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On inner parallel bodies. From the Steiner polynomial to Poincaré inequality

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INFORMA: Que la Tesis Doctoral titulada “On inner parallel bodies. From the Steiner polynomial to Poincaré inequality”, ha sido realizada por Dª Eugenia Saorín Gómez, bajo la inmediata dirección y supervisión de D. Bernardo Cascales Salinas y Dª María de los Ángeles Hernández Cifre, y que el Departamento ha dado su conformidad para que sea presentada ante la Comisión de Doctorado.

En Murcia, a 16 de Junio de 2008

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AUTORIZAN: La presentación de la Tesis Doctoral titulada "On inner parallel bodies. From the Steiner polynomial to Poincaré inequality", realizada por Dª Eugenia Saorín Gómez, bajo nuestra inmediata dirección y supervisión, en el Departamento de Matemáticas, y que presenta para la obtención del Grado de Doctor por la Universidad de Murcia.

En Murcia, a 16 de Junio de 2008

Fdo: Bernardo Cascales Salinas   Fdo: María de los Ángeles Hernández Cifre

These few words I intend to write here are telling me I got a part of the way. I started this way both plenty of fear and plenty of illusion. The path has not been smooth... there have been vertices and edges... and even holes. Nevertheless I was never going along through: I had four eyes, four ears and four hands... I have had a hand hanging up outside the holes, a mattress under the slippery edges and pushing illusion and courage when the vertices were near. There are thousands of things my words are not able to express: I have dared, failed and accepted; I have learnt and grown: it has been my pleasure and my pride to work side by side. Now it is the moment to thank, from heart, the invaluable help and support, and every of the shared things and moments. I would like to express my deep gratitude to María de los Ángeles Hernández Cifre.

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Research continually passes one through the eyes of needles. And at this point of the way, I owe a debt of gratitude to many people who helped me to walk this path and arrive here. I would like to acknowledge the help of all. Specially I wish to thank Prof. Schneider from University Freiburg and Prof. Colesanti from University of Florence, for their kind welcome, their inestimable help and support and their help to grow as a Mathematician as well as a person.

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A mi madre, a mi hermano y a mi hermana.

Für Ralf, meinen Deutschen Guanaquico.

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Preface/Introducción

Brunn-Minkowski Theory is considered the classical center of the Geometry of Convex Bodies and it can be said that it had its origin in the Ph.D dissertation of H. Brunn in 1887. Nevertheless the core of the theory is due to H. Minkowski, going back to around the turn of century: in December of 1900, Minkowski wrote to D. Hilbert informing him that his study on the volume in $\mathbb{R}^3$ was complete. His most important development to this respect was the introduction of a new concept associated to three convex bodies (compact convex sets) that he named, provisionally, their mixed volumes.

Brunn-Minkowski Theory may be briefly defined as the result of combining two elementary notions for sets in the Euclidean space: *vectorial addition*, $+$, and *volume*, V. The vectorial addition, or Minkowski addition, when combined with the volume, leads to the notion of mixed volumes and to the fundamental Brunn-Minkowski inequality, which is maybe the most well-known inequality relating the volume of convex bodies. However, the starting point of this theory can be placed at the moment in which J. Steiner made the "discovery" of an, in principle, amazing fact: the volume of the Minkowski sum of a convex body K and an Euclidean ball λB^n (the so called *outer parallel body of K at distance λ*) can be always expressed as a polinomial in the variable λ, of degree the dimension of the space and whose coefficients are, up to constants, the so called *quermassintegrals* of K, $\mathrm{W}_i(K)$:

$$\mathrm{V}(K+\lambda B^n)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K)\lambda^i.$$

The quermassintegrals are important functionals associated to the original convex body K and amongst them, so well-known magnitudes as the volume or the surface area can be found.

An analogous result is obtained in the more general context of the so called *Minkowski Relative Geometry*, i.e., when the Euclidean ball B^n is replaced by an arbitrary convex body (with non-empty interior) E. In this case, the previous notions of outer parallel body and quermassintegrals are now rewritten relative to the fixed body E and, in particular, the relative Steiner polynomial provides the volume of the Minkowski sum $K+\lambda E$:

$$\mathrm{V}(K+\lambda E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\lambda^i; \tag{1a}$$

the functionals $\mathrm{W}_i(K;E)$ are called *relative quermassintegrals* of K with respect to E.

It is natural to consider an operation somehow "opposite" to the addition of convex bodies, which leads to the so called *Minkowski difference*, $\sim$. Thus the *inner parallel body of K at distance λ* relative to E is defined as

$$K \sim \lambda E = \{x \in \mathbb{R}^n : x + \lambda E \subseteq K\}.$$

Now, on the contrary to what happens with the outer parallel bodies, neither the boundary structure nor the volume of the inner parallel bodies can be "controlled", as can be seen in Figure 1a.

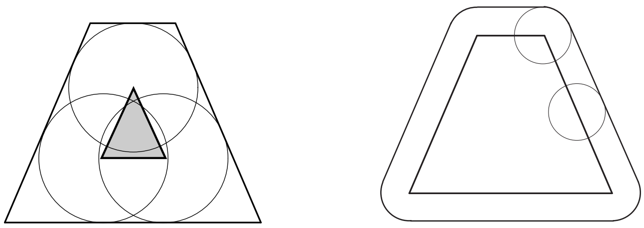

Figure 1a: Inner (left) and outer (right) parallel body of a trapezoid relative to B^2.

This "geometric" difference allows to pose two questions, in principle of different nature, but deep down closely related, as we will point out throughout the work gathered in this dissertation. On one hand, we study the behavior, with regard to the boundary, of the inner parallel body with respect to the original body (it turns out to be the study of the outer normal vectors at the boundary points) and, on the other hand, we try to better understand the behavior of the inner parallel bodies with respect to the quermassintegrals, in general, and to the volume, in particular. The latter suggests two kinds of problems:

i) Since it is not possible to express the volume (nor the quermassintegrals) of the inner parallel body by means of a precise formula involving the magnitudes of the original body, it is natural to look for bounds (from below and from above) for these magnitudes.

ii) In second place, the set of outer and inner parallel bodies of a convex body E (with respect to E) determines a one-parameter family (with parameter λ) of sets which allows to consider quermassintegrals as functions depending on one (real) variable,

$$\mathrm{W}_i(\lambda) = \begin{cases} \mathrm{W}_i\big(K \sim |\lambda|\, E; E\big) & \text{if } \; -\mathrm{r}(K;E) \leq \lambda \leq 0, \\ \mathrm{W}_i(K + \lambda E; E) & \text{if } \quad 0 \leq \lambda < \infty, \end{cases}$$

where $\mathrm{r}(K;E) = \max\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + rE \subseteq K\}$ is the (relative) inradius of K. Clearly, the functions $\mathrm{W}_i(\lambda)$ are continuous. But, are they differentiable for any convex body K? And, in the case it is so, does the derivative take the same value inside the interval $\big[-\mathrm{r}(K;E), 0\big]$ –inner parallel bodies– and inside the interval $[0,\infty)$ –outer parallel bodies? The answer is going to be, obviously, negative, which allows to consider the problem of classifying the convex bodies depending on the differentiability of their quermassintegrals (in the above mentioned sense). This problem was originally posed by Hadwiger in $\mathbb{R}^3$ and for $E = B^3$.

On the other hand, the fact that a convex body is uniquely determined by its support function h_K allows to introduce analytic techniques in order to study many problems in Convexity. So, given a convex body K verifying some particular properties, and a function $\psi : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ defined on the unit sphere $\mathbb{S}^{n-1}$ and of class $\mathcal{C}^2$, it is possible to find an $\varepsilon > 0$ small enough so that $h_K + \lambda\psi$ is the support function of some convex body K_λ for every $\lambda \in (-\varepsilon, \varepsilon)$ (this could be thought as an inner/outer parallel body with respect to ψ). Thus, the volume, the quermassintegrals and any other functional in this context can be seen as a function $F : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ on the real variable λ by considering $F(\lambda) = F(h_K + \lambda\psi)$.

We can say, roughly speaking, that this dissertation is devoted to the study of the inner parallel bodies, as well as their relation with the quermassintegrals, in general, and the volume, in particular, both from a geometric point of view (Chapters 1-4) and from an analytic point of view (Chapter 5). Next we describe the specific contents of each chapter in which this dissertation has been organized.

The work starts with an introductory *first chapter* in which we establish the notation and introduce the concepts and results that will be needed further on, both about general Convexity and, in particular, about mixed volumes and surface area measures. Thus, in a first section, the important notions, such as Minkowski addition, polytopes, supporting hyperplane, support function, polar body... are recalled. Next, mixed volumes and mixed surface area measures are introduced, and we devote a paragraph to the most important inequalities relating them: Aleksandrov-Fenchel inequality, Brunn-Minkowski inequality, Minkowski inequalities, isoperimetric inequality... Then, the chapter is focussed on the in-depth study of the (relative) Steiner polynomial, more precisely, on the study of its roots. The main idea is to look for the geometric properties of a convex body which are "hidden" behind the roots of its Steiner polynomial. We prove that the convex bodies in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be characterized in a precise way depending on the type of roots of their Steiner polynomial, which constitutes the first original work contained in the dissertation. The above mentioned classification has a straightforward translation in the Blaschke diagram, providing a significative progress on the so called *Blaschke problem* which is, nowadays, still open.

The *second chapter* is devoted to the study of the original problem by Hadwiger on inner parallel bodies in the 3-dimensional Euclidean space: to classify the convex bodies in $\mathbb{R}^3$ depending on the differentiability of their quermassintegrals (relative to B^3) with respect to the one-parameter depending family consisting of their inner and outer parallel bodies. Although this problem is going to be tackled in the most general case (dimension n and with respect to an arbitrary fixed convex body E) in the next chapter, we devote this second part of the dissertation to the particular case of $\mathbb{R}^3$ and the Euclidean ball B^3 because of two concrete reasons: in first place, due to the fact that this is the original problem that was posed by Hadwiger in 1955; in second place, because of the close relation of it with the behavior of the roots of the Steiner polynomial. Since in dimension 3 we have at our disposal a precise characterization of convex bodies depending on the type of the roots of their Steiner polynomial, it is going to make possible to exclude sets from the classes in which Hadwiger classifies the convex bodies in $\mathbb{R}^3$. In short, the main results in this chapter deal with the behavior of the inner parallel bodies of a convex body K whose Steiner polynomial has a particular type of roots.

Convex bodies can be classified, depending on the differentiability of their quermassintegrals, in the following way. Given a fixed convex body E with non-empty interior, a convex body K is said to *belong to the class* $\mathcal{R}_p$, $0 \leq p \leq n-1$, if for any $0 \leq i \leq p$ and for every $-\mathrm{r}(K;E) \leq \lambda < \infty$ it holds

$$'\mathrm{W}_i(\lambda) = \mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda),$$

where $'\mathrm{W}_i(\lambda)$ and $\mathrm{W}_i'(\lambda)$ denote, respectively, the left and right derivatives of W_i. In the *third chapter* of this dissertation the convex bodies belonging to the smallest class, $\mathcal{R}_{n-1}$, are characterized, and necessary conditions for a convex body K to lie in the remaining classes $\mathcal{R}_p$, $1 \leq p \leq n-2$, are obtained. These conditions are expressed in terms of some mixed surface area measures and depending on the type of outer normal vectors at the boundary points of K. Specially important are the so called 0-extreme outer normal vectors, i.e., those normal vectors which cannot be written as a linear combination of two linearly independent outer normal vectors at one and the same boundary point of K.

Thus, the first section of the chapter is devoted to the careful study of the relation between the 0-extreme normal vectors of the original body K and the ones of, either its inner parallel bodies, or the so called form body of K, which will play an outstanding role throughout this work. These results will be fundamental for the following sections in the chapter (as well as in the fourth chapter), where we fully deal with the in-depth study of the classes $\mathcal{R}_p$. Finally, there is an especially important class of convex bodies, the so called p-tangential bodies; the last section of Chapter 3 is devoted to them. As it will be noticed throughout the chapter, these bodies appear in a natural way as extremal sets in many inequalities and relations.

As we have already mentioned, it is not possible to give an explicit formula for the volume (analogously, the quermassintegrals) of the inner parallel body of a convex body K. The *fourth chapter* in this dissertation is devoted to determine the best possible upper and lower bounds for the volume of the inner parallel body in terms of the magnitudes of the original body. In the particular case when K is an outer parallel body of E, it can be proved that

$$\mathrm{V}(K \sim \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)(-\lambda)^i$$

in the suitable range for λ. So it is natural to look for upper/lower bounds for the volume of $K \sim \lambda E$ in terms of the so called *alternating Steiner polynomial*, i.e., the polynomial obtained replacing λ by $-\lambda$ in (1a). Moreover, in [34] Matheron conjectures that the alternating Steiner polynomial provides always a lower bound for $\mathrm{V}(K \sim \lambda E)$. Using the classes $\mathcal{R}_p$ studied in the previous chapter, it is proved that the best possible bounds for the volume of the inner parallel body of a convex body $K \in \mathcal{R}_p$ are given by functionals of the type

$$\sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(K;E)(-\lambda)^i \pm \binom{n}{p+2} (n-p-2) \int_0^\lambda \frac{(\lambda - s)^{p+2}}{\mathrm{r}(K;E) - s} \mathrm{W}_{p+2}(-s)\, ds$$

in which, as it can be noticed, a *part* of the alternating Steiner polynomial appears; but not only this one. In fact, in the second section of the chapter it is proved that it is not possible to bound $\mathrm{V}(K \sim \lambda E)$ in terms of precisely the alternating Steiner polynomial, which shows the non-validity of Matheron's conjecture. In the last two sections analogous results are obtained for the quermassintegrals of a convex body.

Finally, we make a brief comment on the *fifth* and last *chapter* in this dissertation. The possibility of identifying a convex body with its support function allows to consider many problems in Convexity from an analytic point of view. Specifically, given a convex body $K \subset \mathbb{R}^n$ of class $\mathcal{C}^2_+$, i.e., such that its boundary is a hypersurface of class $\mathcal{C}^2$ with Gauss curvature strictly positive at every point, and a function $\psi : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^2$, we consider the quermassintegrals W_i as functions on the real variable $\lambda \in (-\varepsilon, \varepsilon)$, for $\varepsilon > 0$ small enough such that $h_K + \lambda\psi$ is a support function. Then Brunn-Minkowski inequality, namely

$$\mathrm{W}_i\big((1-t)K + tL\big)^{1/(n-i)} \geq (1-t)\mathrm{W}_i(K)^{1/(n-i)} + t\mathrm{W}_i(L)^{1/(n-i)}$$

for any convex bodies $K, L \subset \mathbb{R}^n$, $t \in [0,1]$ and $i = 0, 1, \ldots, n$, ensures that $\mathrm{W}_i^{1/n-i}$ is concave in the variable λ. The concavity of these functions will allow us to obtain certain *Poincaré type* inequalities, which is the main aim in this last chapter. First we will work with convex bodies of class $\mathcal{C}^2_+$ and, since every convex body can be approximated (in the Hausdorff metric) by convex bodies of this type, in the last section of Chapter 5 we will obtain Poincaré type inequalities for arbitrary convex bodies of $\mathbb{R}^n$.

The original results which are contained in this dissertation can be found in the papers [13, 26, 27, 28, 29, 30, 46].

Podría decirse que la Teoría de Brunn-Minkowski, centro clásico de la Geometría de los Cuerpos Convexos, tuvo su origen, como tal, en la Tesis de H. Brunn en 1887, siendo, en su parte más esencial, creación de H. Minkowski alrededor del cambio de siglo: en Diciembre de 1900, Minkowski escribió a D. Hilbert informándole de que su estudio sobre el volumen en $\mathbb{R}^3$ estaba completo, siendo su avance más importante la introducción de un nuevo concepto asociado a tres cuerpos convexos (conjuntos convexos y compactos) que él denominó, provisionalmente, los *volúmenes mixtos*.

Si queremos definir brevemente la Teoría de Brunn-Minkowski, podríamos decir que ésta es el resultado de combinar y asociar dos nociones elementales para los conjuntos del espacio euclídeo: la *suma vectorial*, $+$, y el *volumen*, V. La suma vectorial o de Minkowski, combinada con el volumen, nos conduce a la noción de volúmenes mixtos y a la desigualdad fundamental de Brunn-Minkowski, quizá la desigualdad más conocida relacionando el volumen de cuerpos convexos. Sin embargo, podría considerarse como inicio real de toda esta teoría el "descubrimiento" de J. Steiner, en 1840, de un hecho en principio sorprendente: el volumen de la suma de Minkowski de un cuerpo convexo K y una bola euclídea λB^n, lo que se conoce como *conjunto paralelo exterior de K a distancia λ*, viene dado siempre por un polinomio en la variable λ, de grado la dimensión del espacio y cuyos coeficientes son, salvo constantes, las llamadas *quermassintegrales* de K, $\mathrm{W}_i(K)$:

$$\mathrm{V}(K+\lambda B^n)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K)\lambda^i.$$

Las quermassintegrales son funcionales de gran relevancia asociados al cuerpo original K, entre las que se encuentran medidas tan conocidas como el volumen o el área de superficie.

Un resultado análogo se obtiene en el contexto más general de la llamada *Geometría Relativa* o *de Minkowski*, es decir, cuando la bola euclídea B^n se sustituye por un cuerpo convexo arbitrario E (con interior no vacío); en tal caso, los conceptos anteriores de cuerpo paralelo exterior y quermassintegrales se reescriben ahora relativos al cuerpo fijo E y, en particular, el polinomio relativo de Steiner nos da el valor del volumen de la suma de Minkowski $K+\lambda E$:

$$\mathrm{V}(K+\lambda E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\lambda^i; \tag{1b}$$

los funcionales $\mathrm{W}_i(K;E)$ reciben el nombre de *quermassintegrales relativas* de K respecto a E.

Por otro lado, resulta natural considerar una operación "opuesta" a la suma de cuerpos convexos, lo que da lugar a la llamada *diferencia de Minkowski*, $\sim$. Así, se define el *conjunto paralelo interior de K a distancia* $\lambda \geq 0$ respecto a E como

$$K \sim \lambda E = \bigl\{x \in \mathbb{R}^n : x + \lambda E \subseteq K\bigr\}.$$

Sin embargo, al contrario de lo que ocurre con los conjuntos paralelos exteriores, ni la estructura de la frontera ni el volumen de los cuerpos paralelos interiores pueden "controlarse", tal y como puede apreciarse en la figura 1b.

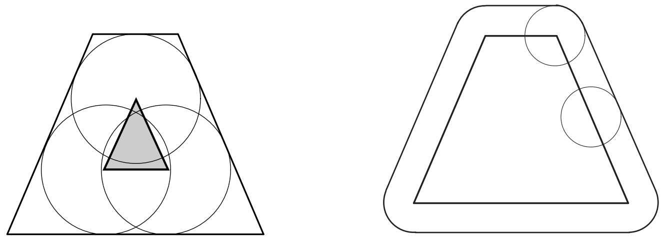

Figura 1b: Cuerpos paralelos interior y exterior de un trapezoide respecto a B^2.

