$ for every $C \subseteq E$.

The following theorem is a version for radii of a celebrated result by Asplund, Namioka and Bourgain [3, p. 51, theorem 3.4.1] for diameters. Since the proof is totally analogue we omit it.

Theorem 4.2. *Suppose that $\varepsilon > 0$. Let J, K_0, K_1 be closed bounded convex subsets of E and $z \in E$ such that*

(i) $J \subseteq \overline{\mathrm{co}}(K_0 \cup K_1)$
(ii) $K_0 \subset J$ *and* $K_0 \subset B(z, \varepsilon)$
(iii) $J \setminus K_1 \neq \emptyset$

Then for every $\varepsilon' > \varepsilon$ there is a slice of J which contains a point of K_0 and is contained in $B(z, \varepsilon')$.

The following lemma is a version for radius of [10, p. 482, Proposition 2.3].

Lemma 4.3. *Suppose that C is a bounded subset of E that admits a slice $S = C \cap H$ contained in $\bigcup_{i=1}^{n} B(x_i, \varepsilon)$. Then for every $\varepsilon' > \varepsilon$ there is a slice S' of C contained in $B(x_k, \varepsilon')$ for some $k \in \{1, ..., n\}$.*

Proof. For $n = 1$ it is clear. If $n = 2$ we can assume that $C \subseteq \overline{\text{co}}(C \setminus B(x_1, \varepsilon))$, since otherwise we can use the separation's theorem to obtain a new slice S' of C contained in $B(x_1, \varepsilon)$. Fix $\varepsilon' > \delta > \varepsilon$ and write $K_0 := \overline{\text{co}}(C \cap B(x_2, \varepsilon))$, $K_1 := \overline{\text{co}}(C \cap H^c)$ and $J := \overline{\text{co}}(C)$. These are closed convex sets verifying the following conditions: (i) $J \subseteq \overline{\text{co}}(C \setminus B(x_1, \varepsilon)) \subseteq \overline{\text{co}}(K_0 \cup K_1)$, (ii) $K_0 \subseteq J$ and $K_0 \subseteq B(x_2, \delta)$, (iii) $J \setminus K_1 \neq \emptyset$ since if $J \setminus K_1 = \emptyset$ then $J \subseteq K_1 \subseteq H^c$ so $S = C \cap H = \emptyset$, which is absurd. By theorem 4.2 there exists a slice S' of J contained in $B(x_2, \varepsilon')$. Thus $S' \cap C$ is a (nonempty) slice of C contained in the same ball.

Suppose now that $n > 1$ and the result is valid for slices contained in less than n balls of radius ε. Fix $\varepsilon < \delta < \varepsilon'$. By the induction hypothesis $C \setminus B(x_n, \varepsilon)$ has a slice $(C \setminus B(x_n, \varepsilon)) \cap H$ contained in some $B(x_k, \delta)$ for some $k \in \{1, ..., n-1\}$. Then $C \cap H$ is a slice of C contained in $B(x_k, \delta) \cup B(x_n, \delta)$. By the case $n = 2$ we conclude the result. $\square$

In some references [3, 5] the dentability of C is defined as the infimum of all $\varepsilon > 0$ such that there exists $x_0 \in C$ with $C \nsubseteq \overline{\text{co}}(C \setminus B(x_0, \varepsilon))$. This motivates another obvious definition of an index of dentability taking the infimum of all $\varepsilon > 0$ for which the previous assertion is true. It can be shown that this new index does not coincide with the one we introduced, although it can be established an equivalence between them in terms of an inequality. However, if we allow that the center x_0 can be out of C then both formulations coincide, as the following lemma shows. This lemma also includes a quantitative version of [10, p. 482-83, Corollary 2.4]

Lemma 4.4. *Let C be a bounded subset of E. Then*

$$\begin{aligned}\mathrm{Dent}(C) &= \inf\{\varepsilon > 0 : \text{ there is } D \subseteq C \text{ with } \mathrm{rad}(D) < \varepsilon \text{ and } C \nsubseteq \overline{\text{co}}(C \setminus D)\} \\ &= \inf\{\varepsilon > 0 : \text{ there is a slice } S \text{ of } C \text{ with } \alpha(S) < \varepsilon\}. \\ &= \inf\{\varepsilon > 0 : \text{ there is } D \subseteq C \text{ with } \alpha(D) < \varepsilon \text{ and } C \nsubseteq \overline{\text{co}}(C \setminus D)\}\end{aligned}$$

where α is the Kuratowski measure of noncompactness

$$\alpha(D) = \inf\{\varepsilon > 0 : \text{ there exists } (D_i)_{i=1}^n \text{ with } D = \bigcup_{i=1}^n D_i \text{ and } \mathrm{rad}(D_i) < \varepsilon\}$$

Proof. Denote by L_1, L_2 and L_3 the infimums from top to bottom, respectively. The equality $\mathrm{Dent}(A) = L_1$ is easy: if D is a subset of C with $\mathrm{rad}(D) < \varepsilon$ and $C \nsubseteq \overline{\text{co}}(C \setminus D)$ then using separation's theorem we can find a slice S of C contained in $C \setminus \overline{\text{co}}(C \setminus D) \subseteq D$. Conversely, if S is a slice of C with $\mathrm{rad}(S) < \varepsilon$ simply take $D = S$.

