Universidad de Murcia | Departamento Matemáticas

Radon-Nikodým indexes and measures of weak noncompactness

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Contents:

2. Three results related to RNP
    - Related with separable dual spaces: RNP and the Lindelöf property.
    - Related to small slices: RNP and norm attaning operators.
    - Related to the definition RNP: indexes.

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1. Vector integration, RNP, dentability and fragmentability.
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    - Related to small slices: RNP and norm attaning operators.
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2. Three results related to RNP
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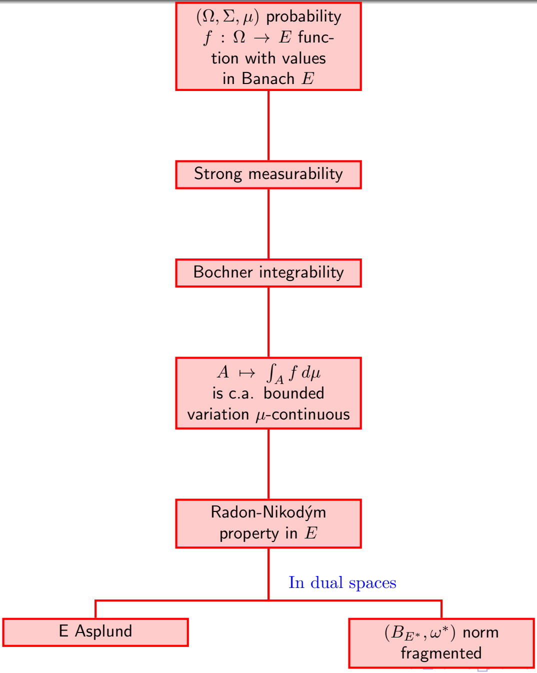
    - Related to the definition RNP: indexes.
3. Further comments: filters, ultrafilters, applications.

Vector integration, RNP, dentability and fragmentability


(Ω, Σ, μ) probability $f : \Omega \to E$ function with values in Banach E
Strong measurability
Bochner integrability
$A \mapsto \int_A f \, d\mu$ is c.a. bounded variation μ-continuous
Radon-Nikodým property in E
In dual spaces
E Asplund
(B_{E^*}, ω^*) norm fragmented

Measurability: $f : (\Omega, \Sigma, \mu) \to E$

Simple function.- $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where $\alpha_i \in E$, $A_i \in \Sigma$, disjoints.

Measurable function.- $\lim_n \| s_n(w) - f(w) \| = 0$, μ a.e. $w \in \Omega$.

Scalarly measurable function.- $x^* f$ is measurable for $x^* \in E^*$.

Pettis's Theorem

For a function $f : \Omega \longrightarrow E$, TFAE:

1. f is measurable.
2. (a) There is $A \in \Sigma$ with $\mu(A) = 0$ such that $f(\Omega \setminus A)$ is separable.
   (b) For each $x^* \in X^*$, the function $x^* f$ is measurable.

Measurable≠scalarly measurable

$$f : [0,1] \to \ell^2([0,1]) \; t \to e_t$$

Bochner integral

The integral of simple functions.- If $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, with $A_i \in \Sigma$ and $\alpha_i \in E$, is a simple function, we define

$$\int_A s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A \cap A_i),$$

Bochner integral.- A μ-measurable $f : \Omega \longrightarrow E$ is *Bochner integrable*, if there is a sequence of simple functions $(s_n)_n$ such that

$$\lim_n \int_\Omega \| s_n - f \| \, d\mu = 0.$$

The vector $\int_A f\, d\mu = \lim_n \int_A s_n d\mu$ is called Bochner integral of f.

A first property of Bochner integral

Theorem

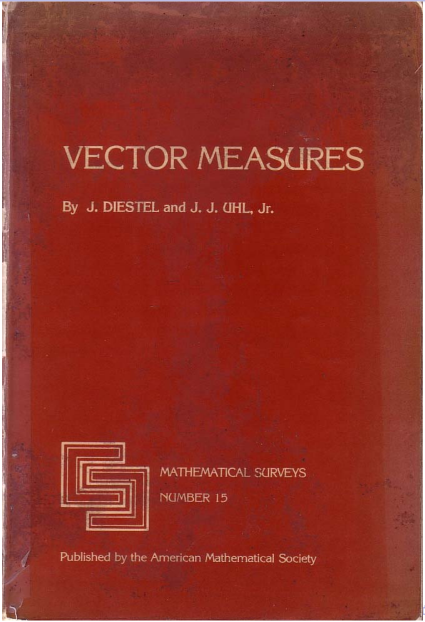
If $f : \Omega \longrightarrow E$ is measurable. TFAE:

1. f is Bochner integrable.
2. $\| f \|$ Lebesgue integrable.