Esta diferencia "geométrica" permite plantear dos cuestiones, en principio de distinta naturaleza, pero en el fondo estrechamente relacionadas, como se pondrá de manifiesto a lo largo del trabajo recogido en esta memoria: estudiar, por un lado, el comportamiento de la frontera del cuerpo paralelo interior respecto al original (lo que viene a ser lo mismo que estudiar cómo son los vectores normales exteriores en los puntos de la misma) y, por otro, intentar conocer cómo se comportan los paralelos interiores con respecto a las quermassintegrales en general, y el volumen en particular. Esta última cuestión sugiere a su vez dos tipos de problemas:

i) Dado que no es posible expresar el volumen (o quermassintegrales) del conjunto paralelo interior mediante una fórmula precisa que involucre las medidas del cuerpo original, la pregunta natural que se plantea es de qué forma acotar (superior e inferiormente) dichas magnitudes.

ii) En segundo lugar, el conjunto de los paralelos interiores y exteriores de un cuerpo convexo K (respecto a E) es una familia uniparamétrica (con parámetro λ) de conjuntos que permite ver las quermassintegrales como funciones dependientes de una variable,

$$\mathrm{W}_i(\lambda) = \begin{cases} \mathrm{W}_i\bigl(K \sim |\lambda|\, E; E\bigr) & \text{si} \quad -\mathrm{r}(K;E) \leq \lambda \leq 0, \\ \mathrm{W}_i(K + \lambda E; E) & \text{si} \qquad 0 \leq \lambda < \infty, \end{cases}$$

donde $\mathrm{r}(K;E) = \max\{r : \exists\, x \in \mathbb{R}^n \text{ con } x + rE \subseteq K\}$ es el inradio (relativo) de K. Claramente, las funciones $\mathrm{W}_i(\lambda)$ son continuas. Pero, ¿son diferenciables para cualquier cuerpo convexo? Y, en caso de que lo sean, ¿toma la derivada el mismo valor en el intervalo $\bigl[-\mathrm{r}(K;E), 0\bigr]$ –paralelos interiores– y en el intervalo $[0,\infty)$ –paralelos exteriores? La respuesta va a ser, obviamente, negativa, lo que permite plantear el problema de clasificar los cuerpos convexos dependiendo de la diferenciabilidad de sus quermassintegrales (en el sentido anterior), problema que fue propuesto originalmente por Hadwiger en $\mathbb{R}^3$ y cuando $E = B^3$.

Por otro lado, el hecho de que un cuerpo convexo K venga determinado de forma única por su función soporte, h_K, permite introducir técnicas puramente analíticas a la hora de estudiar estos problemas. Así, dados un cuerpo convexo K (verificando ciertas condiciones adicionales) y, de forma general, una función $\psi : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ definida sobre la esfera unidad $\mathbb{S}^{n-1}$, de clase $\mathcal{C}^2$, podemos encontrar un $\varepsilon > 0$ suficientemente pequeño de modo que $h_K + \lambda\psi$ es la función soporte de algún cuerpo convexo K_λ para todo $\lambda \in (-\varepsilon, \varepsilon)$ (lo que vendría a ser un cuerpo paralelo exterior/interior respecto a ψ). Así, el volumen, las quermassintegrales, o cualquier otro funcional que interese estudiar, puede verse como una función $F : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ en la variable λ de la forma $F(\lambda) = F\big(h_K + \lambda\psi\big)$.

Podríamos decir, a grandes rasgos, que esta memoria está dedicada al estudio de los cuerpos paralelos interiores, así como su relación con las quermassintegrales en general, y el volumen en particular, tanto desde un punto de vista puramente geométrico (capítulos 1–4) como analítico (capítulo 5). A continuación vamos a proceder a describir el contenido específico de cada uno de los cinco capítulos en que se ha estructurado este trabajo.

La memoria comienza con un *primer capítulo* introductorio, en el que se establece la notación a seguir y se presentan brevemente los conceptos y resultados que serán fundamentales en el posterior desarrollo de los contenidos, tanto de convexidad general, como de los volúmenes mixtos y medidas de área en particular. Así, en una primera sección, se recuerdan nociones importantes como suma de Minkowski, politopos, función e hiperplano soportes, etc. A continuación, se introducen los volúmenes mixtos y las medidas de área mixtas, dedicando un apartado a las desigualdades más importantes que los relacionan: desigualdad de Aleksandrov-Fenchel, desigualdad de Brunn-Minkowski, desigualdades de Minkowski, desigualdad isoperimétrica... Seguidamente, el capítulo se centra en el estudio en profundidad del polinomio (relativo) de Steiner, y más concretamente, en el estudio de sus raíces. La idea es buscar qué propiedades geométricas de un cuerpo convexo "esconden" las raíces de su polinomio de Steiner. Así, en lo que es el primer trabajo original recogido en la memoria, se demuestra que los cuerpos convexos de $\mathbb{R}^2$ y $\mathbb{R}^3$ pueden caracterizarse de forma precisa según el tipo de raíces que tenga su polinomio de Steiner. Esta clasificación tiene una traducción inmediata en el diagrama de Blaschke, lo que nos ha permitido avanzar de forma significativa en el llamado *problema de Blaschke*, que aún se encuentra abierto en la actualidad.

El *segundo capítulo* está dedicado al estudio del problema original de Hadwiger en dimensión 3: clasificar los cuerpos convexos de $\mathbb{R}^3$ dependiendo de la diferenciabilidad de sus quermassintegrales asociadas (relativas a B^3) con respecto a la familia (dependiente de un parámetro) de los cuerpos paralelos interiores y exteriores. Aunque este problema va a ser tratado en su mayor generalidad (en dimensión n arbitraria y respecto a un cuerpo fijo E) en el capítulo siguiente, dedicamos esta segunda parte de la memoria al caso particular de $\mathbb{R}^3$ y la bola euclídea por dos razones concretas: en primer lugar, por ser éste el problema original planteado por Hadwiger en 1955, y en segundo, debido a la estrecha relación del mismo con el comportamiento de las raíces del polinomio de Steiner. Dado que en dimensión 3 disponemos de una caracterización precisa de los cuerpos convexos según el tipo de raíces de su polinomio de Steiner, ésta va a permitir excluir conjuntos de las diversas clases en que Hadwiger clasifica los cuerpos convexos de $\mathbb{R}^3$. En definitiva, los resultados principales que se recogen en este capítulo versan sobre cómo se comportan los paralelos interiores de un cuerpo convexo cuyo polinomio de Steiner tenga un determinado tipo de raíces.

Los cuerpos convexos pueden clasificarse, dependiendo de la diferenciabilidad de sus quermassintegrales, del siguiente modo. Fijado E con interior no vacío, se dice que un cuerpo convexo $K \subset \mathbb{R}^n$ *pertenece a la clase* $\mathcal{R}_p$, $0 \le p \le n-1$, si para cualquier $0 \le i \le p$ y para todo $-\mathrm{r}(K;E) \le \lambda < \infty$ se tiene que

$$'\mathrm{W}_i(\lambda) = \mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda),$$

donde $'\mathrm{W}_i$ y W_i' representan, respectivamente, las derivadas por la izquierda y por la derecha de W_i. En el *tercer capítulo* de la memoria se caracterizan los conjuntos convexos pertenecientes a la clase no trivial más pequeña, $\mathcal{R}_{n-1}$, y se obtienen condiciones necesarias para que un cuerpo K esté en cada una de las clases restantes $\mathcal{R}_p$, $1 \le p \le n-2$. Estas condiciones vienen dadas en términos de ciertas medidas de área de superficie mixtas y en función del tipo de vectores normales que tienen los puntos de la frontera de K; de particular interés son los vectores normales 0-extremos, i.e., aquéllos que no pueden ponerse como combinación lineal de otros dos vectores normales en el mismo punto, que sean linealmente independientes.

Así, la primera sección del capítulo está dedicada a estudiar con detenimiento la relación existente entre los vectores 0-extremos del cuerpo original y los de, o bien sus paralelos interiores, o bien el llamado cuerpo forma asociado a K, que jugará un papel relevante en esta memoria. Estos resultados serán fundamentales en las siguientes secciones del capítulo (así como en el capítulo cuarto) donde se entra de lleno en el estudio minucioso de las clases $\mathcal{R}_p$. Finalmente, una clase de conjuntos de especial relevancia son los llamados cuerpos p-tangenciales, a los que está dedicada la última de las secciones del capítulo 3; como se verá a lo largo del mismo, estas figuras van a aparecer de forma natural como conjuntos extremales en numerosas desigualdades y relaciones.

Como ya hemos comentado, no es posible dar una fórmula explícita para el volumen (análogamente, las quermassintegrales) del paralelo interior de un cuerpo convexo K. Así, el *cuarto capítulo* de esta memoria está dedicado a la determinación de las mejores cotas posibles, tanto superiores como inferiores, del volumen del paralelo interior en función de las medidas del cuerpo original. En el caso particular de que K sea, precisamente, un paralelo exterior de E, se puede demostrar que

$$\mathrm{V}(K \sim \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)(-\lambda)^i,$$

en un rango adecuado. Así pues, resulta natural intentar encontrar cotas superiores y/o inferiores para el volumen de $K \sim \lambda E$ en términos del llamado *polinomio alternado de Steiner*, es decir, el polinomio obtenido al sustituir λ por $-\lambda$ en (1b). Es más, en [34], Matheron conjetura que el polinomio alternado de Steiner proporciona siempre una cota inferior para $\mathrm{V}(K \sim \lambda E)$. Utilizando las clases $\mathcal{R}_p$ estudiadas en el capítulo anterior, se demuestra que las mejores cotas posibles para el volumen del paralelo interior de un cuerpo convexo $K \in \mathcal{R}_p$, vienen dadas por funcionales del tipo

$$\sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(K;E)(-\lambda)^i \pm \binom{n}{p+2}(n-p-2)\int_0^\lambda \frac{(\lambda-s)^{p+2}}{\mathrm{r}(K;E)-s}\mathrm{W}_{p+2}(-s)\,ds$$

en los que, como puede verse, interviene parte del polinomio alternado de Steiner, pero no éste exclusivamente; de hecho, en la segunda sección del capítulo se demuestra que es imposible acotar $\mathrm{V}(K \sim \lambda E)$ utilizando sólo el polinomio alternado, lo que prueba, en particular, la no-veracidad de la conjetura de Matheron. Resultados análogos se obtienen para las quermassintegrales de un cuerpo convexo en las dos últimas secciones de este capítulo.

Finalmente, un breve comentario sobre el *quinto* y último *capítulo* de esta memoria. El hecho de poder identificar un cuerpo convexo con su función soporte permite dar un tratamiento analítico a muchos problemas en Convexidad. Concretamente, dados un cuerpo convexo $K \subset \mathbb{R}^n$ de clase $\mathcal{C}^2_+$, es decir, tal que su frontera es una hipersuperficie de clase $\mathcal{C}^2$ con curvatura de Gauss estrictamente positiva en todo punto, y una función $\psi : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ de clase $\mathcal{C}^2$, si consideramos las quermassintegrales W_i como funciones de la variable real $\lambda \in (-\varepsilon, \varepsilon)$, para $\varepsilon > 0$ tal que $h_K + \lambda\psi$ es una función soporte, la desigualdad de Brunn-Minkowski, a saber,

$$\mathrm{W}_i\big((1-t)K + tL\big)^{1/(n-i)} \geq (1-t)\mathrm{W}_i(K)^{1/(n-i)} + t\mathrm{W}_i(L)^{1/(n-i)}$$

para cuerpos convexos cualesquiera $K, L \subset \mathbb{R}^n$, $t \in [0,1]$ e $i = 0, \ldots, n$, nos va a asegurar que $\mathrm{W}_i^{1/(n-i)}$ es cóncava, ahora en la variable λ. La concavidad de estas funciones va a permitir obtener ciertas desigualdades de *tipo Poincaré*, principal objetivo de este último capítulo. En primer lugar se trabajará con cuerpos convexos de clase $\mathcal{C}^2_+$ y, dado que todo cuerpo convexo puede aproximarse (en la métrica de Hausdorff) por cuerpos convexos de este tipo, en la última sección del capítulo 5 se van a obtener desigualdades de tipo Poincaré para cuerpos convexos arbitrarios.

Los resultados originales que se encuentran recogidos en esta memoria pueden encontrarse en nuestros trabajos [13, 26, 27, 28, 29, 30, 46].

Chapter 1

Preliminaries. The Steiner polynomial

This first chapter is devoted to the study of the Steiner polynomial of a convex body K focussed on the behavior and nature of its roots. In the first section we make a brief survey of the main definitions, properties and results of convex bodies which will be needed for the further study of the mentioned polynomial.

1.1 Convex bodies and their properties

Throughout this dissertation, we will use the following standard notation. We write $\mathbb{R}^n$ to denote the n-dimensional Euclidean space, endowed with the standard inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|$. The closure of a set $\Omega \subseteq \mathbb{R}^n$ is denoted by $\operatorname{cl} \Omega$, its boundary by $\operatorname{bd} \Omega$ and its interior by $\operatorname{int} \Omega$. The *dimension* of a set $\Omega \subseteq \mathbb{R}^n$, i.e., the dimension of the smallest affine subspace containing Ω (its *affine hull*, $\operatorname{aff} \Omega$) is denoted by $\dim \Omega$. Regarding the dimension of a convex set Ω, we write $\operatorname{relint} \Omega$ and $\operatorname{relbd} \Omega$ to denote, respectively, the relative interior and the relative boundary of Ω, i.e., the interior and the boundary of the set Ω relative to its affine hull.

The following definitions and properties are well known and can be found in any book on Convexity, for instance [8, 15, 21, 49, 54, 56]. We would like to mention also the work [1].

Definition 1.1.1. *A (non-empty) set $\Omega \subseteq \mathbb{R}^n$ is said to be* convex *if, whenever two points $x, y \in \Omega$, then the convex combination $\lambda x + (1 - \lambda) y \in \Omega$, for $0 \leq \lambda \leq 1$.*

Definition 1.1.2. *A* convex body *$K \subset \mathbb{R}^n$ is a compact convex set. Moreover, a convex body is called* strictly convex *if its boundary does not contain a segment.*

From now on $\mathcal{K}^n$ will denote the set of all convex bodies in $\mathbb{R}^n$. The subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior is denoted by $\mathcal{K}^n_0$. Let $B^n(p,\mathrm{r}) = \bigl\{x \in \mathbb{R}^n : |x-p| \leq \mathrm{r}\bigr\}$ be the (closed) ball of radius $\mathrm{r} > 0$ centered at $p \in \mathbb{R}^n$; in particular, we will write $B^n = B^n(0,1)$ for the n-dimensional unit ball. Finally, let $\mathbb{S}^{n-1} = \bigl\{u \in \mathbb{R}^n : |u| = 1\bigr\}$ be the $(n-1)$-dimensional unit sphere. The *Minkowski sum* of two convex bodies $K, L \in \mathcal{K}^n$ is defined by

$$K + L = \{x + y : x \in K \text{ and } y \in L\},$$

which is clearly a convex body, and we write $\lambda K = \{\lambda\, x : x \in K\}$, for $\lambda \in \mathbb{R}$.

For every set $\Omega \subseteq \mathbb{R}^n$, there exists a convex set containing it. The intersection of all convex sets containing Ω is the *convex hull* of Ω, and it will be denoted by $\operatorname{conv}\Omega$; thus $\operatorname{conv}\Omega$ is the smallest convex set containing Ω. The convex hull of a compact set is always a convex body; in particular, the convex hull of a finite number of points is so and the family of all of them represents a very important class of convex bodies:

Definition 1.1.3. *A* polytope *is the convex hull of finitely many points in $\mathbb{R}^n$ (its* vertices*).*

The space of convex bodies $\mathcal{K}^n$ is endowed with the *Hausdorff metric*, namely

$$\delta(K,L) = \min\bigl\{\lambda \geq 0 : K \subseteq L + \lambda B^n,\, L \subseteq K + \lambda B^n\bigr\} \quad \text{for } K, L \in \mathcal{K}^n,$$

which allows to consider continuity of functionals and approximation. In this respect, it is well-known that any convex body can be approximated by polytopes (see [49, Theorems 1.8.13, 1.8.15]), as well as by convex bodies with differentiable boundaries. Regarding the differentiability, the following notation will be used. We will say that $K \in \mathcal{K}^n$ is of *class* $\mathcal{C}^k$, $k \in \mathbb{N}$, if its boundary hypersurface is a regular submanifold (in the sense of differential geometry) that is k-times continuously differentiable. The following assumption, stronger than $\mathcal{C}^k$, will be important: we say that K is of class $\mathcal{C}^k_+$ if $\operatorname{bd} K \in \mathcal{C}^k$ (with $k \geq 2$) and its Gauss curvature is strictly positive at every point. The following approximation result will be needed in the last chapter of this work; in fact, much stronger properties are satisfied by the convex bodies involved there (see [49, Notes to Section 2.5, p. 119]), but for our purposes the differentiability condition will be enough.

Proposition 1.1.4. *For any convex body $K \in \mathcal{K}^n$ there exists a sequence $(K_m)_{m\in\mathbb{N}}$ of convex bodies of class $\mathcal{C}^2_+$ converging to K in the Hausdorff metric.*

In spite of the fact that many of the following properties and definitions are valid for closed convex sets, in order to simplify the exposition we will restrict them to compact ones, since we will always work under the hypothesis of compactness. An important notion is the following one:

Definition 1.1.5. *Let $K \in \mathcal{K}^n$. A hyperplane H is called a* supporting hyperplane *of K if $H \cap K \neq \emptyset$ and K is contained in one of the two halfspaces determined by H, which is called its* supporting halfspace*. For each $u \in \mathbb{S}^{n-1}$, the supporting hyperplane and the supporting halfspace to K with outer normal vector u will be denoted, respectively by $H(K,u)$ and $H^-(K,u)$.*

The following classical results concerning supporting hyperplanes will be needed in the following.

Theorem 1.1.6. *At every point of the boundary of a convex body $K \in \mathcal{K}^n$ there exists a supporting hyperplane of K. Furthermore, for every $u \in \mathbb{S}^{n-1}$ there is a supporting hyperplane of K with outer normal vector u.*

Supporting hyperplanes can be used to characterize convexity, since if $K \subset \mathbb{R}^n$ is a compact set with non-empty interior, then K is convex if and only if for every $x \in \operatorname{bd} K$ there exists a supporting hyperplane to K. As a consequence, we get that any convex body is the intersection of its supporting halfspaces.

There is no doubt that convex functions play an important role in the theory of convex bodies. In this context, the most important one is the so called support function.