For the second equality note that $L_2 \leq \mathrm{Dent}(C)$ obviously. The converse inequality is a direct consequence of lemma 4.3.

Finally, $L_3 \leq L_1 = \mathrm{Dent}(C)$ by definition. On the other hand, if $L_3 < \delta$ and $D \subseteq C$ verifies $\alpha(D) < \delta$ and $C \nsubseteq \overline{\text{co}}(C \setminus D)$, then by separation's theorem we can find a slice S of C contained in D, so $\alpha(S) < \delta$. This shows that $\mathrm{Dent}(C) = L_2 \leq L_3$. $\square$

Proposition 4.5. *If C is a bounded subset of E then*

$$\mathrm{Dent}(C) = \mathrm{Dent}(\mathrm{co}(C)) = \mathrm{Dent}(\overline{\text{co}}(C)).$$

Proof. It is clear that $\mathrm{Dent}(C) \leq \mathrm{Dent}(\mathrm{co}(C)) \leq \mathrm{Dent}(\overline{\text{co}}(C))$ since every slice of $\overline{\text{co}}(C)$ contains a nonempty slice of $\mathrm{co}(C)$ by density, and the last one contains a nonempty slice of C by convexity. We just have to show that $\mathrm{Dent}(\overline{\mathrm{co}}(C)) < \varepsilon$ if $\mathrm{Dent}(C) < \varepsilon$. Fix a slice $S = H \cap C$ of C with $\mathrm{rad}(H \cap C) < \varepsilon$. The closed convex sets $K_0 := \overline{\mathrm{co}}(H \cap C)$, $K_1 := \overline{\mathrm{co}}(H^c \cap C)$ and $J := \overline{\mathrm{co}}(C)$ verify: (i) $J \subseteq \overline{\mathrm{co}}(K_0 \cup K_1)$, (ii) $K_0 \subseteq J$ and $\mathrm{rad}(K_0) < \varepsilon$, (iii) $J \backslash K_1 \neq \emptyset$, since otherwise $J = K_1 \subseteq H^c$ implies $C \cap H = \emptyset$.

Hence we can apply theorem 4.2 and deduce that there exists a slice S of J which contains a point of K_0 and with radius less than ε, so $\mathrm{Dent}(\overline{\mathrm{co}}(C)) < \varepsilon$. □

Proposition 4.6. *If C is a bounded subset of E then*

$$\mathrm{Dent}(C) \leq 2 \sup \{\mathrm{Dent}(D) : D \subseteq C,\ D \textit{ is countable}\} \tag{7}$$

Proof. Suppose that $\mathrm{Dent}(C) > \delta$. We are going to construct a sequence $(D_n)_n$ of finite subsets of C satisfying for each $n \in \mathbb{N}$

(a) $D_{n+1} \cap B(z, \delta) = \emptyset$ if $z \in \bigcup_{i=1}^n D_i$.

(b) $\mathrm{co}(D_{n+1}) \cap B(z, \frac{1}{n+1}) \neq \emptyset$ whenever $z \in \bigcup_{i=1}^n D_i$.

This will finish the proof since the countable set $D = \bigcup_n D_n$ verifies $\mathrm{Dent}(D) \geq \delta/2$. To see this, take any subset F of D with radius less than $\delta/2$. If $z \in F$ then F is contained in $B(z, \delta)$. Suppose that $z \in D_n$, then

$$D \cap B(z, \delta) \subseteq \bigcup_{i=1}^{n} D_i \subseteq \mathrm{co}(\bigcup_{i=n+1}^{\infty} D_i) \subseteq \mathrm{co}(D \setminus B(z, \delta)) \subseteq \mathrm{co}(D \setminus F).$$

by properties (a) and (b), and therefore $D \subseteq \mathrm{co}\,(D \setminus F)$.

We will construct such family by induction. Fix an arbitrary point $D_1 = \{x\}$ of C. Since $x \in C \subseteq \overline{\mathrm{co}}(C \setminus B(x, \delta))$ we can find a finite family of points in $C \setminus B(x, \delta)$ whose convex hull intersects $B(x, 1/2)$. Take D_2 as the finite family of points.