If $f : \Omega \longrightarrow E$ Bochner integrable, then, for each $A \in \Sigma$, we have

$$\left\| \int_A f \, d\mu \right\| \leq \int_A \| f \| \, d\mu .$$

You can carry on reading Diestel-Uhl, 1977



THEOREM 3 (DOMINATED CONVERGENCE THEOREM). *Let* (Ω, Σ, μ) *be a finite measure space and* (f_n) *be a sequence of Bochner integrable X-valued functions on* Ω. *If* $\lim_n f_n = f$ *in* μ*-measure,* (i.e., $\lim_n \mu\{\omega \in \Omega : \|f_n - f\| \geqq \varepsilon\} = 0$ *for every* $\varepsilon > 0$) *and if there exists a real-valued Lebesgue integrable function g on* Ω *with* $\|f_n\| \leqq g$ μ*-almost everywhere, then f is Bochner integrable and* $\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$ *for each* $E \in \Sigma$. *In fact,* $\lim_n \int_\Omega \|f - f_n\| \, d\mu = 0$.

THEOREM 4. *If f is a μ-Bochner integrable function, then*

(i) $\lim_{\mu(E)\to 0} \int_E f\, d\mu = 0$;

(ii) $\left\| \int_E f\, d\mu \right\| \leqq \int_E \|f\|\, d\mu$, *for all* $E \in \Sigma$;

(iii) *if* (E_n) *is a sequence of pairwise disjoint members of* Σ *and* $E = \bigcup_{n=1}^{\infty} E_n$, *then*

$$\int_E f\, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\, d\mu,$$

where the sum on the right is absolutely convergent;

(iv) *if* $F(E) = \int_E f\, d\mu$, *then* F *is of bounded variation and*

$$|F|(E) = \int_E \|f\|\, d\mu \quad \text{for all } E \in \Sigma.$$

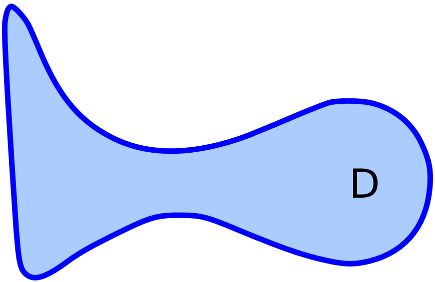
Then, at Berkeley, California, Rieffel taught a real analysis course in which he opted to present the Bochner integral instead of the classical Lebesgue theory. As rumor has it, all went smoothly until he came to the Radon-Nikodým theorem and its attendant difficulties in infinite dimensional Banach spaces.

EXAMPLE 1. *The failure of the Radon-Nikodým theorem for a c_0-valued measure.* Let $\Omega = [0, 1]$ and μ be Lebesgue measure on Σ, the σ-field of Lebesgue measurable subsets of $[0, 1]$. Define a measure $G: \Sigma \to c_0$ by

$$G(E) = \left(\int_E \sin(2^n \pi t)\, d\mu(t)\right).$$

Definition (Rieffel, 1967)

$D \subset E$ is dentable if for each $\varepsilon > 0$ there is a point $x \in D$ such that $x \notin \overline{\text{co}}(D \setminus U_\varepsilon(x))$



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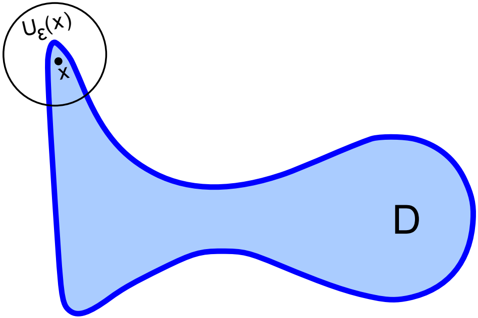

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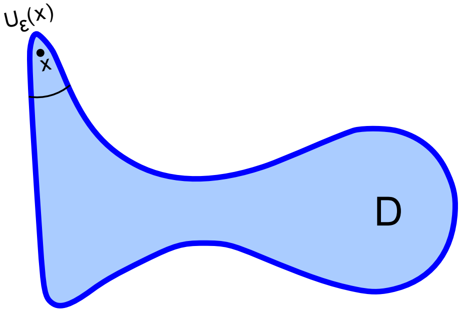