Definition 1.1.7. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is* convex *if for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,*

$$f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y).$$

A function f is concave *if $-f$ is convex; or equivalently, if for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,*

$$f\big((1-\lambda)x + \lambda y\big) \geq (1-\lambda)f(x) + \lambda f(y).$$

The following properties of convex functions will be needed later. For references and further study we refer for instance to [39, 49].

Proposition 1.1.8. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex (concave) function. Then,*

i) *f is continuous in $\operatorname{int}\operatorname{dom} f$ and*

ii) *if $n = 1$, the left and right derivatives, denoted respectively by $'f$ and f', do exist at every point and they are increasing (decreasing) functions. Moreover, $'f \leq f'$ ($'f \geq f'$).*

Definition 1.1.9. *The* support function *of a convex body $K \in \mathcal{K}^n$ in the direction $u \in \mathbb{S}^{n-1}$, denoted by $h(K, u)$ or $h_K(u)$, is the real valued function defined on the sphere by*

$$h(K, u) = \max\big\{\langle x, u\rangle : x \in K\big\}.$$

The radial function *of a convex body $K \in \mathcal{K}^n_0$ with $0 \in \operatorname{int} K$, denoted by ρ_K, is the real valued function defined on $\mathbb{R}^n \backslash \{0\}$ by*

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

The support function and the radial function of a convex body $K \in \mathcal{K}^n_0$ with $0 \in \operatorname{int} K$ are related by means of the *polar body*, $K^\circ = \big\{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } y \in K\big\}$, namely

$$h(K^\circ, \cdot) = \frac{1}{\rho_K(\cdot)}. \tag{1.1}$$

The support function of a convex body may be introduced on $\mathbb{R}^n$, but for our purposes we consider it defined on $\mathbb{S}^{n-1}$. It has many useful properties; here we detail just the ones we will need further on.

Proposition 1.1.10. *Let $K, L \in \mathcal{K}^n$ and $u, v \in \mathbb{S}^{n-1}$.*

i) $h(K+L,u) = h(K,u) + h(L,u)$ *and* $h(\lambda K, u) = \lambda h(K,u)$ *for all* $\lambda \geq 0$.

ii) *If* $K \subseteq L$ *then* $h(K,u) \leq h(L,u)$.

iii) $h(K, \lambda u) = \lambda h(K,u)$ *for all* $\lambda \geq 0$.

iv) $h(K, u+v) \leq h(K,u) + h(K,v)$.

In particular, $h(K,\cdot)$ is convex. The above last two properties are usually expressed by saying that $h(K,\cdot)$ is *sublinear*, i.e., *positively homogeneous* –(iii)– and *subadditive* –(iv). In fact, it turns out that they characterize support functions:

Theorem 1.1.11. *Let h be a sublinear (real-valued) function defined on $\mathbb{S}^{n-1}$. Then there exists a unique convex body $K \in \mathcal{K}^n$ such that $h = h(K,\cdot)$. Moreover, any convex body $K \in \mathcal{K}^n$ is determined by its support function.*

On the other hand, if K is a convex body in $\mathbb{R}^n$ then

$$K = \Big\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h(K,u) \text{ for every } u \in \mathbb{S}^{n-1} \Big\},$$

or equivalently,

$$K = \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h(K,u) \Big\}.$$

We finish this section by formulating the famous *Blaschke selection theorem*, which provides a very useful tool in proving the existence of convex bodies with specific properties.

Theorem 1.1.12 (Blaschke selection theorem). *Any bounded sequence of convex bodies in $\mathbb{R}^n$ contains a convergent subsequence (in the Hausdorff metric).*

1.2 Mixed volumes and surface area measures. The Steiner formula

The so called (relative) *Steiner formula* of a convex body $K \in \mathcal{K}^n$ is nothing else but a polynomial of degree n, which expresses the volume of the (vectorial) sum of K with an homothetic copy of a fixed body $E \in \mathcal{K}^n$ with factor λ (the variable of the polynomial). This section is focussed on the study of this polynomial mostly in the three dimensional case, from an algebraic point of view, in the sense of searching and studying the geometric properties of its algebraic roots.

In order to introduce the Steiner polynomial and the general setting involving the so called mixed volumes, we need the following definitions.

Definition 1.2.1. *Given a convex body $K \in \mathcal{K}^n$, the* volume *of K is defined as its Lebesgue measure and will be denoted by $\mathrm{V}(K)$.*

Proposition 1.2.2. *Let $K, L \in \mathcal{K}^n$. The following holds:*

i) *If* $\dim K = n$ *then* $\mathrm{V}(K) > 0$. *If* $\dim K \leq n-1$ *then* $\mathrm{V}(K) = 0$.

ii) $\mathrm{V}(\lambda K) = \lambda^n \mathrm{V}(K)$ *for* $\lambda \geq 0$.

iii) *The volume* $\mathrm{V} : \mathcal{K}^n \longrightarrow \mathbb{R}^+$ *is a continuous function on the space of convex bodies.*

iv) $\kappa_n := \mathrm{V}(B^n) = \pi^{n/2}/\Gamma\big((n/2)+1\big)$, *where* Γ *denotes the usual* gamma function.

Combining the notions of volume and Minkowski sum, the concept of mixed volume appears (as well as the notion of mixed surface area measure). For a deep study of mixed volumes and mixed surface area measures we refer mainly to [49, Section 5.1].

Theorem 1.2.3. *Let $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_i \geq 0$ for $i = 1, \dots, m$. The volume of the Minkowski sum $\sum_{i=1}^m \lambda_i K_i$ is given by*

$$\mathrm{V}\left(\sum_{i=1}^m \lambda_i K_i\right) = \sum_{i_1=1}^m \cdots \sum_{i_m=1}^m \lambda_{i_1} \cdots \lambda_{i_n} \mathrm{V}(K_{i_1}, \dots, K_{i_n}).$$

The coefficients $\mathrm{V}(K_{i_1}, \dots, K_{i_n})$ are symmetric in the indices for any permutation, and they are called the mixed volumes *of $K_1, \dots, K_m$.*

A kind of reciprocal says that the mixed volume of n convex bodies can be obtained from the volume of Minkowski sums of those convex bodies:

$$\mathrm{V}(K_1, \dots, K_n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{n+k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathrm{V}(K_{i_1} + K_{i_2} + \cdots + K_{i_k}).$$

For the sake of brevity we will use the abbreviation

$$\big(K_1[r_1], \dots, K_m[r_m]\big) \equiv \big(K_1, \overset{(r_1)}{\dots}, K_1, \dots, K_m, \overset{(r_m)}{\dots}, K_m\big).$$

Surface area measures can be viewed as *local* generalizations of mixed volumes and they can be defined as follows.

Theorem 1.2.4. *Let $K_1, \dots, K_{n-1} \in \mathcal{K}^n$. Then there exists a unique finite Borel measure on $\mathbb{S}^{n-1}$, the so called* mixed surface area measure $\mathrm{S}(K_1, \dots, K_{n-1}; \cdot)$, *such that*

$$\mathrm{V}(K, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u) \, d\mathrm{S}(K_1, \dots, K_{n-1}; u) \tag{1.2}$$

for any convex body $K \in \mathcal{K}^n$.

In particular, $\mathrm{S}_i(K; \cdot) := \mathrm{S}\big(K[i], B^n[n-i-1]; \cdot\big)$ is called the *i-th order surface area measure* of K, for $i = 0, \dots, n-1$.

Some useful properties of the mixed volumes and mixed surface area measures are listed in the following proposition; they will be needed throughout this work.

Proposition 1.2.5. *Let $K, L, K_1, \dots, K_n \in \mathcal{K}^n$. The following properties hold:*

i) $\mathrm{V}(K,\dots,K) = \mathrm{V}\big(K[n]\big) = \mathrm{V}(K)$.

ii) $\mathrm{S}(K_1,\dots,K_{n-1};\cdot)$ *is symmetric in the indices for any permutation.*

iii) *For all* $i = 0,\dots,n-1$, $\mathrm{S}_i(K;\mathbb{S}^{n-1}) = n\mathrm{V}\big(K[i], B^n[n-i]\big)$.

iv) *For all* $i = 0,\dots,n-1$, $\mathrm{S}_0(K;\cdot) = \mathrm{S}_i(B^n;\cdot)$ *is the usual spherical Lebesgue measure.*

v) $\mathrm{V}(\alpha K + \beta L, K_2,\dots,K_n) = \alpha\mathrm{V}(K,K_2,\dots,K_n) + \beta\mathrm{V}(L,K_2,\dots,K_n)$ *for every* $\alpha,\beta \geq 0$, *and* $\mathrm{S}(\alpha K + \beta L, K_2,\dots,K_{n-1};\cdot) = \alpha\mathrm{S}(K,K_2,\dots,K_{n-1};\cdot) + \beta\mathrm{S}(L,K_2,\dots,K_{n-1};\cdot)$, *i.e., mixed volumes and surface area measures are linear in each argument.*

vi) *If* $K \subseteq L$ *then* $\mathrm{V}(K,K_2,\dots,K_n) \leq \mathrm{V}(L,K_2,\dots,K_n)$, *i.e., they are monotonous (in each argument).*

vii) $\mathrm{V}(K_1,\dots,K_n) \geq 0$. *Moreover,* $\mathrm{V}(K_1,\dots,K_n) > 0$ *if and only if there are segments* $\sigma_i \subset K_i$, $i = 1,\dots,n$, *with linearly independent directions.*

viii) *Mixed volumes are continuous functions on* $(\mathcal{K}^n)^n$ *and mixed surface area measures are weakly continuous on* $(\mathcal{K}^n)^{n-1}$.

Besides the volume (see Proposition 1.2.5 (i)), other well-known measures of a convex body K are also particular cases of mixed volumes, namely: $n\mathrm{V}\big(K[n-1], B^n\big) = \mathrm{S}(K)$ is the usual *surface area* of K, $(2/\kappa_n)\mathrm{V}\big(K, B^n[n-1]\big) = \mathrm{b}(K)$ is its *mean width* and $n\mathrm{V}\big(K[n-2], B^n[2]\big) = \mathrm{M}(K)$ is the *integral mean curvature* of K. Notice that $\mathrm{S}(K) = \mathrm{S}_{n-1}(K;\mathbb{S}^{n-1})$ (cf. Proposition 1.2.5, iii)), hence the name of surface area measure.

In the particular case of two convex bodies $K, E \in \mathcal{K}^n$, the mixed volumes $\mathrm{V}\big(K[n-i], E[i]\big)$, for $i = 0,\dots,n$, are called *relative quermassintegrals* of K (with respect to E) and they are denoted by $\mathrm{W}_i(K;E)$. Moreover, when $E = B^n$ the i-th quermassintegral $\mathrm{W}_i(K;B^n) := \mathrm{W}_i(K)$ is just called i-th *quermassintegral* of K. Taking into account the following definition, the so called *relative Steiner formula* or *Minkowski-Steiner formula* is obtained (cf. Theorem 1.2.3).

Definition 1.2.6. *For* $K \in \mathcal{K}^n$, *the* outer parallel body (relative to E) *of* K *at distance* $\lambda \geq 0$ *is the Minkowski sum* $K + \lambda E$.

Theorem 1.2.7 (The relative Steiner formula). *Let* $K, E \in \mathcal{K}^n$. *The volume of the outer parallel body of* K *with respect to* E *at distance* $\lambda \geq 0$ *can be expressed as*

$$\mathrm{V}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^i. \tag{1.3}$$

Notice that if $E \in \mathcal{K}^n_0$ then the polynomial in the right-hand side of (1.3), the so called *relative Steiner polynomial*, has degree n, i.e., the dimension of the space. Steiner [52] derived this result in $\mathbb{R}^2$ and $\mathbb{R}^3$ when $E = B^n$ (then the classical Steiner polynomial is obtained), for polytopes and convex bodies with boundary of class $\mathcal{C}^2_+$.

In fact, taking into account that quermassintegrals are particular cases of mixed volumes, the following *Steiner formulae* for the relative quermassintegrals can be obtained.

Theorem 1.2.8 (Steiner formulae for relative quermassintegrals). *Let $K, E \in \mathcal{K}^n$ and let λ be a positive number. The relative i-th quermassintegral, $i = 0, \dots, n$, of the outer parallel body of K (relative to E), $K + \lambda E$, can be expressed as a polynomial in the parameter λ,*

$$\mathrm{W}_i(K + \lambda E; E) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K; E)\lambda^k. \tag{1.4}$$

If $E \in \mathcal{K}^n_0$, then the polynomial in the right-hand side of (1.4) has degree $n - i$. Throughout this text the relative Steiner polynomial/formula will be simply called Steiner polynomial/formula.

Before recalling some famous inequalities regarding mixed volumes/quermassintegrals, we include here a couple of additional definitions that we will use often.

Definition 1.2.9. *The* relative inradius $\mathrm{r}(K; E)$ *and* relative circumradius $\mathrm{R}(K; E)$ *of K relative to E are defined, respectively, by*

$$\mathrm{r}(K; E) = \max\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\,E \subseteq K\},$$
$$\mathrm{R}(K; E) = \min\{R : \exists\, x \in \mathbb{R}^n \text{ with } K \subseteq x + R\,E\}.$$

Notice that it always holds

$$\mathrm{r}(K; E)\mathrm{R}(E; K) = 1. \tag{1.5}$$

In the particular case when $E = B^n$ the classical inradius $\mathrm{r}(K) = \mathrm{r}(K; B^3)$ and circumradius $\mathrm{R}(K) = \mathrm{R}(K; B^3)$ are obtained.

1.2.1 Inequalities for mixed volumes

Mixed volumes satisfy several inequalities. Here we collect some of the most relevant ones, which will be needed throughout this work. Notice first that

$$\mathrm{r}(K; E)\, \mathrm{W}_{i+1}(K; E) \le \mathrm{W}_i(K; E) \le \mathrm{R}(K; E)\, \mathrm{W}_{i+1}(K; E), \tag{1.6}$$

for $i \in \{0, \dots, n-1\}$; indeed since, up to translations, $\mathrm{r}(K; E)E \subseteq K$ and $K \subseteq \mathrm{R}(K; E)E$ these inequalities are a direct consequence of the monotonicity of the mixed volumes (see Proposition 1.2.5, part vi)). The equality case in these inequalities will be considered and studied in Chapter 3 (see Theorem 3.1.5).

We dare to say that the most important inequality relating mixed volumes is the *Aleksandrov-Fenchel inequality*. For a deep study of this inequality we refer to [49, Sections 6.3, 6.6].

Theorem 1.2.10 (Aleksandrov-Fenchel inequality). *Let $K_1,\dots,K_n\in\mathcal{K}^n$. Then*

$$\mathrm{V}(K_1,K_2,K_3,\dots,K_n)^2\geq \mathrm{V}(K_1,K_1,K_3,\dots,K_n)\mathrm{V}(K_2,K_2,K_3,\dots,K_n). \tag{1.7}$$

Clearly, equality holds in (1.7) if K_1 and K_2 are homothetic. However, the complete classification of the equality case has not yet been settled. Only in several special cases the solution is known.

As particular cases of the most general Aleksandrov-Fenchel inequality (1.7) we get the so called Aleksandrov-Fenchel inequalities for quermassintegrals: for all $i=1,\dots,n-1$,

$$\mathrm{W}_i(K;E)^2\geq \mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E). \tag{1.8}$$

The potent extensions of the classical Brunn-Minkowski inequality (some of them very recent) have a great impact on many different fields in Mathematics (and even further on –the study of gases, crystals...). Its statement is rather simple: it ensures the concavity of the n-th root of the volume functional $\mathrm{V}:\mathcal{K}^n\longrightarrow\mathbb{R}$.

Theorem 1.2.11 (Brunn-Minkowski inequality). *For convex bodies $K,L\in\mathcal{K}^n$ and $t\in[0,1]$,*

$$\mathrm{V}\bigl((1-t)K+tL\bigr)^{1/n}\geq (1-t)\mathrm{V}(K)^{1/n}+t\mathrm{V}(L)^{1/n}. \tag{1.9}$$

If $t\in(0,1)$ then equality holds if and only if either K and L are homothetic or they lie in parallel hyperplanes.

Brunn-Minkowski inequality has a more general version for measurable sets, as well as an integral version usually called *Prékopa-Leindler inequality.* This theorem can be found in any of the already mentioned books of classical Convexity; but we want to refer specially to [19], a beautiful survey on this inequality.

There exists also a general Brunn-Minkowski theorem stating an analogous inequality for mixed volumes (see [49, p. 339]), and in particular, for every (relative) quermassintegral, which we will need later.

Theorem 1.2.12 (Brunn-Minkowski inequality for quermassintegrals). *For convex bodies $K,L,E\in\mathcal{K}^n$, $t\in[0,1]$ and any $i\in\{0,\dots,n\}$,*

$$\mathrm{W}_i\bigl((1-t)K+tL;E\bigr)^{1/(n-i)}\geq (1-t)\mathrm{W}_i(K;E)^{1/(n-i)}+t\mathrm{W}_i(L;E)^{1/(n-i)}, \tag{1.10}$$

i.e., the $(n-i)$-th root of the i-th quermassintegral $\mathrm{W}_i(\,\cdot\,;E):\mathcal{K}^n\longrightarrow\mathbb{R}$ is a concave function. If $t\in(0,1)$ then equality holds for $0\leq i<n-1$ if and only if either K and L are homothetic or they lie in parallel $(n-i-1)$-planes.

These inequalities can be also obtained as consequences of the Aleksandrov-Fenchel inequality (see [49, Section 6.4]).

By using the concavity of the n-th root of the volume (i.e., Brunn-Minkowski inequality) and the fact that the volume of the Minkowski convex combination $(1-t)K+tL$ is a polynomial in $t\in[0,1]$, another two important inequalities can be obtained, namely, the *first* and the *second Minkowski inequalities* (see [49, Section 6.2]).

Theorem 1.2.13 (Minkowski inequalities). *Let $K,L\in\mathcal{K}^n$. Then*

$$\begin{aligned} \mathrm{V}\bigl(K[n-1],L\bigr)^n &\geq \mathrm{V}(K)^{n-1}\mathrm{V}(L),\\ \mathrm{V}\bigl(K[n-1],L\bigr)^2 &\geq \mathrm{V}(K)\mathrm{V}\bigl(K[n-2],L[2]\bigr). \end{aligned} \tag{1.11}$$

*For $K,L\in\mathcal{K}_0^n$, equality holds in the first inequality if and only if K and L are homothetic. For $L\in\mathcal{K}_0^n$, equality holds in the second inequality if and only if either $\dim K<n-1$ or K is homothetic to an $(n-2)$-*tangential body *of L.*

The so called p-tangential bodies, $p=0,\dots,n-1$, will play an important role in this work. In order to state the definition of these sets we need further notions, and hence we omit it here. They will be defined and studied in Chapter 3. Notice that second Minkowski inequality is a particular case of the Aleksandrov-Fenchel inequality (1.7).