By induction, suppose that we have constructed $(D_i)_{i=1}^n$ verifying conditions (a) and (b) and call $F_n = \bigcup_{i=1}^n D_i$. Since $\alpha\left(\bigcup_{z \in F_n} B(z, \delta) \cap C\right) < \delta$ we have that

$$C \subseteq \overline{\mathrm{co}}\left(C \setminus \bigcup_{z \in F_n} B(z, \delta)\right).$$

For every $z_0 \in F_n$ there exists a finite family of points in $C \setminus \bigcup_{z \in F_n} B(z, \delta)$ whose convex hull intersects $B(z_0, 1/(n+1))$. Consider D_{n+1} the union of all those finite families. This is a finite set that satisfies (a) and (b) by construction. □

The following example shows that constant 2 in (7) is sharp even if C is closed and convex.

Example 4.7. Let $\ell^\infty([0, 1])$ be the family of all real-valued bounded functions on $[0, 1]$ endowed with the supremum norm. Consider the closed subspace E of $\ell^\infty([0, 1])$ made up of all the functions with countable support. We note that $(E, \|\cdot\|_\infty)$ is a Banach space.

Take the convex and closed set $C \subseteq E$ which consists of all functions f whose range is contained in $[0, 2]$.

Every countable subset D of C verify $\mathrm{Dent}(D) \leq 1$: if S is the union of the supports of the functions in D then S must be countable. Then, the characteristic function χ_S verify that $D \subseteq B[\chi_S, 1]$.

On the other hand $\mathrm{Dent}(C) \geq 2$ since if $D \subseteq C$ has radius $\mathrm{rad}(D) < \delta < 2$ then $C \subseteq \overline{\mathrm{co}}(C \setminus D)$. To prove this last claim suppose that $D \subseteq B(f, \delta)$, $f \in E$.

Then for each $t \notin \mathrm{support}(f)$ we have that $g(t) \in [0, \delta]$ whenever $g \in D$. Now take an arbitrary $g \in D$ and fix $m \in \mathbb{N}$. Since $\mathrm{support}(f)$ is countable, we can take m points $t_1, ..., t_m \in [0,1] \setminus (\mathrm{support}(f) \cup \mathrm{support}(g))$. Define m functions $g_1, ..., g_m$ as $g_i = g + 2\chi_{\{t_i\}}$ for every $i = 1, ..., m$. Then

$$\left| \frac{g_1(t) + ... + g_m(t)}{m} - g(t) \right| \leq \frac{2}{m} \text{ for every } t \in [0,1]$$

and $g_i \notin D$ since $g_i(t_i) = 2 > \delta$ and $t_i \notin \mathrm{support}(f)$.

The connection between representability of measures and dentability is exhibited in the following theorem which is a quantitative version of [20, p.71, theorem 1] and it shows that we can replace rad by Dent in the definition of index of representability. The proof is inspired on [3, p. 21, lemma 2.2.5].

Theorem 4.8. *Let $m : \Sigma \to E$ be a μ-continuous vector measure of bounded variation. Then*

$$\mathcal{R}(m) = \inf\{\varepsilon > 0 : \textit{for every } A \in \Sigma^+ \textit{ there exists } B \in \Sigma_A^+ \textit{ with } \mathrm{Dent}(\Gamma_B) < \varepsilon\}$$

Proof. Let us denote by L the infimum of the right hand side of the the expression above. It is clear that $L \leq \mathcal{R}(m)$ since $\mathrm{Dent}(\Gamma_B) \leq \mathrm{rad}(\Gamma_B)$ for every $B \in \Sigma^+$. Let us prove the converse inequality. If $L = \infty$ then the equality holds; otherwise fix an arbitrary $\varepsilon > L$. Given $A \in \Sigma^+$ there exists $B \in \Sigma_A^+$ and a ball $B(z, \varepsilon)$ such that $\Gamma_B \not\subseteq \overline{\mathrm{co}}(\Gamma_B \setminus B(z, \varepsilon))$. Take $B' \in \Sigma_B^+$ with

$$\frac{m(B')}{\mu(B')} \notin \overline{\mathrm{co}}(\Gamma_B \setminus B(z, \varepsilon)). \tag{8}$$