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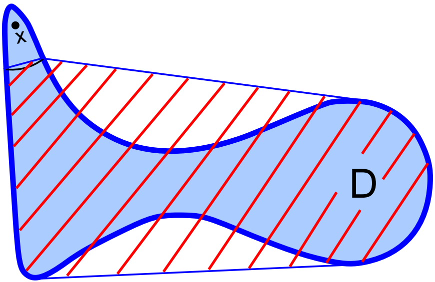
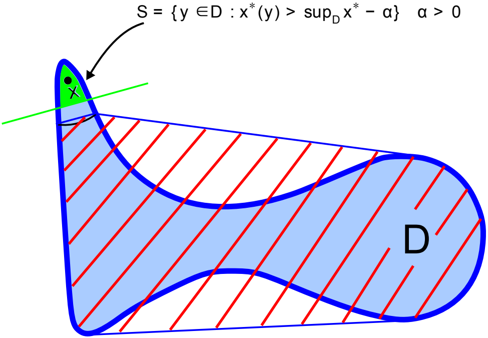


Proposition (Small slices)

$D \subset E$ is dentable if, and only if, D has slices of arbitrarily small diameter.



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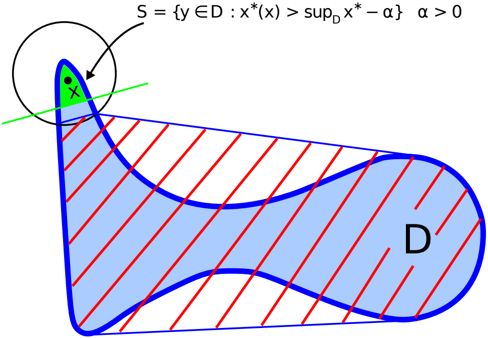


Theorem 7 (Rieffel-Maynard-Huff-Davis-Phelps). *Any one of the following statements about a Banach space X implies all the others.*

(a) *Every bounded subset of X is dentable.*

(b) *Every bounded subset of X is σ-dentable.*

(c) *The space X has the Radon-Nikodým property.*

Fragment from Phelps' memorial article

"As I recall, there was a period of excitement involving Rieffel's generalization of the classical Radon-Nikodým Theorem to Banach space valued functions provided the Banach space has the Radon-Nikodým Property (RNP) which is a geometric property. The RNP immediately attracted widespread attention, and Bob was not immune from the RNP-bug. He published in 1974-75 three papers[7] (one jointly with Davis) on RNP. By 1975 it was shown that the RNP, which was shown to be necessary by Rieffel for his generalized Radon-Nikodým theorem, is also sufficient by the combined efforts of Chatterji, Davis, Huff, Maynard and Phelps."

I. Namiola (August 2013).

Fragment from Phelps' memorial article

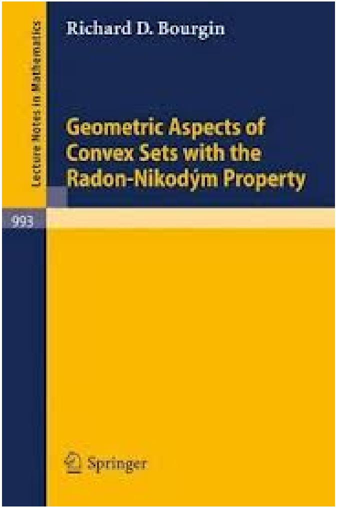
"While visiting Seattle, Edgar Asplund from Sweden wrote a very interesting paper Fréchet differentiability of convex functions[8]*, where he introduced a new class of Banach spaces called strong differentiability space(or SDS). Around 1974, Bob got interested in SDS space again. First he proposed to rename SDS spaces Asplund spaces in honor of Edgar Asplund, who had died shortly before."*

. . .

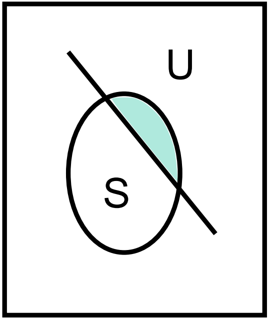
"In 1983, Bob's former Ph.D. student, Richard Bourgin, published an excellent exposition of the matters related to the RNP[11]*, and parts of the Rainwater seminar notes were incorporated into the book."*

I. Namiola (August 2013).

Bourgin's book



A very remarkable result



$\|\cdot\| - \text{diam}(U \cap S) \leq \varepsilon$

Namioka, Phelps and Stegall

Let E be a Banach space. Then the following conditions are equivalent:

(i) E is an Asplund space, *i.e.*, whenever f is a convex continuous function defined on an open convex subset U of E, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U.