By considering the special case where L is the unit ball, first Minkowski inequality in (1.11) reduces to the famous *isoperimetric inequality*,

$$\mathrm{S}(K)^n\geq n^n\kappa_n\mathrm{V}(K)^{n-1}. \tag{1.12}$$

1.3 Steiner polynomials in $\mathbb{R}^2$ and $\mathbb{R}^3$

From now on $E\in\mathcal{K}_0^n$ will be a fixed convex body (with interior points, in order to avoid trivial situations), and everything will be made relative to E. We will point out when E is some particular convex body or an additional property of E is needed. We will also write $f_E(K,\lambda)$ to denote the Steiner polynomial of a convex body K relative to E.

Regarding the Steiner polynomial as a formal polynomial in the complex variable λ, it is a natural question to wonder about the geometric meaning and the behavior of its algebraic roots. In this section we give a classification of the 3-dimensional convex bodies in terms of relations, i.e., equations and inequalities amongst $\mathrm{V}(K)$, $\mathrm{W}_1(K;E)$, $\mathrm{W}_2(K;E)$ and $\mathrm{V}(E)$ with respect to the algebraic type of the roots. As we will see later, this kind of classification seems to be a rather useful approach to a well-known open problem in Convexity: the Blaschke problem. The results we present here can be found in [27], in the case when $E=B^3$.

1.3.1 The Steiner polynomial for a planar convex body

The Steiner polynomial for a planar convex body K is given by $\mathrm{A}(K)+2\mathrm{W}_1(K;E)\lambda+\mathrm{A}(E)\lambda^2$; notice that now $\mathrm{V}(K)$ is just the *area* of K, and we denote it in the usual way $\mathrm{A}(K)$. In this case the retrieval of the roots is trivial,

$$\lambda_1 = \frac{-\mathrm{W}_1(K;E) - \sqrt{\mathrm{W}_1(K;E)^2 - \mathrm{A}(K)\mathrm{A}(E)}}{\mathrm{A}(E)}, \quad \lambda_2 = \frac{-\mathrm{W}_1(K;E) + \sqrt{\mathrm{W}_1(K;E)^2 - \mathrm{A}(K)\mathrm{A}(E)}}{\mathrm{A}(E)},$$

and then, the following very nice properties can be established easily (see [20], where the case $E = B^2$ is considered):

(P1) First Minkowski inequality in the plane, namely $\mathrm{W}_1(K;E)^2 \geq \mathrm{A}(K)\mathrm{A}(E)$ (see Theorem 1.2.13), is equivalent to the fact that all the roots of the Steiner polynomial are *real*.

(P2) $\lambda_1 \leq \lambda_2 \leq 0$.

(P3) *Characterization of the body* E: $\lambda_1 = \lambda_2$, i.e., the Steiner polynomial has a double real root, if and only if the body $K = -\lambda_1 E$.

(P4) Since $\min\bigl\{f_E(K,\lambda) : \lambda \in \mathbb{R}\bigr\} = f_E\bigl(K, -\mathrm{W}_1(K;E)/\mathrm{A}(E)\bigr) = \mathrm{A}(K) - \mathrm{W}_1(K;E)^2/\mathrm{A}(E) \leq 0$, we can say that first Minkowski inequality is equivalent to $\min\bigl\{f_E(K,\lambda) : \lambda \in \mathbb{R}\bigr\} \leq 0$.

In the planar case the relative inradius, circumradius and quermassintegrals are related by the well-known *Bonnesen inequality*

$$\mathrm{W}_1(K;E)^2 - \mathrm{A}(K)\mathrm{A}(E) \geq \frac{\mathrm{A}(E)^2}{4}\Bigl(\mathrm{R}(K;E) - \mathrm{r}(K;E)\Bigr)^2. \tag{1.13}$$

Bonnesen [7] proved this result for $E = B^2$, whereas the proof of the general case is due to Blaschke [4, pp. 33–36]. In fact (1.13) is an immediate consequence of the stronger relations

$$\mathrm{A}(K) + 2\mathrm{W}_1(K;E)\lambda + \mathrm{A}(E)\lambda^2 \leq 0 \ \text{ if } \ -\mathrm{R}(K;E) \leq \lambda \leq -\mathrm{r}(K;E) \tag{1.14}$$

(see [4, pp. 33–36] and also [18]). Thus:

(P5) $\lambda_1 \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \lambda_2$.

Bonnesen's inequality sharpens (in the plane) the Aleksandrov-Fenchel inequality (1.7) and the first Minkowski inequality, and there is no known generalization of it to higher dimensions. Notice that if $E = B^2$ then $2\mathrm{W}_1(K) = \mathrm{p}(K)$ is the *perimeter* of K and the classical and well-known isoperimetric and Bonnesen inequalities are obtained:

$$\mathrm{p}(K)^2 \geq 4\pi\mathrm{A}(K) \quad \text{and} \quad \mathrm{p}(K)^2 - 4\pi\mathrm{A}(K) \geq \frac{\pi^2}{4}\Bigl(\mathrm{R}(K;E) - \mathrm{r}(K;E)\Bigr)^2.$$

In this case, the fact that $f_{B^2}(K,\lambda) = \mathrm{A}(K) + \mathrm{p}(K)\lambda + \pi\lambda^2$ has a double real root provides a characterization of the ball, and the isoperimetric inequality is equivalent to all its roots are real.

In the following subsection we give a complete characterization of the 3-dimensional convex bodies, depending on the algebraic type of roots that their Steiner's polynomial presents. In the case $K = B^3$ this characterization has a very precise interpretation in the Blaschke problem, which we will introduce later; from this, interesting consequences can be obtained for the problem. We also use it for studying the above properties **(P1)**–**(P5)**.

1.3.2 The Steiner formula for a convex body in $\mathbb{R}^3$

Throughout the rest of this section $K \in \mathcal{K}^3$ and $E \in \mathcal{K}^3_0$ will denote convex bodies in $\mathbb{R}^3$, with Steiner polynomial

$$f_E(K,\lambda) = \mathrm{V}(K) + 3\mathrm{W}_1(K,E)\lambda + 3\mathrm{W}_2(K,\lambda)\lambda^2 + \mathrm{V}(E)\lambda^3.$$

It is clear that now all the possibilities are feasible for its roots: $\lambda_1, \lambda_2, \lambda_3$ can be real or complex numbers, which states a first difference with the planar case. Since all the coefficients of the Steiner polynomial are non-negative, Routh-Hurwitz criterion (see e.g. [33, p. 181]) ensures that $f_E(K,\lambda)$ is a *Hurwitz* polynomial, i.e., all its roots lie in the left half plane, if and only if the determinant

$$\begin{vmatrix} 3\mathrm{W}_2(K;E) & \mathrm{V}(K) & 0 \\ \mathrm{V}(E) & 3\mathrm{W}_1(K;E) & 0 \\ 0 & 3\mathrm{W}_2(K;E) & \mathrm{V}(K) \end{vmatrix} = \mathrm{V}(K)\Big[9\mathrm{W}_1(K;E)\mathrm{W}_2(K;E) - \mathrm{V}(E)\mathrm{V}(K)\Big]$$

is positive. This is obtained as an easy consequence of inequalities (1.8). Notice that if $\mathrm{V}(K) = 0$ then one root is zero and the two remaining ones lie in the left half plane, since the second order determinant given by the first minor is also positive; and analogously when $\mathrm{W}_1(K;E) = 0$. In fact, using the Routh-Hurwitz criterion and the inequalities (1.8) it can be proved that the Steiner polynomial is Hurwitz for $n \leq 5$, see [25] and [53, p. 103]. There are however counterexamples in higher dimensions, see [25, 31]. Thus we get that all roots of the 3-dimensional Steiner polynomial have negative real part, which extends property **(P2)**.

Concerning the algebraic type of its roots, since there always exists a real one, the following possibilities appear: a triple real root, a double real root and a simple real root, three different real roots, or two complex roots and a simple real root.

(i) A triple real root. Let us denote by a the triple real root of $f_E(K,\lambda)$. Then, from the identity

$$\mathrm{V}(K) + 3\mathrm{W}_1(K,E)\lambda + 3\mathrm{W}_2(K,\lambda)\lambda^2 + \mathrm{V}(E)\lambda^3 = \mathrm{V}(E)(\lambda - a)^3 = \mathrm{V}(E)(\lambda^3 - 3a\lambda^2 + 3a^2\lambda - a^3),$$

we obtain that $\mathrm{V}(K) = -\mathrm{V}(E)\,a^3$, $\mathrm{W}_1(K;E) = \mathrm{V}(E)\,a^2$ and $\mathrm{W}_2(K;E) = -\mathrm{V}(E)\,a$; hence, $f_E(K,\lambda)$ will have a triple real root if and only if

$$a = -\frac{\mathrm{W}_2(K;E)}{\mathrm{V}(E)}, \quad \mathrm{W}_2(K;E)^2 = \mathrm{V}(E)\mathrm{W}_1(K;E) \quad \text{and} \quad \mathrm{W}_2(K;E)^3 = \mathrm{V}(E)^2\mathrm{V}(K).$$

The convex body E and its homothets are the only sets for which the above two equalities (on the right) hold, see Theorem 1.2.13. Since $a = -\mathrm{W}_2(K;E)/\mathrm{V}(E)$ we obtain that the homothecy factor is precisely $-a$ and then we can state a characterization of this figure in terms of the roots of the Steiner polynomial, which extends property **(P3)**:

Proposition 1.3.1 ([27]). *The Steiner polynomial of a convex body $K \in \mathcal{K}^3$ (relative to $E \in \mathcal{K}^3_0$) has a triple real root a if and only if $K = -aE$ (up to translations).*

In arbitrary dimension the characterization of the "fixed body" E as the only one whose Steiner polynomial has an n-fold real root can be also obtained. In order not to lose the thread of the argument we are developing here and for the sake of completeness, a proof of this fact will be given at the end of this section.

(ii) A double real root and a simple real root. If $f_E(K,\lambda)$ has a double real root, this one is, either its minimum or its maximum. Then, we can find two possible situations (see Figure 1.1), which will be called TYPE 1 and TYPE 2, respectively.

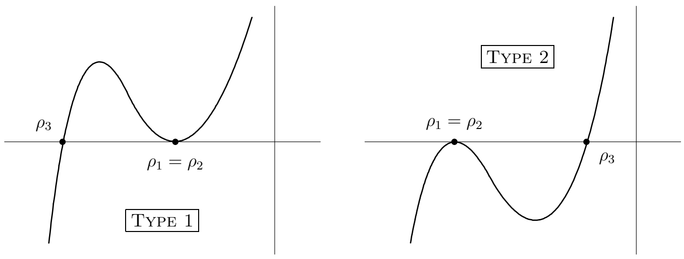

Figure 1.1: The Steiner polynomial has a double root and a simple one.

If we want to compute explicitly the roots of the Steiner polynomial, we just have to find its local extreme values: from $f'_E(K,\lambda) = 3\mathrm{W}_1(K;E) + 6\mathrm{W}_2(K;E)\lambda + 3\mathrm{V}(E)\lambda^2$, we get that $f_E(K,\lambda)$ attains its relative minimum and maximum in

$$\begin{aligned} \lambda_m &= \frac{-\mathrm{W}_2(K;E) + \sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{W}_1(K;E)\mathrm{V}(E)}}{\mathrm{V}(E)} \qquad \text{and} \\ \lambda_M &= \frac{-\mathrm{W}_2(K;E) - \sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{W}_1(K;E)\mathrm{V}(E)}}{\mathrm{V}(E)}, \end{aligned} \tag{1.15}$$

respectively. So, the Steiner polynomial will have a double real root of TYPE 1 (respectively, TYPE 2) if and only if λ_m (respectively, λ_M) is that root; this is also equivalent to $f_E(K,\lambda_m) = 0$ (respectively, $f_E(K,\lambda_M) = 0$). But it is easy to see that

$$f_E(K,\lambda_m) = \frac{\mathrm{V}(E)^2\mathrm{V}(K) - 3\mathrm{V}(E)\mathrm{W}_1(K;E)\mathrm{W}_2(K;E) + 2\mathrm{W}_2(K;E)^3 - 2\Big[\mathrm{W}_2(K;E)^2 - \mathrm{V}(E)\mathrm{W}_1(K;E)\Big]^{3/2}}{\mathrm{V}(E)^2}. \tag{1.16}$$

Therefore, $f_E(K,\lambda_m) = 0$ if and only if

$$2\mathrm{W}_2(K;E)^3 + \mathrm{V}(E)\Big[\mathrm{V}(E)\mathrm{V}(K) - 3\mathrm{W}_1(K;E)\mathrm{W}_2(K;E)\Big] - 2\Big[\mathrm{W}_2(K;E)^2 - \mathrm{V}(E)\mathrm{W}_1(K;E)\Big]^{3/2} = 0. \tag{1.17}$$

Analogously, we obtain $f_E(K,\lambda_M) = 0$ if and only if

$$2\mathrm{W}_2(K;E)^3 + \mathrm{V}(E)\Big[\mathrm{V}(E)\mathrm{V}(K) - 3\mathrm{W}_1(K;E)\mathrm{W}_2(K;E)\Big] + 2\Big[\mathrm{W}_2(K;E)^2 - \mathrm{V}(E)\mathrm{W}_1(K;E)\Big]^{3/2} = 0. \tag{1.18}$$

From now on, these two functionals will appear quite often; so, for the sake of brevity, they will be denoted by ϕ_- and ϕ_+, respectively, i.e.,

$$\begin{aligned}\phi_\mp(K;E) := {} & 2\mathrm{W}_2(K;E)^3 + \mathrm{V}(E)\Big[\mathrm{V}(E)\mathrm{V}(K) - 3\mathrm{W}_1(K;E)\mathrm{W}_2(K;E)\Big] \\ & \mp 2\Big[\mathrm{W}_2(K;E)^2 - \mathrm{V}(E)\mathrm{W}_1(K;E)\Big]^{3/2}.\end{aligned} \tag{1.19}$$

It is a long and tedious calculation to compute the simple real root of the Steiner polynomial in both cases Type 1 and Type 2: it can be checked that

$$\begin{aligned}f_E(K,\lambda) = (\lambda-\lambda_m)\Bigg(& \mathrm{V}(E)\lambda^2 + \Big[2\mathrm{W}_2(K;E) + \sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{V}(E)\mathrm{W}_1(K;E)}\Big]\lambda \\ & + \frac{2\mathrm{V}(E)\mathrm{W}_1(K;E) - \mathrm{W}_2(K;E)^2 + \mathrm{W}_2(K;E)\sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{V}(E)\mathrm{W}_1(K;E)}}{\mathrm{V}(E)}\Bigg),\end{aligned}$$

since the remainder of $f_E(K,\lambda)/(\lambda-\lambda_m)$ is given by (1.16), which vanishes in our case provided that λ_m is a root of $f_E(K,\lambda)$. Thus, the roots of the second degree polynomial at the right-hand side in the previous expression are λ_m, which is a double root of $f_E(K,\lambda)$, as needed, and the third required root,

$$\lambda_3 = \frac{-\mathrm{W}_2(K;E) - 2\sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{V}(E)\mathrm{W}_1(K;E)}}{\mathrm{V}(E)}.$$

Analogously, in the case of a double real root of Type 2, which necessarily is λ_M, the simple real one is

$$\lambda_3 = \frac{-\mathrm{W}_2(K;E) + 2\sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{V}(E)\mathrm{W}_1(K;E)}}{\mathrm{V}(E)}.$$

(iii) Three different real roots. Following the notation stated in formula (1.19), it is clear that $f_E(K,\lambda)$ has three *different* real roots only if its local maximum is strictly positive, i.e., if $\phi_+(K;E) > 0$, *and* its local minimum strictly negative, i.e., $\phi_-(K;E) < 0$.

(iv) Two conjugate complex roots and a simple real root. It is clear that this case occurs if and only if either the local minimum of $f_E(K,\lambda)$ is strictly positive, or its local maximum is strictly negative. So, we find again two possible situations (see Figure 1.2), which will be called, following the analogy with the case of the double roots, Type 1 and Type 2, respectively.

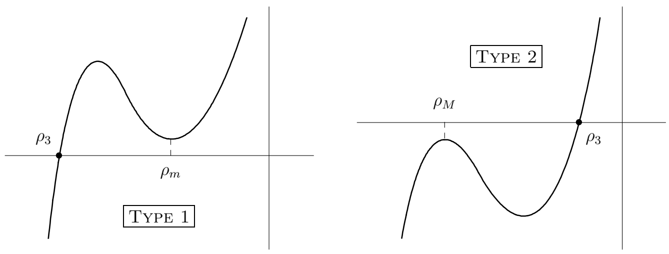

Figure 1.2: The Steiner polynomial has two conjugate complex roots and a simple one.

Therefore the following result has been proved:

Theorem 1.3.2 ([27]). *Let $K \in \mathcal{K}^3$. Then its Steiner polynomial has:*

- *A double real root of* TYPE 1 *if and only if* $\phi_-(K;E) = 0$.
- *A double real root of* TYPE 2 *if and only if* $\phi_+(K;E) = 0$.
- *Two conjugate complex roots of* TYPE 1 *if and only if* $\phi_-(K;E) > 0$.
- *Two conjugate complex roots of* TYPE 2 *if and only if* $\phi_+(K;E) < 0$.
- *Three simple (*different*) real roots if and only if* $\phi_+(K;E) > 0$ *and* $\phi_-(K;E) < 0$.