If we show that there exists $B'' \in \Sigma_{B'}^+ \subseteq \Sigma_A^+$ with $\Gamma_{B''} \subseteq B(z, \varepsilon)$ then the proof is over. We will check this by contradiction: suppose that for every $B'' \in \Sigma_{B'}^+$ there exists $B''' \in \Sigma_{B''}^+$ with $\left\| \frac{m(B''')}{\mu(B''')} - z \right\| \geq \varepsilon$. Construct a maximal family $\mathcal{C} = \{B_i''' : i \in \mathbb{N}\}$ of disjoint subsets of B' having positive measure (it has to be countable since $\mu(B') < \infty$) verifying

$$\left\| \frac{m(B_i''')}{\mu(B_i''')} - z \right\| \geq \varepsilon, \text{ for every } i \in \mathbb{N}$$

If we write $B_0''' = \bigcup_{n \in \mathbb{N}} B_n'''$ then $\mu(B' \setminus B_0''') = 0$ by maximality. Hence

$$\frac{m(B')}{\mu(B')} = \sum_{i=1}^{\infty} \frac{\mu(B_i''')}{\mu(B_0''')} \frac{m(B_i''')}{\mu(B_i''')} \in \overline{\mathrm{co}}(\Gamma_B \setminus B(z, \varepsilon))$$

which contradicts (8). □

Corollary 4.9. *Let $m : \Sigma \to E$ be a μ-continuous vector measure of bounded variation. Then*

$$\mathcal{R}(m) \leq \sup\{\mathrm{Dent}(C) : C \subseteq \mathrm{AR}(m)\}$$

In particular, if every subset of E is dentable then m is representable.

4.1. **Fragmentability in dual spaces.** In [14], Jayne and Rogers introduced the following concept: a topological space (X, τ) is *fragmented by a metric ρ on X* if for every subset C of X and $\varepsilon > 0$ there exists a τ-open subset U of X such that $C \cap U$ has ρ-diameter less than ε. The connection between RNP and ω^*-fragmentability in Banach spaces was originally established by Namioka, Phelps and Stegall (see [18] for more details and references). They proved that every subset of E^* is dentable if and only if every subset of (E^*, ω^*) is fragmentable. We are going to quantify this result in terms of indexes.

Definition 4.10. *For each subset C of E^* define*

$$\mathrm{Frag}(C) = \inf \{\varepsilon > 0 : \textit{ there is } U \in \omega^* \textit{ such that } \mathrm{rad}\,(C \cap U) < \varepsilon\}$$

See [1] for related indexes.

A ω^*-slice of a subset C of E^* is a nonempty set $S = H \cap C$ where H is a ω^*-open halfspace of the form

$$H = \{x^* \in E^* : \langle x, x^* \rangle > \alpha\}$$

for some $\alpha \in \mathbb{R}$ and $x \in E$.

The following theorem is a version for the ω^*-topology of theorem 4.2 with identical proof. See also [3, p. 52, theorem 3.4.1] for the original enunciate.

Theorem 4.11. *Suppose that $\varepsilon > 0$. Let J, K_0, K_1 be ω^*-compact convex subsets of E^* and $z^* \in E^*$ such that*

(i) $J \subseteq \mathrm{co}\,(K_0 \cup K_1)$
(ii) $K_0 \subset J$ *and* $K_0 \subset B(z^*, \varepsilon)$
(iii) $J \setminus K_1 \neq \emptyset$

Then for every $\varepsilon' > \varepsilon$ there is a ω^*-slice S of J which contains a point of K_0 and is contained in $B(z^*, \varepsilon')$.

An analogue result to lemma 4.3 for dual spaces with the ω^*-topology can be proved.

Lemma 4.12. *Suppose that C is a subset of E^* that admits a ω^*-slice $S = H \cap C$ contained in $\bigcup_{i=1}^n B(x_i^*, \varepsilon)$. Then for every $\varepsilon' > \varepsilon$ there exists a ω^*-slice S' of C contained in $B(x_k^*, \varepsilon')$ for some $k \in \{1, ..., n\}$.*

The following proposition is based on the proof of [19, p. 737, lemma 3].

Proposition 4.13. *Let C be a bounded subset of E^*. Then*

$$\begin{aligned}\mathrm{Frag}(\mathrm{Ext}(\overline{\mathrm{co}}^{\omega^*}(C))) &= \inf \{\varepsilon > 0 : \textit{there is } \omega^*\textit{-slice } S \textit{ of } C \textit{ with } \mathrm{rad}(S) < \varepsilon\} \\ &= \inf \{\varepsilon > 0 : \textit{there is } \omega^*\textit{-slice } S \textit{ of } C \textit{ with } \alpha(S) < \varepsilon\}\end{aligned}$$

where α is the Kuratowski measure of noncompactness (see lemma 4.4).

Proof. The proof of the equality between the two infimums of the right hand of the expression is totally analogue to the one of lemma 4.4. We will denote this common value by L.