(ii) every w^*-compact subset of (E^*, w^*) is fragmented by the norm;

(iii) each separable subspace of E has separable dual;

(iv) E^* has the Radon-Nikodým property.

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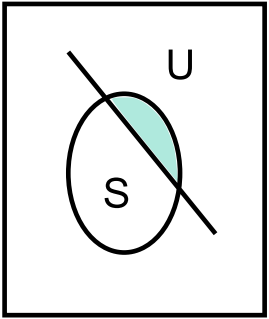

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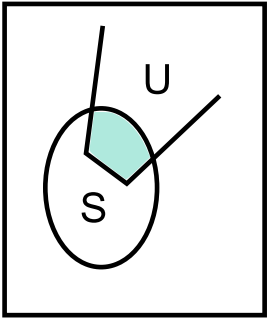

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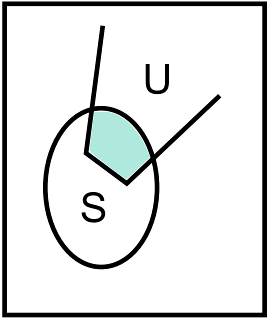

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Three results

RNP and the Lindelöf property

Theorem (Namioka-Orihuela-Cascales, 2003)

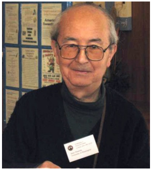
K compact subset of M^D, (M,ρ) metric space. T.F.A.E:

(a) *The space (K,τ_p) is fragmented by d.*

(b) *For each $A\in\mathscr{C}$, the pseudo-metric space (K,d_A) is separable.*

(c) *$(K,\gamma(D))$ is Lindelöf.*

(d) *$(K,\gamma(D))^{\mathbb{N}}$ is Lindelöf.*



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Corollary

E^* has the RNP if, and only if, $(E^*,\gamma(B_X))$ is Lindelöf.

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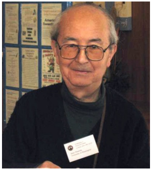


Corollary

E^* has the RNP if, and only if, $(E^*,\gamma(B_X))$ is Lindelöf.

Theorem (Solution to a problem by Corson)

If (E^*,w) is Lindelöf, then $(E^*,w)^2$ is Lindelöf

RNP and norm attaning operators

Theorem (Guirao-Kadets-Cascales, 2013)

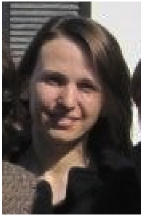
Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T} u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

This gives

- for $C(K)$ an example of the BPBp for c_0 as domain and an infinite dimensional Banach space as range (answer a question by Acosta-Aron-García-Maestre, 2008);
- new cases, in particular, disk algebra as range.

Co-authors


R. M. Aron, B. Cascales and O. Kozhushkina,
The Bishop-Phelps-Bollobás theorem and Asplund operators,
Proc. Amer. Math. Soc. 139 (2011), no. 10, 3553–3560.

B. Cascales, A. J. Guirao and V. Kadets,
A Bishop-Phelps-Bollobás type theorem for uniform algebras,
Advances in Mathematics 240 (2013) 370-382

A Urysohn type lemma for uniform algebras

Proposition 2.8. *Let* $A \subset C(K)$ *be a unital uniform algebra,* $\Omega \subset \mathbb{C}$ *a bounded simply connected region such that all points in its boundary* $\partial\Omega$ *are simple. Let us fix two different points* a *and* b *with* $b \in \partial\Omega$, $a \in \overline{\Omega}$ *and a neighborhood* $V_a \subset \overline{\Omega}$ *of* a*. Then, for every open set* $U \subset K$ *with* $U \cap \Gamma_0 \neq \emptyset$ *and for every* $t_0 \in U \cap \Gamma_0$*, there exists* $f \in A$ *such that*

(i) $f(K) \subset \overline{\Omega}$;
(ii) $f(t_0) = b$;
(iii) $f(K \setminus U) \subset V_a$.