In the case of double real roots, the zeros are explicitly given by

$$\text{TYPE 1}: \quad \lambda_m = \frac{-\mathrm{W}_2(K;E) + \sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{W}_1(K;E)\mathrm{V}(E)}}{\mathrm{V}(E)} \qquad \textit{(double)},$$
$$\lambda_3 = \frac{-\mathrm{W}_2(K;E) - 2\sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{W}_1(K;E)\mathrm{V}(E)}}{\mathrm{V}(E)};$$

$$\text{TYPE 1}: \quad \lambda_M = \frac{-\mathrm{W}_2(K;E) - \sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{W}_1(K;E)\mathrm{V}(E)}}{\mathrm{V}(E)} \qquad \textit{(double)},$$
$$\lambda_3 = \frac{-\mathrm{W}_2(K;E) + 2\sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{W}_1(K;E)\mathrm{V}(E)}}{\mathrm{V}(E)}.$$

Another characterization of the different types of roots in terms of the functionals $\mathrm{V}(K)$, $\mathrm{W}_1(K;E)$ and $\mathrm{W}_2(K;E)$ can also be given. If we denote by λ_3 the real root of the Steiner polynomial which always exists, we can rewrite $f_E(K,\lambda)$ in the following way:

$$\begin{aligned} f_E(K,\lambda) &= \mathrm{V}(E)(\lambda-\lambda_3)\left[\lambda^2 + \left(\lambda_3 + \frac{3\mathrm{W}_2(K;E)}{\mathrm{V}(E)}\right)\lambda + \left(\lambda_3^2 + \frac{3\mathrm{W}_2(K;E)}{\mathrm{V}(E)}\lambda_3 + \frac{3\mathrm{W}_1(K;E)}{\mathrm{V}(E)}\right)\right] \\ &= \mathrm{V}(E)(\lambda-\lambda_3)Q(\lambda). \end{aligned} \tag{1.20}$$

The discriminant Δ_Q of the second degree polynomial $Q(\lambda)$ is given by

$$\Delta_Q = -3\mathrm{V}(E)\left(\mathrm{V}(E)\lambda_3^2 + 2\mathrm{W}_2(K;E)\lambda_3 + 4\mathrm{W}_1(K;E) - \frac{3\mathrm{W}_2(K;E)^2}{\mathrm{V}(E)}\right). \tag{1.21}$$

Consequently, we can assure that:

- If $\Delta_Q = 0$, then $f_E(K,\lambda)$ has a double real root, besides λ_3. This is the only case in which a triple real root can appear.
- If $\Delta_Q > 0$, $f_E(K,\lambda)$ has two different simple real roots, besides λ_3.
- If $\Delta_Q < 0$, $f_E(K,\lambda)$ has two complex roots, and of course, the real one λ_3.

So, we just have to check the meaning of the condition $\Delta_Q = 0$ in terms of λ_3. But it is clear that $\mathrm{V}(E)\lambda_3^2 + 2\mathrm{W}_2(K;E)\lambda_3 + 4\mathrm{W}_1(K;E) - 3\mathrm{W}_2(K;E)^2/\mathrm{V}(E) = 0$ if and only if

$$\lambda_3 = \frac{-\mathrm{W}_2(K;E) \pm 2\sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{W}_1(K;E)\mathrm{V}(E)}}{\mathrm{V}(E)},$$

from which we can derive the required characterization:

Theorem 1.3.3 ([27]). *Let $K \in \mathcal{K}^3$ be a convex body whose Steiner polynomial has λ_3 as real root, and we denote by $N_+(K;E)$ and $N_-(K;E)$ the values*

$$N_\pm(K;E) := \frac{-\mathrm{W}_2(K;E) \pm 2\sqrt{\mathrm{W}_2(K;E)^2 - \mathrm{W}_1(K;E)\mathrm{V}(E)}}{\mathrm{V}(E)}.$$

Then, the Steiner polynomial of K has:

i) *Three different real roots if and only if $N_-(K;E) < \lambda_3 < N_+(K;E)$; in this case, also the other two real roots, $\lambda_1, \lambda_2 \in \big(N_-(K;E), N_+(K;E)\big)$.*

ii) *Two conjugate complex roots of* Type 1 *if and only if $\lambda_3 < N_-(K;E)$.*

iii) *Two conjugate complex roots of* Type 2 *if and only if $\lambda_3 > N_+(K;E)$.*

If one of the roots of the Steiner polynomial is equal to one of the values $N_-(K;E)$ or $N_+(K;E)$, then we can assure that $f_E(K,\lambda)$ has double real roots of Type 1 or Type 2, respectively. So, it suffices to find a real root lying in one of the intervals $-\infty < N_-(K;E) < N_+(K;E) < 0$, to ensure that the other roots are, respectively, complex of Type 1, real, or complex of Type 2.

1.3.3 The planar properties (P1), (P4) and (P5)

In this section we generalize properties **(P1)** and **(P4)** which were stated for the roots of the Steiner polynomial of a planar convex body; for property **(P5)** it will be not possible.

In the planar case, the main property was first Minkowski inequality, which is equivalent to the fact that the Steiner polynomial has real roots; now, since there is no unique inequality relating the corresponding 3-dimensional magnitudes V, W_1 and W_2 (besides, we know that, at least, one is missing), the fundamental property for us will be that the Steiner polynomial has only real roots. So, we will look for characterizations of this fact.

Property **(P4)** has a trivial generalization. Just considering the values given in (1.15) where the local minimum and maximum of the Steiner polynomial are reached, and evaluating, the following characterization is obtained, equivalent to Theorem 1.3.2:

Proposition 1.3.4 ([27]). *All roots of the Steiner polynomial of a convex body are* real *if and only if* $\min\big\{f_E(K,\lambda) : \lambda \in \mathbb{R}\big\} \leq 0$ *and* $\max\big\{f_E(K,\lambda) : \lambda \in \mathbb{R}\big\} \geq 0$*, simultaneously; the equalities* $\min\big\{f_E(K,\lambda) : \lambda \in \mathbb{R}\big\} = \max\big\{f_E(K,\lambda) : \lambda \in \mathbb{R}\big\} = 0$ *are attained precisely when K is an homothetic copy of E.*

Now we study property **(P1)**, which stated the equivalence between first Minkowski inequality and the fact that the Steiner polynomial has no complex roots. If we denote, as usual, by λ_3 the real root of the Steiner polynomial which always exists, $f_E(K,\lambda)$ can be rewritten again as in (1.20). Now, we wonder: when is $Q(\lambda)$ the Steiner polynomial of a *planar* convex body $\overline{K}\in\mathcal{K}^2$ relative to an $\overline{E}\in\mathcal{K}^2_0$? If this is the case, we need to write $f_E(K,\lambda)=\bigl(\mathrm{V}(E)/\mathrm{A}(\overline{E})\bigr)(\lambda-\lambda_3)Q_0(\lambda)$, where

$$Q_0(\lambda)=\mathrm{A}(\overline{E})\lambda^2+\mathrm{A}(\overline{E})\left(\lambda_3+\frac{3\mathrm{W}_2(K;E)}{\mathrm{V}(E)}\right)\lambda+\mathrm{A}(\overline{E})\left(\lambda_3^2+\frac{3\mathrm{W}_2(K;E)}{\mathrm{V}(E)}\lambda_3+\frac{3\mathrm{W}_1(K;E)}{\mathrm{V}(E)}\right),$$

and then the above question is equivalent to ask, *when does it hold that*

$$\mathrm{A}(\overline{E})\left(\lambda_3+\frac{3\mathrm{W}_2(K;E)}{\mathrm{V}(E)}\right)=2\mathrm{W}_1^{(2)}(\overline{K};\overline{E})\quad\text{and}$$
$$\mathrm{A}(\overline{E})\left(\lambda_3^2+\frac{3\mathrm{W}_2(K;E)}{\mathrm{V}(E)}\lambda_3+\frac{3\mathrm{W}_1(K;E)}{\mathrm{V}(E)}\right)=\mathrm{A}(\overline{K})$$

for a planar convex body $\overline{K}$ relative to $\overline{E}\in\mathcal{K}^2_0$? Here $\mathrm{W}_1^{(2)}$ denotes the 1-st quermassintegral computed in $\mathbb{R}^2$. Obviously, it holds if and only if these numbers satisfy first Minkowski inequality, i.e., if and only if $\mathrm{W}_1^{(2)}(\overline{K};\overline{E})^2-\mathrm{A}(\overline{K})\mathrm{A}(\overline{E})\geq 0$. Notice that the factor $\mathrm{A}(\overline{E})$ is cancelled, and then the set $\overline{E}$ will have no influence in the final result. An easy computation allows to check that this planar Minkowski inequality is equivalent to the discriminant $\Delta_Q\geq 0$ (cf. (1.21)), and hence to $N_-(K;E)\leq\lambda_3\leq N_+(K;E)$ (see argument for Theorem 1.3.3). Now, Theorem 1.3.3 shows that this condition is equivalent to the fact that the Steiner polynomial has only real roots. Thus, the following result has been established:

Theorem 1.3.5 ([27]). *All roots of the Steiner polynomial $f_E(K,\lambda)$ of a convex body $K\in\mathcal{K}^3$ are real if and only if the second degree polynomial $Q(\lambda)$, obtained by the decomposition of $f_E(K,\lambda)$ by means of its real root, is the Steiner polynomial of a planar convex body (relative to* any *convex body).*

Finally we deal with property **(P5)**. This one can not be however extended to dimension 3. In [53] Teissier studied Bonnesen-Type inequalities in Algebraic Geometry and raised the problem to find extensions of this two dimensional property to higher dimensions (see also [35, p. 103]). In view of the properties derived from (1.14) in the planar case, in [42] and [45, p. 65] the following conjecture was posed:

Conjecture 1.3.6. *Let $K,E\in\mathcal{K}^n$. If γ_i, $i=1,\dots,n$, are the roots of the Steiner polynomial $f_E(K,\lambda)$ with $\mathrm{Re}(\gamma_1)\leq\cdots\leq\mathrm{Re}(\gamma_n)$, then*

$$\mathrm{Re}(\gamma_1)\leq-\mathrm{R}(K;E)\leq-\mathrm{r}(K;E)\leq\mathrm{Re}(\gamma_n)\leq 0.$$

Here $\mathrm{Re}(\lambda)$ denotes the real part of the complex number λ. We have already mentioned that the "negativity property" of the roots does not hold in general. In [25] counterexamples are also obtained for both the inradius and the circumradius bounds in dimension 3, which shows that it is impossible to extend property **(P5)** to $\mathbb{R}^3$.

1.3.4 Appendix: a property of the n-dimensional Steiner polynomial

In this brief appendix, we state the characterization of the fixed body $E \in \mathcal{K}_0^n$ in terms of the type of roots of its Steiner polynomial. More precisely, we prove the following result:

Proposition 1.3.7 ([26]). *The Steiner polynomial of a convex body $K \in \mathcal{K}^n$ (relative to $E \in \mathcal{K}_0^n$) has an n-fold real root a if and only if $K = -aE$ (up to translations).*

This result provides also a characterization of the Euclidean balls, just taking $E = B^n$. In [26] a proof similar to the one of the 3-dimensional case (see Proposition 1.3.1) is developed, in the sense of assuming that we have $\sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^i = (\lambda - a)^n \mathrm{V}(E)$, then identifying the corresponding coefficients, and finally applying some known inequalities between the relative quermassintegrals. Here we present however an elegant and shorter proof using Brunn-Minkowski inequality, which is due to M. Henk (private communication).

Proof. The Steiner polynomial of a convex body $K \in \mathcal{K}^n$ has an n-fold real root a if and only if $f_E(K,\lambda) = \mathrm{V}(E)(\lambda - a)^n$. In particular, when $\lambda \geq 0$, we have

$$\mathrm{V}(E)^{1/n}(\lambda - a) = f_E(K,\lambda)^{1/n} = \mathrm{V}(K + \lambda E)^{1/n} \geq \mathrm{V}(K)^{1/n} + \lambda \mathrm{V}(E)^{1/n}$$

by Brunn-Minkowski inequality (see Theorem 1.2.11), i.e., $-\mathrm{V}(E)^{1/n} a \geq \mathrm{V}(K)^{1/n}$. On the other hand, since $\mathrm{V}(K + \lambda E)^{1/n} = \mathrm{V}(E)^{1/n}(\lambda - a)$ for any $\lambda \geq 0$, for $\lambda = 0$ we get $-a = \mathrm{V}(K)^{1/n}/\mathrm{V}(E)^{1/n}$. Hence we have equality in the previous inequality, or equivalently, we have equality in Brunn-Minkowski inequality, which implies that K and E are homothetic. Since $-a = \mathrm{V}(K)^{1/n}/\mathrm{V}(E)^{1/n}$, it is clear that the homothecy factor is $-a$. $\square$

1.4 An application: the Blaschke diagram

As we mentioned before, in the case $E = B^3$ the characterization of the convex bodies in terms of the roots of their Steiner polynomial $f_{B^3}(K,\lambda)$ has an interesting application to the *Blaschke problem.* This section is devoted to study this question. First, we will start by introducing the so called Blaschke problem.

1.4.1 The Blaschke problem

In 1916 Blaschke [3] asked for a characterization of the set of all points in $\mathbb{R}^3$ taking the form $(\mathrm{V}(K), \mathrm{S}(K), \mathrm{M}(K))$ as K ranges over $\mathcal{K}^3$ or, equivalently, for a characterization of the set of all points in $\mathbb{R}^2$ of the form

$$x(K) = \frac{4\pi \mathrm{S}(K)}{\mathrm{M}(K)^2} \quad \text{and} \quad y(K) = \frac{48\pi^2 \mathrm{V}(K)}{\mathrm{M}(K)^3}.$$

The latter set is called *Blaschke diagram*, and the map $\mathcal{K}^3 \longrightarrow [0,1]^2$, given by $K \rightsquigarrow \big(x(K), y(K)\big)$, *Blaschke map.* Notice that Blaschke map is not injective: because of the choice of the coordinates, all homothetic copies of a convex body K have the same image; and moreover, except in a few cases (the balls, the segments and the circles), each point in the diagram is the image of infinite many *different* convex bodies. One of the main problems in this context is how to describe the Blaschke diagram. According to the known inequalities relating V, S and M, i.e.,

Minkowski inequalities (cf. (1.11)):

$$\mathrm{S}(K)^2 \geq 3\mathrm{V}(K)\mathrm{M}(K) \qquad \text{(equality for cap-bodies)}, \tag{1.22}$$

$$\mathrm{M}(K)^2 \geq 4\pi\mathrm{S}(K) \qquad \text{(equality for balls)}, \tag{1.23}$$

$$\mathrm{M}(K)^3 \geq 48\pi^2\mathrm{V}(K) \qquad \text{(equality for balls)}, \tag{1.24}$$

isoperimetric inequality in $\mathbb{R}^3$ (cf. (1.12)):

$$\mathrm{S}(K)^3 \geq 36\pi\mathrm{V}(K)^2 \qquad \text{(equality for balls)}, \tag{1.25}$$

isoperimetric inequality for planar sets in $\mathbb{R}^3$*:*

$$2\mathrm{M}(K)^2 \geq \pi^3\mathrm{S}(K), \;\text{ where }\; \mathrm{V}(K) = 0 \qquad \text{(equality for discs)}, \tag{1.26}$$

and translating them in terms of the (x, y)-coordinates, it is easy to see that the Blaschke diagram *contains* the shaded region in Figure 1.4. We recall that a cap-body (see Figure 1.3) is the convex hull of the ball (in general, any convex body E) and countably many points such that the line segment joining any pair of those points intersects the ball (in general, E); the limit cases of the line segment and the ball are included. These sets are obtained as extremal sets of second Minkowski inequality in $\mathbb{R}^3$ (see Theorem 1.2.13) since they are just 1-tangential bodies, as will be seen in Chapter 3. About inequality (1.26), it is obtained from the classical isoperimetric inequality $\mathrm{p}(K)^2 \geq 4\pi\mathrm{A}(K)$ just taking into account that if K is a 2-dimensional convex body in $\mathbb{R}^3$, its volume is $\mathrm{V}(K) = 0$, its surface area $\mathrm{S}(K) = 2\mathrm{A}(K)$ and its integral mean curvature $\mathrm{M}(K) = (\pi/2)\mathrm{p}(K)$ (see [45, Property 3.1]).

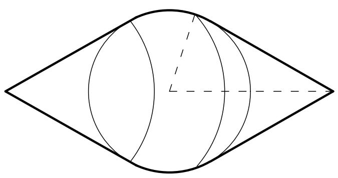

Figure 1.3: A cap-body of B^3.

For instance, inequality (1.22) corresponds to $y \leq x^2$. Since the cap-bodies are the extremal sets for this inequality, they are mapped to the points of the parabola $y = x^2$, from $(0,0)$ –the segments– to $(1,1)$ –the balls. Analogously, all planar convex bodies in 3-space are mapped on the interval $[0, 8/\pi^2]$ of the x-axis because they have volume $\mathrm{V} = 0$, and by inequality (1.26), we get the upper bound of $8/\pi^2$ –the discs. Hence, if $8/\pi^2 < x \leq 1$ then y must be strictly positive. At the moment, however, it is not known which inequality the bodies in this range have to satisfy. The corresponding missing curve is known in the literature as *the missing boundary of the Blaschke* *diagram*. The problem and its higher dimensional versions remain open. Nowadays, there are two different conjectures, posed by Bieri and Sangwine-Yager. See [23, Section 28] and [44] for a more detailed explanation.

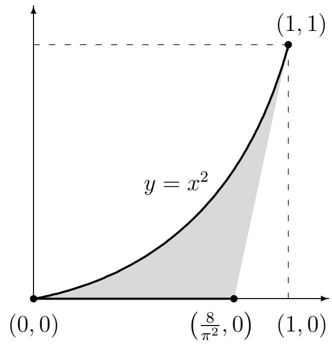

Figure 1.4: The Blaschke diagram.

Regarding the interior of the Blaschke diagram, it can be proved that it is, not only connected, but also simply connected: denoting by (x_0, y_0) the coordinates, by Blaschke map, of a convex body K, an easy computation leads to the coordinates (x_λ, y_λ) of its outer parallel body at distance λ, namely

$$x_\lambda = \frac{x_0 + 2c + c^2}{(1+c)^2}, \qquad y_\lambda = \frac{y_0 + 3x_0c + 3c^2 + c^3}{(1+c)^3}$$

where $c = 4\pi\lambda/\mathrm{M}(K)$, which gives the equation of an algebraic curve connecting, the given point (x_0, y_0) of the diagram, with the point $(1, 1)$ (see [23, p. 73–74]). This curve depends continuously on the initial point (x_0, y_0), which shows that the Blaschke diagram is simply connected and the following fact:

Corollary 1.4.1. *The interior of the Blaschke diagram can be "filled" with the images of outer parallel bodies.*

From this moment on we will say that a curve or region is *filled* with the images, by Blaschke map, of a certain family of sets, if every point of the region can be obtained as the image of a member of that family. In the following subsection we give some applications to the Blaschke problem of the classification of the 3−dimensional convex bodies that we have developed in the previous section.

1.4.2 The Steiner polynomial and the Blaschke diagram

Notice that all the conditions that we have obtained in Section 1.3 for the roots of the Steiner polynomial of a convex body $K \in \mathcal{K}^3$, now relative to the ball B^3, are given in this case in terms of its magnitudes V, S and M; hence they have an interesting translation in the Blaschke diagram.

Thus, if we rewrite the relations $\phi_-(K;B^3)=0$, $\phi_+(K;B^3)=0$ by means of the Blaschke map, this is, using the coordinates $x=4\pi\mathrm{S}(K)/\mathrm{M}(K)^2$ and $y=48\pi^2\mathrm{V}(K)/\mathrm{M}(K)^3$, we obtain, respectively,

$$y=3x+2(1-x)^{3/2}-2, \tag{1.27}$$

$$y=3x-2(1-x)^{3/2}-2; \tag{1.28}$$

these equations correspond to the curves represented in Figure 1.5. All convex bodies whose Steiner polynomial has double real roots of TYPE 1 and TYPE 2, will be mapped to the points of those curves, since, according to Theorem 1.3.2, this kind of sets satisfy the equalities $\phi_-(K;B^3)=0$ and $\phi_+(K;B^3)=0$, respectively.