We start by showing that $\mathrm{Frag}(\mathrm{Ext}(\overline{\mathrm{co}}^{\omega^*}(C))) \leq L$. Suppose that $H \cap C$ is a ω^*-slice of C with radius less than ε. The convex ω^*-compact sets $J = \overline{\mathrm{co}}^{\omega^*}(C)$, $K_0 = \overline{\mathrm{co}}^{\omega^*}(H \cap C)$ and $K_1 = \overline{\mathrm{co}}^{\omega^*}(H^c \cap C)$ verify: (i) $J \subseteq \overline{\mathrm{co}}^{\omega^*}(K_0 \cup K_1)$, (ii) $K_0 \subseteq J$ and $\mathrm{rad}(K_0) = \mathrm{rad}(H \cap C) < \varepsilon$, (iii) $J \setminus K_1 \neq \emptyset$ since otherwise $C \subseteq J \subseteq K_1 \subseteq H^c$ implies that $C \cap H = \emptyset$.

By theorem 4.11 we deduce that there exists a ω^*-slice $S' = H' \cap J$ of J with $\operatorname{rad}(S') < \varepsilon$. Therefore $H' \cap \operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C)) \neq \emptyset$ is a relatively ω^*-open subset of $\operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C))$ with radius less than ε.

To prove that the equality $\operatorname{Frag}(\operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C))) = L$ holds we will suppose that $\operatorname{Frag}(\operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C))) < L$ and get a contradiction. There exists a ω^*-open set U with $\operatorname{rad}(U \cap \operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C))) < L$. If $x_0^* \in U \cap \operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C))$, by Choquet's lemma [8, p.111 , lemma 3.6.9], there is a ω^*-open halfspace H such that x_0^* belongs to $H \cap \operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C)) \subseteq U \cap \operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C))$. Note that $H \cap \operatorname{Ext}(\overline{\operatorname{co}}^{\omega^*}(C))$ has radius less than L, so we can reason as above and get a ω^*-slice $H' \cap J$ of $J = \overline{\operatorname{co}}^{\omega^*}(C)$ with radius less than L. The intersection $H' \cap C$ is a nonempty slice of C with $\operatorname{rad}(H' \cap C) < L$. □

Proposition 4.14. *Suppose that U is a relatively ω^*-open subset of $\operatorname{Ext}(C)$ for some ω^*-compact convex subset $C \subseteq E^*$ with $\operatorname{Frag}(\operatorname{Ext}(C)) > \delta$. We can find sequences $(x_n^*)_n \subseteq U$, $(x_n)_n \subseteq B_E$ and $(\alpha_n)_n \subseteq \mathbb{R}$ such that for every $n \in \mathbb{N}$*

$$\langle x_n, x_n^* \rangle > \alpha_n + \delta > \alpha_n > \langle x_n, x_k^* \rangle \quad \textit{whenever } k \neq n \tag{9}$$

Proof. We construct this sequence by induction. Fix $x_1^* \in U$. We have that $U \not\subseteq B(x_1^*, \delta)$ since $\operatorname{Frag}(\operatorname{Ext}(C)) > \delta$, so we can find $x_2^* \in U$, $\alpha_1 \in \mathbb{R}$, $a \in B_E$ such that $\langle a, x_1^* \rangle > \alpha + \delta > \alpha > \langle a, x_2^* \rangle$ by Hanh-Banach. Take $x_1 = a$, $\alpha_1 = \alpha$ and $x_2 = -a$, $\alpha_2 = -\alpha - \delta$.

Suppose that $n \geq 2$ and we have constructed $(x_i^*)_{i=1}^n \subseteq U$, $(x_i)_{i=1}^n \subseteq B_E$ and $(\alpha_i)_{i=1}^n \subseteq \mathbb{R}$ for $i = 1, ..., n$ satisfying

$$\langle x_i, x_i^* \rangle > \alpha_i + \delta > \alpha_i > \sup_{k \neq i} \{\langle x_i, x_k^* \rangle\}$$

for every $i = 1, ..., n$.

Consider the relatively ω^*-open subset of $\operatorname{Ext}(C)$ given by

$$W_n := \bigcap_{i=1}^{n-1} \{x^* \in U : \alpha_i > \langle x_i, x^* \rangle\} \cap \{x^* \in U : \langle x_n, x^* \rangle > \alpha_n + \delta\}$$

Note that $x_n^* \in W_n$, so by Choquet's lemma we can find a slice S_n of $\operatorname{Ext}(C)$ with $x_n^* \in S_n \subseteq W_n$. Moreover S_n is not contained in $\operatorname{co}(x_1^*, .., x_n^*) + \delta B_{E^*}$, since otherwise $\alpha(S_n) \leq \delta$ would imply, by lemma 4.12, that $\operatorname{Frag}(\operatorname{Ext}(C)) \leq \delta$ which is a contradiction.