Comments about indexes related to RNP

A bit of notation

- $m : \Sigma \to E$ vector measure c. a. and μ-continuous;
- For every $B \in \Sigma^+$ the average range of $m|_B$ is denoted by

$$\Gamma_B := \left\{ \frac{m(C)}{\mu(C)} : C \in \Sigma_B^+ \right\}.$$

- $D \subset E$,

$$\mathrm{rad}(D) = \inf\{\delta > 0 : \exists x \in E \text{ such that } D \subseteq B(x, \delta)\}.$$

Index of representability vs. dentability

Definition

Given a μ-continuous vector measure of bounded variation $m : \Sigma \to E$, the *index of representability* $\mathscr{R}(m)$ of m is defined as

$$\mathscr{R}(m) := \inf\{\varepsilon > 0 : \forall A \in \Sigma^+, \exists B \in \Sigma_A^+ \text{ with } \operatorname{rad}(\Gamma_B) < \varepsilon\}.$$

Characterization

$\mathscr{R}(m)$ is characterized as the infimum of the constants δ for which there exists $g \in L^1(\mu, E)$ satisfying

$$\left\| m(A) - \int_A g \, d\mu \right\| \leq \delta\, \mu(A), \text{ for every } A \in \Sigma,$$

With the help of the lifting theorem, it is possible to prove that if (Ω, Σ, μ) is a finite measure space and $T\colon L_1(\mu) \to X^*$ is a bounded linear operator, then there exists a function $g\colon \Omega \to X^*$ that is weak*-measurable and such that for each $x \in X$ and $f \in L_1(\mu)$, one has $(Tf)(x) = \int_\Omega f(\omega)g(\omega)(x)\,d\mu(\omega)$. On the surface, this is a vast generalization of Theorem 3.1 which it includes. Unfortunately, this generalization is mostly an esthetic generalization because the measurability properties of the kernel g are not, in general, strong enough to exhibit structural properties of the operator under representation.

Fix $\rho : \Sigma \to \Sigma$ a lifting, and write:

- $\mathscr{F} := \rho(\Sigma) \setminus \{\emptyset\}$.
- $(\mho, \succ)$ is the directed set of all finite partitions of Ω into elements of $\mathscr{F}$ ordered by refinement $\succ$.

Lemma (Folklore + a little thing)

The net

$$\left\{ \sum_{A \in \pi} \frac{m(A)}{\mu(A)} \chi_A : \pi \in \mho \right\}$$

converges pointwise μ-almost everywhere in the ω^*-topology to a function $\psi : \Omega \to E^*$ which is ω^*-measurable and that satisfies the following properties for every $A \in \Sigma$:

(I) $\langle x, m(A) \rangle = \int_A \langle x, \psi(t) \rangle \, d\mu$

(II) $\psi(t) \in \overline{\Gamma_A}^{\omega^*}$ for μ-almost every $t \in A$.

Theorem

Let $m : \Sigma \to E \hookrightarrow E^{**}$ be a μ-continuous measure of bounded variation and $\psi : \Omega \to E^{**}$ a Gelfand derivative of m. Then

$$\mathscr{R}(m) \leq \text{meas}(\psi) \leq 2\mathscr{R}(m)$$

and there exists a μ-null set D such that

$$\hat{d}(\psi(\Omega \setminus D), E) \leq \mathscr{R}(m).$$

For subsets of E define:

- $\text{Dent}(C) = \inf\{\varepsilon > 0 : \exists S \text{ slice of } C \text{ with } \text{rad}(S) < \varepsilon\}$
- $\text{dent}(D) = \sup\{\text{Dent}(C) : C \subseteq D\}$

Theorem

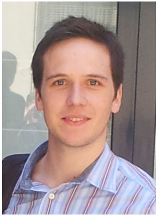
Let $m : \Sigma \to E$ be a μ-continuous vector measure of bounded variation. Then

$$\mathscr{R}(m) \leq \text{dent}(AR(m)).$$

Theorem

Let (Ω, Σ, μ) be a finite measure space and $T : L^1(\mu) \to E$ a continuous linear operator. Then

$$d(T, \mathscr{L}_{rep}(L^1(\mu), E)) \leq 2\,\gamma(T(B_{L^1(\mu)})).$$



B. Cascales, A. Pérez and M. Raja,
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Preprint, 2013

One last thing

Good readings for liftings are:

D. H. Fremlin, *Measure theory. Vol. 3*, Torres Fremlin, Colchester, 2004, Measure algebras, Corrected second printing of the 2002 original.

A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the theory of lifting*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 48, Springer-Verlag New York Inc., New York, 1969.

THANKS