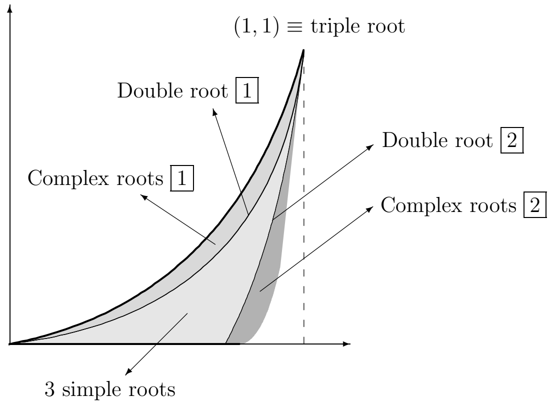

Figure 1.5: Representation in Blaschke diagram of $\mathcal{K}^3$, depending on the roots of their Steiner polynomial.

Furthermore, all convex bodies whose Steiner polynomial has complex roots of TYPE 1 satisfy $\phi_-(K;B^3)>0$, i.e., $y>3x+2(1-x)^{3/2}-2$ in terms of x and y. Thus, these bodies are mapped to the shaded region of the left-hand side, between the (known) boundary of the diagram $y=x^2$ and the curve (1.27) (double real roots of TYPE 1). Analogously, all convex bodies whose Steiner polynomial has complex roots of TYPE 2 have their image by Blaschke map in the shaded region of the diagram at the right-hand side of the curve (1.28) (double real roots of TYPE 2), because this kind of sets satisfies the strict inequality $\phi_+(K;B^3)<0$, and hence, $y<3x-2(1-x)^{3/2}-2$ in terms of x and y (see Figure 1.5). Finally, since all convex bodies whose Steiner polynomial has three *different* simple real roots satisfy *both* (strict) inequalities $\phi_-(K;B^3)<0$ and $\phi_+(K;B^3)>0$, their image by Blaschke map will satisfy simultaneously both relations $y>3x+2(1-x)^{3/2}-2$ and $y>3x-2(1-x)^{3/2}-2$; therefore, all these sets will be mapped to the central region of the diagram, bounded by the curves (1.27) and (1.28) (see Figure 1.5).

The above argument allows to draw a "map" of the Blaschke diagram (Figure 1.5), distinguishing three disjoint regions in which the convex bodies are mapped depending on the roots of their Steiner polynomial. And, as a consequence, we obtain the following assertion, which states a necessary condition for the missing boundary of the Blaschke diagram:

Corollary 1.4.2 ([27]). *If a convex body corresponds to a point on the missing boundary of the Blaschke diagram, then its Steiner polynomial has complex roots of* Type 2*.*

It is interesting to determine families of convex bodies whose Steiner polynomial has, either simple real roots, either complex roots of Type 1 and Type 2, or double real roots of both types. And, in this last case, to find a continuous family of sets which is mapped, by Blaschke map, onto the corresponding curve, $y = 3x + 2(1-x)^{3/2} - 2$ or $y = 3x - 2(1-x)^{3/2} - 2$. To obtain this, we start analyzing carefully the functionals $\phi_+(K; B^3)$ and $\phi_-(K; B^3)$:

Lemma 1.4.3 ([27]). *Let $K \in \mathcal{K}^3$. Then, ϕ_+, ϕ_- are constant on the family of outer parallel bodies of K; i.e., $\phi_+(K_\lambda; B^3) = \phi_+(K; B^3)$ and $\phi_-(K_\lambda; B^3) = \phi_-(K; B^3)$, for all $\lambda \geq 0$.*

Proof. In the particular case of a 3-dimensional convex body, relations (1.4) are translated into the following three equalities:

$$\mathrm{V}(K_\lambda) = \mathrm{V}(K) + \mathrm{S}(K)\lambda + \mathrm{M}(K)\lambda^2 + \frac{4}{3}\pi\lambda^3, \tag{1.29}$$

$$\mathrm{S}(K_\lambda) = \mathrm{S}(K) + 2\mathrm{M}(K)\lambda + 4\pi\lambda^2, \tag{1.30}$$

$$\mathrm{M}(K_\lambda) = \mathrm{M}(K) + 4\pi\lambda. \tag{1.31}$$

Then, it is easy to compute that

$$\mathrm{M}(K_\lambda)^2 - 4\pi\mathrm{S}(K_\lambda) = \big(\mathrm{M}(K) + 4\pi\lambda\big)^2 - 4\pi\big(\mathrm{S}(K) + 2\mathrm{M}(K)\lambda + 4\pi\lambda^2\big) = \mathrm{M}(K)^2 - 4\pi\mathrm{S}(K),$$
$$\mathrm{M}(K_\lambda)^3 - 6\pi\big(\mathrm{M}(K_\lambda)\mathrm{S}(K_\lambda) - 4\pi\mathrm{V}(K_\lambda)\big) = \mathrm{M}(K)^3 - 6\pi\big(\mathrm{M}(K)\mathrm{S}(K) - 4\pi\mathrm{V}(K)\big).$$

Hence,

$$\begin{aligned}\phi_+(K_\lambda; B^3) &= \mathrm{M}(K_\lambda)^3 - 6\pi\big(\mathrm{M}(K_\lambda)\mathrm{S}(K_\lambda) - 4\pi\mathrm{V}(K_\lambda)\big) + \big(\mathrm{M}(K_\lambda)^2 - 4\pi\mathrm{S}(K_\lambda)\big)^{3/2}\\ &= \mathrm{M}(K)^3 - 6\pi\big(\mathrm{M}(K)\mathrm{S}(K) - 4\pi\mathrm{V}(K)\big) + \big(\mathrm{M}(K)^2 - 4\pi\mathrm{S}(K)\big)^{3/2} = \phi_+(K; B^3),\end{aligned}$$

and also, $\phi_-(K_\lambda; B^3) = \phi_-(K; B^3)$, as required. □

Now, before stating the result that fulfills our aim in this section, we deal with the 2-dimensional sets, characterizing the planar convex bodies of $\mathbb{R}^3$ depending on the algebraic type of roots that their Steiner polynomial has.

If K is a 2-dimensional convex body in $\mathbb{R}^3$, we already know that $\mathrm{V}(K) = 0$, $\mathrm{S}(K) = 2\mathrm{A}(K)$ and $\mathrm{M} = (\pi/2)\mathrm{p}(K)$. Then $f_{B^3}(K, \lambda) = \lambda\big(2\mathrm{A}(K) + (\pi/2)\mathrm{p}(K)\lambda + 4/3\pi\lambda^2\big)$, with roots

$$\lambda_1 = 0, \quad \lambda_2 = 3\frac{-\mathrm{p}(K) + \sqrt{\mathrm{p}(K)^2 - \frac{128}{3\pi}\mathrm{A}(K)}}{16}, \quad \lambda_3 = 3\frac{-\mathrm{p}(K) - \sqrt{\mathrm{p}(K)^2 - \frac{128}{3\pi}\mathrm{A}(K)}}{16}.$$

Thus, we obtain the following classification for the family of planar convex bodies of $\mathbb{R}^3$, depending on whether the above discriminant is positive, negative or zero:

Lemma 1.4.4 ([27]). *If K is a planar convex body in $\mathbb{R}^3$, its Steiner polynomial has:*

- *three different real roots if and only if* $\mathrm{p}(K)^2 > 128/(3\pi)\mathrm{A}(K)$ *and* $\mathrm{A}(K) \neq 0$*;*
- *a double real root of* TYPE 1 *if and only if* $\mathrm{A}(K) = 0$*; this is, if and only if K is a line segment. The double real root is $\lambda_1 = \lambda_2 = 0$;*
- *a double real root of* TYPE 2 *if and only if* $\mathrm{p}(K)^2 = 128/(3\pi)\mathrm{A}(K)$*;*
- *complex roots of* TYPE 2 *if and only if* $\mathrm{p}(K)^2 < 128/(3\pi)\mathrm{A}(K)$*.*
- *The Steiner polynomial of K will* never *have complex roots of* TYPE 1*.*

As a consequence of these two lemmas, we can easily prove the following theorem, which determines the existence of families of convex bodies which are mapped, by Blaschke map, to *all* the points of the curves (1.27) and (1.28); and a bit further:

Theorem 1.4.5 ([27]). *If $K \in \mathcal{K}^3$ is a convex body whose Steiner polynomial has a certain type of roots (simple real, double real or complex), then* all *the outer parallel bodies of K verify that their Steiner polynomial has the same type of roots. In particular:*

i) *If K is a planar convex body whose Steiner polynomial has double real roots of* TYPE 2*, then the same occurs for all the outer parallel bodies K_λ, and the family $\{K_\lambda : \lambda \geq 0\}$ is mapped to* all *points of the curve* (1.28)*.*

ii) *Let σ be a line segment. All sets of the family $\{\sigma_\lambda : \lambda \geq 0\}$ satisfy that their Steiner polynomial has double real roots of* TYPE 1*, and are mapped to* all *points of the curve* (1.27)*.*

Proof. It is an immediate consequence of Lemma 1.4.3: if K is a convex body whose Steiner polynomial has a certain type of roots, then the functionals $\phi_\pm(K; B^3)$ will verify a precise condition; since $\phi_\pm(K_\lambda; B^3) = \phi_\pm(K; B^3)$, all the sets K_λ will satisfy the *same condition*, and therefore, their Steiner polynomial will have the same type of roots as the original body K.

In particular, if $K \in \mathcal{K}^3$ is mapped to the point labelled A in Figure 1.6, i.e., it is a planar convex body whose Steiner polynomial has double real roots of TYPE 2, then, $\phi_+(K; B^3) = 0$ holds. Applying Lemma 1.4.3, $\phi_+(K_\lambda; B^3) = \phi_+(K; B^3) = 0$, and hence, all the sets K_λ are mapped to all the points of the curve labelled α in Figure 1.6.

In order to conclude the proof, we just have to show the existence of a planar convex body K whose Steiner polynomial has double real roots of TYPE 2. Lemma 1.4.4 ensures the existence of the required set: it suffices to find a set K verifying the relation $\mathrm{p}(K)^2 = 128/(3\pi)\mathrm{A}(K)$. For instance, the convex hull of two suitable discs with equal radius $\mathrm{r} = \left(4\sqrt{2}/\sqrt{32 - 3\pi^2} - 1\right)/\pi \approx 0.846\ldots$ and centers at distance apart 1, satisfies this equality.

The same reasoning is useful for ii). The line segment verifies $\phi_-(\sigma; B^3) = 0$, i.e., its Steiner polynomial has double roots of TYPE 1. Hence, $\phi_-(\sigma_\lambda; B^3) = \phi_-(\sigma; B^3) = 0$. $\square$

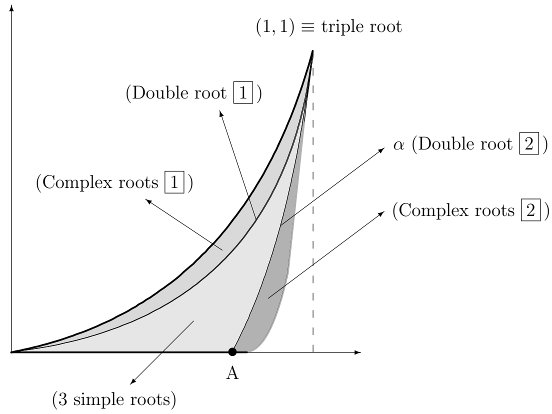

Figure 1.6: Families of convex bodies for each type of roots.

Corollary 1.4.1 ensured that the interior of the Blaschke diagram can be filled with the images of outer parallel bodies; using Theorem 1.4.5, we can say even more:

Corollary 1.4.6 ([27]). *The outer parallel bodies of the planar convex bodies (whose Steiner polynomial can have real roots or complex roots of* Type 2*) fill the part of the diagram which corresponds to the real roots and a part of the complex roots of* Type 2*, till the curve corresponding to the outer parallel bodies of the disc.*

The outer parallel bodies of the cap-bodies (complex roots of Type 1*) fill the part of the diagram corresponding to complex roots of* Type 1*.*

The outer parallel bodies of the sets corresponding to the missing boundary (complex roots of Type 2*) fill the part of the diagram corresponding to the complex roots of* Type 2*, till the curve of the outer parallel bodies of the disc (see Figure 1.7).*

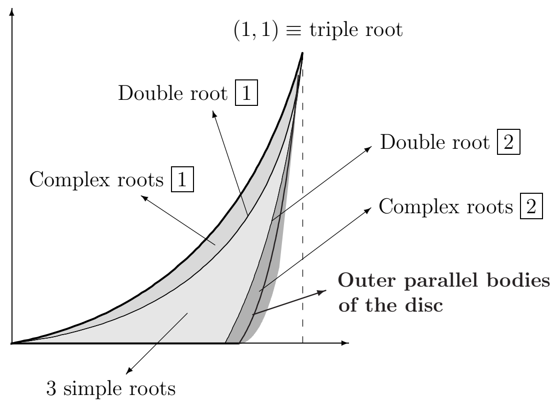

Figure 1.7: How to fill the Blaschke diagram with the outer parallel bodies.

About the last assertion, let us notice that, for the limit case when the cap-body is a segment σ, the image curve of its outer parallel bodies is the one corresponding to the double roots of TYPE 1 (Theorem 1.4.5); it allows us to assure that statement.

Remark 1.1. The different types of roots of the Steiner polynomial of a convex body, lead to a precise classification of the family $\mathcal{K}^3$ of the 3-dimensional convex bodies into three big (mutually disjoint) classes:

- Convex bodies whose Steiner polynomial has only real roots (simple, double or triple); we will denote this class by $\mathfrak{R}$; in particular we write $\mathfrak{R}_1$ for double roots of TYPE 1 and $\mathfrak{R}_2$ for double roots of TYPE 2.
- Convex bodies whose Steiner polynomial has complex roots of TYPE 1; we will denote this class by $\mathfrak{C}_1$.
- Convex bodies whose Steiner polynomial has complex roots of TYPE 2; we will denote this class by $\mathfrak{C}_2$. ♢

It is also easy to determine the type of roots of the Steiner polynomial of particular convex bodies or certain families of convex bodies. To obtain this, it suffices to know the point or the curve in the Blaschke diagram which is the image of that set or family of sets; then, depending on the region where it lies, so the roots of its Steiner polynomial will be. We show now some examples.

Example 1.1. The five platonic solids with circumradius $\mathrm{R} = 1$, have volume, surface area and integral mean curvature as shown in Table 1.1.

	V	S	M
Tetrahedron	$8\sqrt{3}/27$	$8\sqrt{3}/3$	$2\sqrt{6}\arccos(-1/3)$
Cube	$8\sqrt{3}/9$	8	$2\sqrt{3}\pi$
Octahedron	$4/3$	$4\sqrt{3}$	$6\sqrt{2}\arccos(1/3)$
Dodecahedron	$2\sqrt{15}\big(\sqrt{5}+1\big)/9$	$2\sqrt{10(5-\sqrt{5})}$	$5\sqrt{3}\big(\sqrt{5}-1\big)\arctan 2$
Icosahedron	$2\sqrt{2(5+\sqrt{5})}/3$	$2\sqrt{3}\big(5-\sqrt{5}\big)$	$3\sqrt{10(5-\sqrt{5})}\arcsin(2/3)$

Table 1.1: Volume, surface area and integral mean curvature of the regular polyhedrons with $\mathrm{R} = 1$.

Then it is easy to check that the Steiner polynomial of the tetrahedron, the cube and the octahedron have three *different* real roots; the one of the dodecahedron has complex roots of TYPE 2; and the one of the icosahedron, complex roots of TYPE 1. ♢

Example 1.2. The sets of constant width in $\mathbb{R}^3$, i.e., those sets with the same width in any direction, are mapped to the line $y = 3x - 2$. It follows from an identity due to Blaschke: if K has constant width b then $\mathrm{M}(K) = 2\pi\mathrm{b}$ and $8\pi^2\mathrm{V}(K) = 2\pi\mathrm{M}(K)\mathrm{S}(K) - \mathrm{M}(K)^3/3$ (see [10, p. 66]). Notice that this line will never touch the x-axis. It is easy to check that the line $y = 3x - 2$ lies strictly in the interior of the central region corresponding to "simple roots"; thus the Steiner polynomials of all constant width sets have three *different* real roots; the exception, of course, are the balls, with just a triple real root. $\diamondsuit$

Example 1.3. The circular cylinders with radius, for instance, 1 and height $h \in [0, \infty)$ have volume $\mathrm{V} = \pi h$, surface area $\mathrm{S} = 2\pi(h + 1)$, and integral mean curvature $\mathrm{M} = \pi(\pi + h)$; hence,

$$x = \frac{8(h+1)}{(\pi + h)^2} \qquad \text{and} \qquad y = \frac{48h}{(\pi + h)^3}.$$

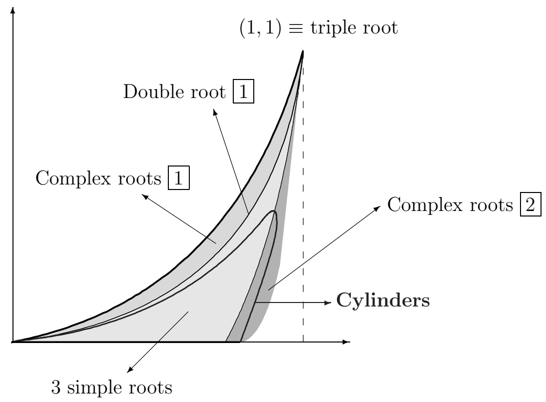

Figure 1.8: Image of the family of cylinders.

If we represent these points in the Blaschke diagram, we obtain the curve shown in Figure 1.8. Hence, we can assure that all cylinders satisfy that their Steiner polynomial has either three real roots, or complex roots of Type 2: the Steiner polynomial of just one cylinder (up to congruences) will have double real roots of Type 2, the one with height $h = h_0 \approx 1.71065$; if $h < h_0$, complex roots of Type 2 appear; if $h > h_0$, three *simple* real roots; and only in the limit case of a segment, double real roots of Type 1 are obtained. The Steiner polynomial of any cylinder will never have complex roots of Type 1. $\diamondsuit$

Chapter 2

Hadwiger's problem on inner parallel bodies

If we are working with respect to the unit ball, the inner parallel body of a convex body K at distance λ (for suitable values of λ) turns out to be the intersection of the closed supporting half-spaces of K moved in a distance λ; although the definition can be stated in a more general way. When the set of inner and outer parallel bodies of K are considered together under a single parameter λ, the so called *full system of parallel bodies* of K arises.