Thus using Hanh-Banach we can find $x_{n+1}^* \in S_n$, $x_{n+1} \in B_E$ and $\alpha_{n+1} \in \mathbb{R}$ such that

$$\langle x_{n+1}, x_{n+1}^* \rangle > \alpha_{n+1} + \delta > \alpha_{n+1} > \langle x_{n+1}, x_i^* \rangle \text{ for } i = 1, ..., n \ .$$

Now consider

$$V_{n+1} := S_n \cap \{x^* : \langle x_{n+1}, x^* \rangle > \alpha_{n+1} + \delta\} \quad (x_{n+1}^* \in V_{n+1})$$
$$V_n := S_n \cap \{x^* : \langle x_{n+1}, x^* \rangle < \alpha_{n+1}\} \quad (x_n^* \in V_n)$$

By Choquet's lemma there is a ω^*-slice S_n' of $\operatorname{Ext}(C)$ with $x_n^* \in S_n' \subseteq V_n$. Since S_n' is not contained in

$$\operatorname{co}(x_1^*, ..., x_{n+1}^*) + \delta B_{E^*}$$

there are elements $y_n^* \in S_n'$, $\beta_n \in \mathbb{R}$ and $y_n \in B_E$ such that

$$\langle y_n, y_n^* \rangle > \beta_n + \delta > \beta_n > \langle y_n, x_i^* \rangle \text{ for every } i = 1, ..., n+1 \ .$$

We then redefine $x_n^* := y_n^*$, $x_n := y_n$ and $\alpha_n := \beta_n$ so that

$$\langle x_i, x_i^* \rangle > \alpha_i + \delta > \alpha_i > \sup_{k \neq i} \{\langle x_i, x_k^* \rangle\}$$

for every $i = 1, ..., n+1$. □

Corollary 4.15. *Suppose that U is a relatively ω^*-open subset of $\operatorname{Ext}(C)$ for some ω^*-compact convex subset $C \subseteq E^*$ with $\operatorname{Frag}(\operatorname{Ext}(C)) > \delta$. Given $m \in \mathbb{N}$ there are m relatively ω^*-open (nonempty) subsets $U_1, ..., U_m$ of U and $x_1, ..., x_m \in B_E$ such that*

$$\inf \left\{ \langle x_j, y_j^* - y^* \rangle : y_j^* \in U_j, y^* \in \cup_{i \neq j} U_i \right\} \geq \delta \tag{10}$$

for every $j = 1, ..., m$

Proof. By proposition 4.14 there exist sequences $(x_n^*)_n \subseteq U$, $(x_n)_n \subseteq B_E$ and $(\alpha_n)_n \subseteq \mathbb{R}$ verifying equation (9). Define for every $j = 1, ..., m$

$$U_j = \{x^* \in U : \langle x_j, x^* \rangle > \alpha_j + \delta\} \cap \bigcap_{k=1, k \neq j}^{m} \{x^* \in U : \langle x_k, x^* \rangle < \alpha_k\}$$

Note that $x_j^* \in U_j$ so they are nonempty. It is clear that they satisfy the condition of the enunciate □

Proposition 4.16. *If C is convex and ω^*-compact with $\operatorname{Frag}(\operatorname{Ext}(C)) > \varepsilon$ then there exists a countable subset $C_0 \subseteq C$ such that every slice of C_0 has diameter greater than ε.*

Proof. Suppose now that $\operatorname{Frag}(\operatorname{Ext}(C)) > \delta > \varepsilon$. The next argument is inspired by a van Dulst and Namioka result [22, p. 8, proposition 2].

Consider $\mathcal{D} = \{\emptyset\} \cup \{1, ..., m\} \cup \{1, ..., m\}^2 \cup ....$ If $d \in \mathcal{D}$ has the form $d = (d_1, ..., d_k)$ then we write $di = (d_1, ..., d_k, i)$ for $i \in \{1, ..., m\}$. The length of d is denoted by $|d|$.

Using corollary 4.15 we can construct a tree of ω^*-open sets $\{U_d : d \in \mathcal{D}\}$ such that $U_{di} \subseteq U_d$ and $\mathrm{d}(\overline{\mathrm{co}}^{\omega^*}(U_{dj}), \overline{\mathrm{co}}^{\omega^*} \cup_{i \neq j} U_{di}) > \delta$ if $d \in \mathcal{D}$ and $j \in \{1, ..., m\}$.