In this chapter we consider the problem of classifying the convex bodies in the 3-dimensional space depending on the differentiability of their associated quermassintegrals with respect to the one-parameter-depending family given by the full system of parallel bodies. It turns out that this problem is closely related to some behavior of the roots of the 3-dimensional Steiner polynomial. The original work we collect in this chapter can be found in [46].

2.1 Full system of parallel bodies of K relative to E

Definition 2.1.1. *Let $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$. For $0 \leq \lambda \leq \mathrm{r}(K;E)$ the* inner parallel body (relative to E) *of K at distance λ is the set*

$$K \sim \lambda E = \left\{x \in \mathbb{R}^n : x + \lambda E \subseteq K\right\}.$$

When $\lambda = \mathrm{r}(K;E)$, $K \sim \mathrm{r}(K;E)E$ is the set of (relative) incenters of K, usually called *kernel* of K and denoted by $\ker(K;E)$. The dimension of $\ker(K;E)$ is strictly less than n (see [8, p. 59]).

Clearly if $\lambda = 0$ the original body K is obtained. Moreover,

$$\mathrm{r}(K_\lambda; E) = \mathrm{r}(K; E) - |\lambda| \,. \tag{2.1}$$

As mentioned above, in order to get a rather more geometrical view of the inner parallel body definition, when $E = B^n$ it can be seen as the intersection of the closed supporting half-spaces of K moved in a distance λ (see [41, p. 6]). In the general case, a similar interpretation can be given but a bit more involved: since we need further definitions for it, we will deal with it in the next chapter. The inner parallel bodies and their properties were studied mainly by Bol [6], Dinghas [14] (see also [23] and [24]) and later by Sangwine-Yager [41].

Definition 2.1.2. *The* full system of parallel bodies *of K (relative to E) is defined by*

$$K_\lambda := \begin{cases} K \sim (-\lambda)E & \text{for } -\mathrm{r}(K;E) \leq \lambda \leq 0, \\ K + \lambda E & \text{for } 0 \leq \lambda < \infty. \end{cases}$$

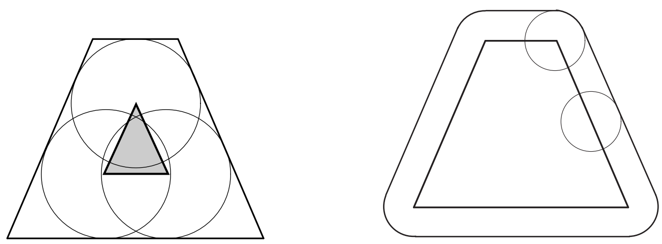

Figure 2.1: Inner and outer parallel body of a trapezoid relative to B^2.

For the sake of brevity we will not write explicitly the convex body E in the notation for the parallel bodies, in spite of the fact they depend on the convex body E which is fixed. When a particular set is used it will be made clear.

The following important property can be found in [49, p. 135]:

Lemma 2.1.3. *The full system of parallel bodies is a concave family, i.e., it satisfies*

$$(1-\mu)K_\lambda + \mu K_\sigma \subseteq K_{(1-\mu)\lambda+\mu\sigma} \tag{2.2}$$

for $\mu \in [0,1]$ and $\lambda, \sigma \in \big[-\mathrm{r}(K), \infty\big)$.

By using the monotonicity of the quermassintegrals in (2.2) and then applying Brunn-Minkowski inequality for quermassintegrals (see Theorem 1.2.12) to the sets K_λ and K_σ, the following theorem is easily obtained.

Theorem 2.1.4. *For all $i = 0, \dots, n$, the $(n-i)$-th root of the i-th quermassintegral $\mathrm{W}_i(K_\lambda; E)$ as a real function on λ is a concave function in $\big[-\mathrm{r}(K;E), \infty\big)$.*

2.2 Hadwiger's problem on inner parallel bodies

2.2.1 Some preliminary facts

We consider here the particular case of 3-dimensional convex bodies and $E = B^3$. Then the definition of full system of parallel bodies of a convex body K (relative to B^3) allows to define the functionals V, S and M as functionals depending on the parameter λ for $-\mathrm{r}(K) \leq \lambda < \infty$. For the sake of brevity we will use the notation $\mathrm{V}(\lambda) = \mathrm{V}(K_\lambda)$ and analogously for S and M. Thus Theorem 2.1.4 leads to the following lemma (see Proposition 1.1.8):

Lemma 2.2.1. *For any $K \in \mathcal{K}^3$, the functionals $\mathrm{V}(\lambda)$, $\mathrm{S}(\lambda)$ and $\mathrm{M}(\lambda)$, for $-\mathrm{r}(K) \leq \lambda < \infty$, have left and right derivatives at each point $-\mathrm{r}(K) \leq \lambda < \infty$ and they satisfy:*

$$'\mathrm{V}(\lambda) \geq \mathrm{V}'(\lambda) \geq \mathrm{S}(\lambda), \tag{2.3}$$

$$'\mathrm{S}(\lambda) \geq \mathrm{S}'(\lambda) \geq 2\mathrm{M}(\lambda), \tag{2.4}$$

$$'\mathrm{M}(\lambda) \geq \mathrm{M}'(\lambda) \geq 4\pi. \tag{2.5}$$

Moreover, in the case of the volume much more can be said:

Theorem 2.2.2 (Bol [6]). *Let $K \in \mathcal{K}^3$. Then the volume functional $\mathrm{V}(\lambda)$ is always differentiable, i.e., $'\mathrm{V}(\lambda) = \mathrm{V}'(\lambda)$, and $\mathrm{V}'(\lambda) = \mathrm{S}(\lambda)$ for $\lambda \in \big[-\mathrm{r}(K), \infty\big)$.*

Of course if $\lambda \geq 0$, i.e., if we restrict ourselves to the family of outer parallel bodies, then all the above functionals are differentiable and we have equalities in all the inequalities. So the interesting problem is to study the behavior of the functionals on the family of inner parallel bodies ($\lambda \leq 0$).

The above lemma and theorem will be established in a more general setting for the n-dimensional case in Chapter 3. We refer to this chapter for a more detailed explanation.

Inequalities (2.4) and (2.5) and the differentiability of the volume are the starting point of the problem we are going to develop in the following section.

2.2.2 Hadwiger's problem

In 1955, Hadwiger [23, Sections 23, 29] posed the problem of classifying the convex bodies in $\mathbb{R}^3$ according to the differentiability of the classical functionals V, S, M, defined, as mentioned above, as functions of the parameter λ of the full system of parallel bodies of K, i.e., $\mathrm{V}(\lambda)$, $\mathrm{S}(\lambda)$, $\mathrm{M}(\lambda)$. He classified the convex bodies in three different classes, denoted by $\mathcal{R}_\alpha$, $\mathcal{R}_\beta$, $\mathcal{R}_\gamma$, depending on whether equalities hold, respectively, in (2.3), (2.3) and (2.4), or (2.3), (2.4) and (2.5) (see Table 2.1).

From Theorem 2.2.2, the volume is always differentiable and $\mathrm{V}'(\lambda) = \mathrm{S}(\lambda)$, which is equivalent to the fact that $\mathcal{R}_\alpha$ is the whole family of convex bodies $\mathcal{K}^3$. So, the question arose to characterize the convex bodies belonging to the classes $\mathcal{R}_\beta$ and $\mathcal{R}_\gamma$.

If	then K lies in
$'\mathrm{V}(\lambda)=\mathrm{S}(\lambda)$	**class** $\boldsymbol{\mathcal{R}_\alpha}\equiv\mathcal{K}^3$
$'\mathrm{V}(\lambda)=\mathrm{S}(\lambda)$, $'\mathrm{S}(\lambda)=2\mathrm{M}(\lambda)$	**class** $\boldsymbol{\mathcal{R}_\beta}$
$'\mathrm{V}(\lambda)=\mathrm{S}(\lambda)$, $'\mathrm{S}(\lambda)=2\mathrm{M}(\lambda)$, $'\mathrm{M}(\lambda)=4\pi$	**class** $\boldsymbol{\mathcal{R}_\gamma}$

Table 2.1: Classification of $\mathcal{K}^3$ in Hadwiger's problem.

Hadwiger provided a partial solution in the sense that, what it is proved is not a characterization of the bodies belonging to each class, but of the *triples of values* $(\mathrm{V},\mathrm{S},\mathrm{M})$ which can be respectively the volume, surface area and integral mean curvature of *some* convex body in each class. More precisely, he proved the following theorem.

Theorem 2.2.3 (Hadwiger [23]). *i) Three positive real numbers* V, S *and* M *are the (respective) magnitudes of* some *convex body belonging to the class* $\mathcal{R}_\gamma$ *if and only if they verify the inequalities*

$$\mathrm{V}\leq\frac{1}{24\pi^2}\left[6\pi\mathrm{MS}-\mathrm{M}^3+(\mathrm{M}^2-4\pi\mathrm{S})^{3/2}\right] \tag{2.6}$$

$$\mathrm{V}\geq\frac{1}{24\pi^2}\left[6\pi\mathrm{MS}-\mathrm{M}^3-(12-\pi^2)\pi\left(\frac{\mathrm{M}^2-4\pi\mathrm{S}}{\pi^2-8}\right)^{3/2}\right]. \tag{2.7}$$

ii) Three positive real numbers V, S *and* M *are the (respective) magnitudes of* some *convex body belonging to the class* $\mathcal{R}_\beta$ *if and only if they verify the inequalities*

$$\mathrm{V}\leq\frac{\mathrm{S}^2}{3\mathrm{M}}$$

$$\mathrm{V}\geq\frac{1}{24\pi^2}\left[6\pi\mathrm{MS}-\mathrm{M}^3-(12-\pi^2)\pi\left(\frac{\mathrm{M}^2-4\pi\mathrm{S}}{\pi^2-8}\right)^{3/2}\right].$$

Notice that the first inequality in ii) is just the Minkowski inequality (1.22), which holds for all convex bodies. So it is proved that if a convex body $K\in\mathcal{K}^3$ lies in one of the classes, then the corresponding inequalities on V, S and M hold for $\mathrm{V}=\mathrm{V}(K),\mathrm{S}=\mathrm{S}(K)$ and $\mathrm{M}=\mathrm{M}(K)$. On the other hand, given $K\in\mathcal{K}^3$ such that the corresponding triple $\bigl(\mathrm{V}(K),\mathrm{S}(K),\mathrm{M}(K)\bigr)$ satisfies the inequalities in i) or ii), then there exists another convex body L (in principle $L\neq K$) with volume $\mathrm{V}(L)=\mathrm{V}(K)$, surface area $\mathrm{S}(L)=\mathrm{S}(K)$ and integral mean curvature $\mathrm{M}(L)=\mathrm{M}(K)$ lying in the class $\mathcal{R}_\gamma$ or $\mathcal{R}_\beta$, depending on the satisfied inequalities. It *does not ensure*, however, that the body K with $\bigl(\mathrm{V}(K),\mathrm{S}(K),\mathrm{M}(K)\bigr)$ satisfying the inequalities in i) or ii) lies in the corresponding class.

Remark 2.1. Notice also that this result does not characterize the convex bodies lying in each class because, as we have already seen in Section 1.4, in general to each triple $(\mathrm{V},\mathrm{S},\mathrm{M})$ correspond several convex bodies. $\diamondsuit$

In the next section we will study the relation between Hadwiger's problem and the behavior of the roots of the Steiner polynomial in dimension three.

2.3 The roots of the Steiner polynomial in Hadwiger's problem

This section is devoted to prove some results concerning Hadwiger's problem provided the relation between it and the behavior of the roots of the Steiner polynomial $f_{B^3}(K,\lambda)$ in $\mathbb{R}^3$ studied in Chapter 1. The work we present here can be found in [46].

In Chapter 3 the question on the differentiability of the quermassintegrals will be studied in the most general way, and all the results we will obtain there will have a translation to the original Hadwiger problem in dimension 3. Our aim in this chapter is just to show some results which establish a nice connection between Hadwiger's problem and the Steiner polynomial: we will provide necessary conditions for a convex body to belong to $\mathcal{R}_\beta$ in terms of the roots of its Steiner polynomial. Using them it will be possible to determine convex bodies not lying in $\mathcal{R}_\beta$. We will focus on this class since the class $\mathcal{R}_\gamma$ can be characterized in a precise way (see Theorem 3.3.1 for the general result as well as for its proof):

Theorem 2.3.1 ([29]). *The only convex bodies lying in $\mathcal{R}_\gamma$ are the outer parallel bodies of planar convex bodies, i.e.,*

$$\mathcal{R}_\gamma = \big\{K + \lambda B^3 : K \textit{ planar convex body}\big\}.$$

Notice that inequality (2.6) is just the condition $\phi_-(K;B^3) \leq 0$ which appeared in the characterization of the real roots of the Steiner polynomial, whereas it is easy to see that the equality in (2.7) corresponds to the curve in Blaschke diagram where all outer parallel bodies of the circle are mapped. It leads to think that there exists a connection between Hadwiger's problem and the behavior of the roots of the Steiner polynomial.

Thus the first question which arises deals with the type of roots of the Steiner polynomial of a body K_λ of the full system of parallel bodies of K. From Lema 1.4.3 we know that the roots of the Steiner polynomial of every outer parallel body of K are of the same type as the ones of K. We will prove that a similar result holds for inner parallel bodies of a convex body $K \in \mathcal{R}_\beta$ but only for certain type of roots.

Notice that given a convex body K, the functionals $\phi_-(K;B^3)$ and $\phi_+(K;B^3)$, as depending on V, S and M, can be defined as functionals on the parameter $\lambda \in \big[-\mathrm{r}(K),\infty\big)$ of the full system of parallel bodies of K. We start by showing the following lemma, which will be used in the proofs of the theorems. For the sake of brevity we write

$$\phi_\pm(\lambda) := \phi_\pm(K_\lambda;B^3).$$

Lemma 2.3.2 ([46]). *Let $K \in \mathcal{K}^3$. If $K \in \mathcal{R}_\beta\backslash\mathcal{R}_\gamma$ then both $\phi_+(\lambda)$ and $\phi_-(\lambda)$ are strictly increasing functions in $\lambda \in \big[-\mathrm{r}(K),0\big]$.*

Proof. Since the convex body K lies in the class $\mathcal{R}_\beta$, we have

$$\mathrm{V}'(\lambda) = \mathrm{S}(\lambda) \quad \text{and} \quad \mathrm{S}'(\lambda) = 2\mathrm{M}(\lambda)$$

for $-\mathrm{r}(K) \leq \lambda \leq 0$. Then an easy computation gives

$$\begin{aligned}\phi'_{\pm}(\lambda) = 3\Big[&\mathrm{M}(\lambda)^2\mathrm{M}'(\lambda) - 2\pi\big(\mathrm{M}'(\lambda)\mathrm{S}(\lambda) + 2\mathrm{M}(\lambda)^2 - 4\pi\mathrm{S}(\lambda)\big)\\ &\pm \big(\mathrm{M}(\lambda)^2 - 4\pi\mathrm{S}(\lambda)\big)^{1/2}\big(\mathrm{M}(\lambda)\mathrm{M}'(\lambda) - 4\pi\mathrm{M}(\lambda)\big)\Big]\\ = 3\Big[&\big(\mathrm{M}(\lambda)^2 - 2\pi\mathrm{S}(\lambda)\big)\big(\mathrm{M}'(\lambda) - 4\pi\big) \pm \mathrm{M}(\lambda)\big(\mathrm{M}(\lambda)^2 - 4\pi\mathrm{S}(\lambda)\big)^{1/2}\big(\mathrm{M}'(\lambda) - 4\pi\big)\Big].\end{aligned}$$

Since it always holds $\mathrm{M}(K)^2 \geq 4\pi\mathrm{S}(K)$ for any convex body $K \in \mathcal{K}^3$ (cf. (1.23)) and since by (2.5) $\mathrm{M}'(\lambda) > 4\pi$ for $K \in \mathcal{R}_\beta \backslash \mathcal{R}_\gamma$, we immediately get $\phi'_+(\lambda) > 0$ for every $\lambda \in \big[-\mathrm{r}(K), 0\big]$, which proves that $\phi_+(\lambda)$ is strictly increasing.

In the case of $\phi_-(\lambda)$ we have

$$\phi'_-(\lambda) = 3\big(\mathrm{M}'(\lambda) - 4\pi\big)\Big[\big(\mathrm{M}(\lambda)^2 - 2\pi\mathrm{S}(\lambda)\big) - \mathrm{M}(\lambda)\big(\mathrm{M}(\lambda)^2 - 4\pi\mathrm{S}(\lambda)\big)^{1/2}\Big].$$

Since $\mathrm{M}(\lambda)^2 > 2\pi\mathrm{S}(\lambda)$, the second term in the above product is positive if and only if

$$\big(\mathrm{M}(\lambda)^2 - 2\pi\mathrm{S}(\lambda)\big)^2 \geq \mathrm{M}(\lambda)^2\big(\mathrm{M}(\lambda)^2 - 4\pi\mathrm{S}(\lambda)\big),$$

i.e., if and only if $4\pi^2\mathrm{S}(\lambda)^2 \geq 0$ which is trivially true. Moreover, $\phi'_-(\lambda) = 0$ if and only if equality holds in the above inequality, i.e., when the surface area $\mathrm{S}(\lambda) = 0$ for any $\lambda \in \big[-\mathrm{r}(K), 0\big]$. Since we have $K \notin \mathcal{R}_\gamma$, we know that K is not a planar convex body and hence $\mathrm{S}(\lambda) > 0$, which implies $\big(\mathrm{M}(\lambda)^2 - 2\pi\mathrm{S}(\lambda)\big)^2 > \mathrm{M}(\lambda)^2\big(\mathrm{M}(\lambda)^2 - 4\pi\mathrm{S}(\lambda)\big)$. Using again the inequality $\mathrm{M}'(\lambda) > 4\pi$ we get $\phi'_-(\lambda) > 0$ for every $\lambda \in \big[-\mathrm{r}(K), 0\big]$, which proves that $\phi_-(\lambda)$ is strictly increasing. □

Using this lemma, we can now state and prove the announced relation between Hadwiger's problem and the behavior of the roots of the Steiner polynomial. We split it into two theorems given the different nature of the statements and proofs (see Remark 1.1 for the used notation).