For $d \in \mathcal{D}$ we write $K_d = \overline{\mathrm{co}}^{\omega^*}(U_d)$. We claim that there exists $(s_d)_{d \in \mathcal{D}}$ with $s_d \in K_d$ and $s_d = \sum_{i=1}^m \frac{1}{m} s_{di}$ for every $d \in \mathcal{D}$. Write for every $e \in \mathcal{D}$

$$A_e := \left\{ (t_d)_{d \in \mathcal{D}} \in \prod_{d \in \mathcal{D}} K_d : t_e = \sum_{i=1}^m \frac{1}{m} t_{ei} \right\}.$$

Note that $(x_d)_{d \in \mathcal{D}}$ has to belong to the intersection of all A_e ($e \in \mathcal{D}$), so by ω^*-compactness we just have to show that $\bigcap \{A_e : e \in \mathcal{D}, |e| \leq n\} \neq \emptyset$ for every $n \in \mathbb{N}$. To see this we are going to construct an element belonging to this finite intersection. Choose for every $d \in \mathcal{D}$ with $|d| > n$ an element $t_d \in K_d$. For the rest we use downward induction. Fix $d \in \mathcal{D}$ and suppose that we have defined t_{di} for each $i \in \{1, ..., n\}$. Then take $t_d := \sum_{i=1}^m \frac{1}{m} t_{di}$. It is clear that the so defined element $(t_d)_{d \in \mathcal{D}}$ belongs to the previous intersection.

Now consider the set $C_0 = \{s_d : d \in \mathcal{D}\}$ and suppose that $x^{**} \in E^{**}$ and $\alpha \in \mathbb{R}$ are elements that determine a nonempty slice $S = \{s_d : \langle x^{**} s_d \rangle > \alpha\}$ of C_0. If $s_d \in S$ then one of the elements s_{di} must also belong to S since $s_d = \sum_{i=1}^m \frac{1}{m} s_{di}$. The diameter of S is then greater than $\|s_d - s_{di}\| \geq \frac{m-1}{m} \delta$. Taking m larger enough so that $\frac{m-1}{m} \delta > \varepsilon$ the proof is over. □

Remark 4.17. *Proposition 4.16 is a strengthening of* [22, p. 8, proposition 2] *in the sense that let us construct m-adic trees for every $m \in \mathbb{N}$.*

Finally, as we announced at the beginning of this section, we are going to show the relationship between the indexes of dentability and fragmentability.

Theorem 4.18. *If C is a convex ω^*-compact subset of E^* then*

$$\begin{aligned} \sup\{\mathrm{Dent}(D) : D \subseteq C\} &\leq \sup\{\mathrm{Frag}(D) : D \subseteq C\} \leq \\ &\leq 2\,\sup\{\mathrm{Dent}(D) : D \subseteq C,\ D \textit{ countable}\} \end{aligned} \tag{11}$$

Proof. For every $D \subseteq C$ the set $\mathrm{Ext}(\overline{\mathrm{co}}^{\omega^*}(D))$ is contained in C. By proposition 4.13 we have that $\mathrm{Dent}(D) \leq \mathrm{Frag}(\mathrm{Ext}(\overline{\mathrm{co}}^{\omega^*}(D)))$ so the first inequality is true. On the other hand, again by proposition 4.13 we can claim that $\mathrm{Frag}(D) \leq \mathrm{Frag}(\mathrm{Ext}(\overline{\mathrm{co}}^{\omega^*}(D)))$. Combining this with proposition 4.16 we deduce the second inequality. □

Remark 4.19. *We do not know if constant 2 in theorem 4.18 is sharp.*

4.2. **Dentability versus index of weak noncompactness.** It is well-known that every weakly compact set in a Banach spaces is dentable (see [3, p. 60, theorem 3.6.1]). We will give a quantitative version of this result using the following measure of weak noncompactness

$$\gamma(H) = \sup\left\{\left|\lim_n \lim_m x_m^*(x_n) - \lim_m \lim_n x_m^*(x_n)\right| : (x_m^*)_m \subseteq B_{E^*}, (x_n)_n \subseteq H\right\}$$

where the supremum is taken over those values for which the previous limits exist.

Proposition 4.20. *If C is a bounded convex subset of E then*

$$\mathrm{Dent}(C) \leq \gamma(C) \tag{12}$$

Proof. Using the identification $E \subseteq E^{**}$ we will write $C' := \mathrm{Ext}(\overline{\mathrm{co}}^{\omega^*}(C)) \subseteq E^{**}$. We point out that when we write $\mathrm{Dent}(C)$ we are looking at C as a subset of E and not of E^{**}.

Fix an element $x_0 \in C$ and suppose that $\delta < \mathrm{Dent}(C)$. There exists $x_1^{**} \in C'$ which does not belong to $B[x_0, \delta]$. Hence we can find an element $x_1^* \in B_{E^*}$ and a real number α_1 such that

$$\langle x_1^{**}, x_1^* \rangle > \alpha_1 + \delta > \alpha_1 > \langle x_0, x_1^* \rangle$$

By density there exists $x_1 \in C$ with

$$\langle x_1, x_1^* \rangle > \alpha_1 + \delta.$$

Suppose now that we have constructed sequences $x_k \in C$, $x_k^* \in B_{E^*}$, $\alpha_k \in \mathbb{R}$ for $k = 1, ..., n$ and an element $x_n^{**} \in C'$ such that

$$\langle x_j, x_k^* \rangle > \alpha_k + \delta > \alpha_k > \langle x_i, x_k^* \rangle \qquad \text{for every } i < k \text{ and } j \geq k$$

$$\langle x_n^{**}, x_k^* \rangle > \alpha_k + \delta \qquad \text{for each } k = 1, ...n.$$

The ω^*-open set

$$W = \bigcap_{k=1}^{n} \{x^{**} : \langle x^{**}, x_k^* \rangle > \alpha_k + \delta\}$$

has nonempty intersection with C' since $x_n^{**} \in C' \cap W$. Using Choquet's lemma we can find a ω^*-slice S of C' such that $x_n^{**} \in S \subseteq W$. We claim that this slice cannot be contained in

$$K_n = \operatorname{co}(x_0, ..., x_n) + \delta B_{E^{**}}$$

Assume that S is contained in K_n so $\alpha(S) \leq \delta$ (recall that α is the Kuratowski measure of noncompactness) and then $\operatorname{Frag}(C') \leq \delta$ by proposition 4.13. But $\operatorname{Frag}(C')$ is equal to $\operatorname{Dent}(C)$ again by proposition 4.13, since a ω^*-slice of C as a subset of E^{**} is a slice of C regarded as a subset of E. This gives a contradiction that proves the claim.

We can then find $x_{n+1}^{**} \in S$ which does not belong to the ω^*-compact set K_n. By Hanh-Banach, there exists $x_{n+1}^* \in B_{E^*}$ and $\alpha_{n+1} \in \mathbb{R}$ such that

$$\langle x_{n+1}^{**}, x_{n+1}^* \rangle > \alpha_{n+1} + \delta > \alpha_{n+1} > \langle x_i, x_{n+1}^* \rangle \text{ for every } i = 1, ..., n$$

Since $x_{n+1}^{**} \in W \cap \{x^{**} : \langle x^{**}, x_{n+1}^* \rangle > \alpha_{n+1} + \delta\}$, by density there is $x_{n+1} \in C$ belonging to the same ω^*-open set (C is convex).

This way, we have constructed by induction sequences $(x_j)_j \subseteq C$, $(x_k^*)_k \subseteq B_E^*$ and $(\alpha_k)_k \subseteq \mathbb{R}$ such that

$$\langle x_j, x_k^* \rangle > \alpha_k + \delta > \alpha_k > \langle x_i, x_k^* \rangle \quad \text{for every } i < k \text{ and } j \geq k$$

Passing to subsequences we can suppose that $\lim_k \alpha_k = \alpha$ exists (note that C is bounded), $\lim_n \langle x_n, x_k^* \rangle$ exists for every $k \in \mathbb{N}$ and $\lim_k \langle x_n, x_k^* \rangle$ exists for every $n \in \mathbb{N}$. Taking further subsequences we may assume that the following double limits exist and by construction they must satisfy

$$\lim_k \lim_n \langle x_n, x_k^* \rangle - \lim_n \lim_k \langle x_n, x_k^* \rangle \geq \delta.$$

□

Remark 4.21. *Inequality (12) is sharp. For $E = c_0$ and $C = B_{c_0}$ we have that $\gamma(B_{c_0}) = 1$ (see* [16, p. 394, example 2.7]*) and* $\operatorname{Dent}(B_{c_0}) = 1$ *(it can be easily seen that every slice of B_{c_0} has diameter equal to* 2*).*

Combining theorem 4.8 with propositions 2.7 and 4.20 we deduce the following corollary.

Corollary 4.22. *Let (Ω, Σ, μ) be a finite measure space and $T : L^1(\mu) \to E$ a continuous linear operator. Then*

$$\mathrm{d}(T, L_{rep}(L^1(\mu), E)) \leq \sup\{\operatorname{Dent}(C) : C \subseteq T(B_{L^1(\mu)})\} \leq \gamma(T(B_{L^1(\mu)})).$$

In particular, every weakly compact operator is representable.

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