Theorem 2.3.3 ([46]). *Let $K \in \mathcal{K}^3$ with $K \in \mathcal{R}_\beta \backslash \mathcal{R}_\gamma$.*

i) *If $K \in \mathfrak{C}_2$ then $K_\lambda \in \mathfrak{C}_2$ for every $\lambda \in \big[-\mathrm{r}(K), 0\big]$.*
ii) *If $K \in \mathfrak{R}$ then $K_\lambda \in \mathfrak{R} \cup \mathfrak{C}_1$ for every $\lambda \in \big[-\mathrm{r}(K), 0\big]$.*

Proof. By Lemma 2.3.2 we know that ϕ_+ and ϕ_- are strictly increasing functions in λ, with $-\mathrm{r}(K) \leq \lambda \leq 0$, which gives $\phi_+(\lambda) \leq \phi_+(0) = \phi_+(K; B^3)$ and $\phi_-(\lambda) \leq \phi_-(0) = \phi_-(K; B^3)$.

i) If $K \in \mathfrak{C}_2$ then by Theorem 1.3.2 we have $\phi_+(K; B^3) < 0$, and hence $\phi_+(\lambda) < 0$. It shows that $K_\lambda \in \mathfrak{C}_2$ for every $\lambda \in \big[-\mathrm{r}(K), 0\big]$.
ii) If $K \in \mathfrak{R}$ then again by Theorem 1.3.2 we get $\phi_+(K; B^3) \geq 0$ and $\phi_-(K; B^3) \leq 0$; hence, in particular we have $\phi_-(\lambda) \leq 0$. It shows that $K_\lambda \in \mathfrak{R} \cup \mathfrak{C}_1$ for every $\lambda \in \big[-\mathrm{r}(K), 0\big]$. □

In the case $K \in \mathfrak{C}_1$ no condition is obtained, since the Steiner polynomial of its inner parallel bodies can have any type of roots.

We recall that $K_{-\mathrm{r}(K)} = \ker(K; B^3)$ and $\dim K_{-\mathrm{r}(K)} \leq 2$; hence Lemma 1.4.4 ensures that either $K_{-\mathrm{r}(K)} \in \mathfrak{R}$ or $K_{-\mathrm{r}(K)} \in \mathfrak{C}_2$.

Theorem 2.3.4 ([46]). *Let $K \in \mathcal{K}^3$ with $K \in \mathcal{R}_\beta \backslash \mathcal{R}_\gamma$.*

i) *If $\dim K_{-\mathrm{r}(K)} \leq 1$ then $K_\lambda \in \mathfrak{C}_1$ for every $\lambda \in \big(-\mathrm{r}(K), 0\big]$.*

ii) If $\dim K_{-\mathrm{r}(K)} = 2$ and $K_{-\mathrm{r}(K)} \in \mathfrak{R}$ then $K_\lambda \notin \mathfrak{C}_2 \cup \mathfrak{R}_2$ for every $\lambda \in \big(-\mathrm{r}(K), 0\big]$.

Proof. We prove first i). By Lemma 2.3.2 we know that ϕ_+ and ϕ_- are strictly increasing functions in $\lambda > -\mathrm{r}(K)$, which gives $\phi_+(\lambda) > \phi_+\big(-\mathrm{r}(K)\big)$ and $\phi_-(\lambda) > \phi_-\big(-\mathrm{r}(K)\big)$. Since $\dim K_{-\mathrm{r}(K)} \leq 1$, $\mathrm{V}\big(-\mathrm{r}(K)\big) = \mathrm{S}\big(-\mathrm{r}(K)\big) = 0$ and hence

$$\begin{aligned}\phi_+(\lambda) &> \phi_+\big(-\mathrm{r}(K)\big) = 2\mathrm{M}\big(-\mathrm{r}(K)\big)^3 = 2\mathrm{M}(K_{-\mathrm{r}(K)})^3 \geq 0,\\ \phi_-(\lambda) &> \phi_-\big(-\mathrm{r}(K)\big) = \mathrm{M}\big(-\mathrm{r}(K)\big)^3 - \mathrm{M}\big(-\mathrm{r}(K)\big)^3 = 0.\end{aligned}$$

From $\phi_+(\lambda) > 0$ and by Theorem 1.3.2 we can assure that $K_\lambda \in \mathfrak{R} \cup \mathfrak{C}_1$ for all $\lambda \in \big(-\mathrm{r}(K), 0\big]$. Moreover, since the inequality is strict, the Steiner polynomial of K_λ cannot have double real roots of Type 2. From $\phi_-(\lambda) > 0$ and by Theorem 1.3.2 we know that $K_\lambda \in \mathfrak{C}_1$ for all $\lambda \in \big(-\mathrm{r}(K), 0\big]$, i.e., the Steiner polynomial of all its inner parallel bodies has only complex roots of Type 1. It shows i).

Analogously we get ii). By the strict monotonicity of the function ϕ_+ (in this case ϕ_- plays no role) we get $\phi_+(\lambda) > \phi_+\big(-\mathrm{r}(K)\big)$. Since $\dim K_{-\mathrm{r}(K)} = 2$ and we assume $K_{-\mathrm{r}(K)} \in \mathfrak{R}$, we know (Theorem 1.3.2) that $\phi_+\big(-\mathrm{r}(K)\big) \geq 0$, which gives $\phi_+(\lambda) > 0$. It shows that $K_\lambda \notin \mathfrak{C}_2 \cup \mathfrak{R}_2$ for every $\lambda \in \big(-\mathrm{r}(K), 0\big]$. □

In the case when $\dim K_{-\mathrm{r}(K)} = 2$ and $K_{-\mathrm{r}(K)} \in \mathfrak{C}_2$, no condition for all inner parallel bodies is obtained, since their Steiner polynomial can have any type of roots. We can only assure that the original body $K \in \mathfrak{C}_1 \cup \mathfrak{C}_2$; in fact, if we suppose that $K \in \mathfrak{R}$ then, by Theorem 2.3.3, part ii), we conclude that $K_\lambda \in \mathfrak{R} \cup \mathfrak{C}_1$ for all $\lambda \in \big[-\mathrm{r}(K), 0\big]$, a contradiction since $K_{-\mathrm{r}(K)} \in \mathfrak{C}_2$.

2.3.1 Some examples of exclusion of convex bodies from the class $\mathcal{R}_\beta$

We show with a couple of examples how some convex bodies can be excluded from the class $\mathcal{R}_\beta$ by using the previous theorems.

Example 2.1. We are going to show, as an application of Theorem 2.3.3, that there are no cylinders in $\mathcal{R}_\beta$. The orthogonal cylinders $C^{(\mathrm{r},h)}$ with circular basis of radius $\mathrm{r} > 0$ and height $h \in [0, \infty)$ have volume $\mathrm{V}\big(C^{(\mathrm{r},h)}\big) = \pi\mathrm{r}^2 h$, surface area $\mathrm{S}\big(C^{(\mathrm{r},h)}\big) = 2\pi\mathrm{r}(\mathrm{r}+h)$, and integral mean curvature

$\mathrm{M}\big(C^{(\mathrm{r},h)}\big)=\pi(\pi\mathrm{r}+h)$. As we have seen in Example 1.3, the Steiner polynomial $f_{B^3}\big(C^{(\mathrm{r},h)},\lambda\big)$ can have either three real roots or complex roots of TYPE 2, depending on the ratio between r and h.

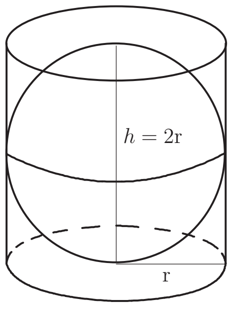

Figure 2.2: The archimedean cylinder.

Moreover, the Steiner polynomial of just one cylinder (up to congruences) will have double real roots of TYPE 2, the one with $h=h_0\mathrm{r}\approx 1.71065\,\mathrm{r}$; if $h<h_0\mathrm{r}$, complex roots of TYPE 2 appear; if $h>h_0\mathrm{r}$, three simple real roots; and only in the limit case of a segment, double real roots of TYPE 1 are obtained. For instance, the *archimedean cylinder* (obtained when $h=2\mathrm{r}$, see Figure 2.2) verifies $C^{(\mathrm{r},2r)}\in\mathfrak{R}\backslash\{\mathfrak{R}_1,\mathfrak{R}_2\}$.

It is easy to check that for any cylinder $C^{(\mathrm{r},h)}$ with $h\in(h_0\mathrm{r},2\mathrm{r})$, its inner parallel bodies $C_\lambda^{(\mathrm{r},h)}$, $\lambda\in\big[-\mathrm{r}\big(C^{(\mathrm{r},h)}\big)=-h/2,0\big]$, have

$$\mathrm{V}\big(C_\lambda^{(\mathrm{r},h)}\big)=\pi(\mathrm{r}+\lambda)^2(h+2\lambda),\quad \mathrm{S}\big(C_\lambda^{(\mathrm{r},h)}\big)=2\pi(\mathrm{r}+\lambda)(\mathrm{r}+h+3\lambda),$$
$$\mathrm{M}\big(C_\lambda^{(\mathrm{r},h)}\big)=\pi\big(\pi\mathrm{r}+h+(\pi+2)\lambda\big),$$

and for $\lambda<-(h-h_0\mathrm{r})/(2-h_0)$, all the inner parallel bodies $C_\lambda^{(\mathrm{r},h)}\in\mathfrak{C}_2$, whereas we already know that $C^{(\mathrm{r},h)}\in\mathfrak{R}$. Hence by Theorem 2.3.3 we can assure that $C^{(\mathrm{r},h)}\notin\mathcal{R}_\beta$. $\diamondsuit$

Example 2.2. Theorem 2.3.4 i) allows to exclude from $\mathcal{R}_\beta$ those convex bodies not lying in $\mathfrak{C}_1$ with kernel a single point. For instance, if we consider the family of cones with circular basis of radius e.g. 1 and height h, it is easy to check that for such a cone K^h,

$$\mathrm{V}(K^h)=\frac{\pi}{3}h,\quad \mathrm{S}(K^h)=\pi\left(1+\sqrt{1+h^2}\right),\quad \mathrm{M}(K^h)=\pi(\pi+h-\arctan h),$$

and that they can have either real roots, complex roots of TYPE 1 or complex roots of TYPE 2 (see Figure 2.3).

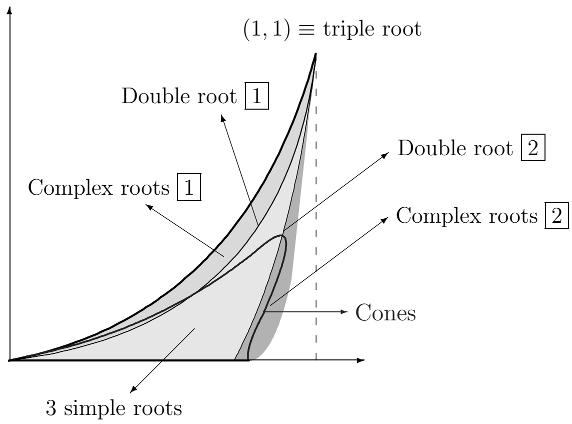

Figure 2.3: Image by Blaschke map of the family of cones.

Notice that the kernel of any of these cones is a point. Hence, if K^h is a cone with $K^h\notin\mathfrak{C}_1$ (which is obtained for any $h\in[0,\overline{h}\approx 3.37108])$, we can assure that $K^h\notin\mathcal{R}_\beta$. We remark moreover that the inner parallel bodies of any cone are homothetic copies of it (cf. Theorem 3.1.4). Then the Steiner polynomial of the inner parallel bodies will have the same type of roots as the original cone K^h, i.e., $K^h_\lambda \notin \mathfrak{C}_1$ for all $\lambda \in \big[-\mathrm{r}(K^h), 0\big]$ if $K^h \notin \mathfrak{C}_1$.

Another illustrative example is provided by the truncated cones. Notice that their inner parallel bodies are cones for sufficiently small λ (see Figure 2.4 and also Figure 2.1). Hence by the previous argument for the family of cones, suitable truncated cones can be also excluded from $\mathcal{R}_\beta$. $\diamondsuit$

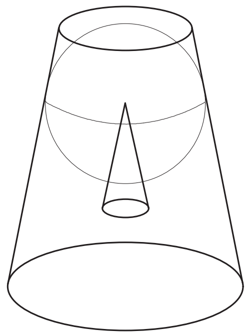

Figure 2.4: The inner parallel body of a truncated cone.

Remark 2.2. These results show the cases in which the type of roots of the Steiner polynomial are "preserved by inner parallel bodies", in the following sense: from Lemma 1.4.3 we know that if K is a convex body whose Steiner polynomial has a certain type of roots (simple real, double real or complex), then *all* the outer parallel bodies of K verify that their Steiner polynomial has the same type of roots. Theorem 2.3.3 and Theorem 2.3.4 show that in the case of the inner parallel bodies, this can be assured only in particular situations. $\diamondsuit$

Chapter 3

On differentiability of quermassintegrals

This chapter is devoted to study the problem of classifying the convex bodies in $\mathbb{R}^n$, depending on the differentiability of their associated quermassintegrals (relative to a convex body $E \in \mathcal{K}_0^n$ which is fixed throughout the full chapter) with respect to the one-parameter-depending family given by the full system of parallel bodies in dimension n.

As mentioned in the previous chapter, this problem was originally posed by Hadwiger in the 3-dimensional space when $E = B^3$. In this chapter we characterize one of the non-trivial classes and give necessary conditions for a convex body to belong to the analogous classes in dimension n and with respect to the general convex body E. The original work we collect in this chapter can be found in [28] and mainly in [29].

3.1 Extreme vectors and related notions

In order to establish most of the results contained in this chapter we need some definitions and known facts about extreme normal vectors and some other related notions associated to the boundary of a convex body $K \in \mathcal{K}^n$ ([49, pp. 74–77]). For a given $K \in \mathcal{K}^n$ we write $N(K, x)$ to denote the *normal cone* of K at $x \in \operatorname{bd} K$, i.e., the set of all outer normal vectors of K at x (with the zero vector). Moreover, for $u \in \mathbb{S}^{n-1}$ let $x \in K \cap H(K, u)$ be a boundary point such that u is an outer normal vector of K at x. Then there exists a unique face of the normal cone $N(K, x)$ that contains u in its relative interior (see [49, Theorem 2.1.2]). Since this face does not depend upon the choice of the point $x \in K \cap H(K, u)$ (the normal cone in all the points of the support set $K \cap H(K, u)$ is the same) the notation $T(K, u)$ will be adopted for it (without reference to the point x). Usually $T(K, u)$ is called the *touching cone* of K at u.

Definition 3.1.1. *A vector $u \in \mathbb{S}^{n-1}$ is an r*-extreme normal vector *of $K \in \mathcal{K}^n$ if we cannot write $u = u_1 + \cdots + u_{r+2}$, with u_i, $i = 1, \ldots, r+2$, linearly independent normal vectors at one and the same boundary point of K. We will say that the supporting hyperplane $H(K,u)$ is an r*-extreme (supporting) hyperplane *if u is an r-extreme normal vector of K.*

Usually 0-extreme normal vectors are called just *extreme normal vectors.* The set of r-extreme normal vectors of K will be denoted by $\mathcal{U}_r(K)$. It follows immediately from the definition that each r-extreme normal vector is also an s-extreme one for $r < s \leq n-1$ and so, for every $K \in \mathcal{K}^n$,

$$\mathcal{U}_0(K) \subseteq \mathcal{U}_1(K) \subseteq \cdots \subseteq \mathcal{U}_{n-1}(K).$$

Remark 3.1. Notice that for $K \in \mathcal{K}^n$, if $x \in \operatorname{bd} K$ is a *regular point* (i.e., if the supporting hyperplane to K at x is unique), then $\dim N(K,x) = 1$ and hence the (only) outer unit normal vector $u \in N(K,x)$ is a 0-extreme normal vector of K. ♢

The following characterization of 0-extreme normal vectors in terms of the support function of K will be needed later on. It can be found in [41, Lemma 2.3].

Lemma 3.1.2. *Let $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$ be an outer normal vector to K. Then $u \in \mathcal{U}_0(K)$ if and only if for any distinct*

$$u_1, u_2 \in \mathbb{S}^{n-1} \quad \text{and} \quad \alpha > 0,\ \beta > 0 \quad \text{such that} \quad u = \alpha u_1 + \beta u_2, \tag{3.1}$$

it holds that

$$h(K,u) < \alpha h(K,u_1) + \beta h(K,u_2).$$

3.1.1 Tangential bodies and form bodies

Extreme normal vectors turn out to be rather useful to approach some properties that we will study for the full system of parallel bodies of $K \in \mathcal{K}^n$. Moreover, extreme (supporting) hyperplanes lead to pairs of convex bodies that have some, but not all supporting hyperplanes in common. Next we define two of these pairs of bodies, which will play a central role in this chapter.

Definition 3.1.3. *A convex body $K \in \mathcal{K}^n$ containing the convex body $E \in \mathcal{K}^n_0$ is called a p*-tangential body *of E, $p \in \{0, \ldots, n-1\}$, if each $(n-p-1)$-extreme supporting hyperplane of K is a supporting hyperplane of E for $p \in \{0, \ldots, n-1\}$.*

From the above definition it follows that a 0-tangential body of E is the body E itself, and each p-tangential body of E is also a q-tangential body for $p < q \leq n-1$. There are intuitive reasons (see [49, p. 76]) for which 1-tangential bodies are usually called cap-bodies, and it can be easily proved the equivalence with the definition given in page 18 (see Figure 1.3). An $(n-1)$-tangential body will be briefly called *tangential body.* For further characterizations and properties of p-tangential bodies we refer to [49, Section 2.2]. The following nice theorem (see [49, pp. 136–137]) shows the close relation existing between inner parallel bodies and tangential bodies.

Theorem 3.1.4 (Schneider [49]). *Let $K \in \mathcal{K}_0^n$ and $-\mathrm{r}(K;E) < \lambda < 0$. Then K_λ is homothetic to K if, and only if, K is homothetic to a tangential body of E.*

Remark 3.2. More precisely, from the proof of Theorem 3.1.4 it is obtained that $K_\lambda = \rho K$ for a certain $\rho \in (0,1)$ if and only if K is homothetic to a tangential body of E with homothecy factor given by $|\lambda|/(1-\rho) = \mathrm{r}(K;E)$. ♢

We will also use the following result, which gives a characterization of n-dimensional p-tangential bodies in terms of quermassintegrals. It was proved by Favard in [16] (see also [49, p. 367]).

Theorem 3.1.5 (Favard [16]). *Let $K \in \mathcal{K}_0^n$ with $E \subseteq K$, and let $p \in \{0,\dots,n-1\}$. Then $\mathrm{W}_{n-p-1}(K;E) = \mathrm{W}_{n-p}(K;E)$ if and only if K is a p-tangential body of E; in this case,*

$$\mathrm{V}(K) = \mathrm{W}_1(K;E) = \cdots = \mathrm{W}_{n-p}(K;E).$$

Remark 3.3. Notice that in the above theorem $\mathrm{r}(K;E) = 1$; if it is not the case, we have that $\mathrm{W}_{n-p-1}(K;E) = \mathrm{r}(K;E)\mathrm{W}_{n-p}(K;E)$ if and only if K is homothetical to a p-tangential body of E and in this case, $\mathrm{V}(K) = \mathrm{r}(K;E)\mathrm{W}_1(K;E) = \cdots = \mathrm{r}(K;E)\mathrm{W}_{n-p}(K;E)$. ♢

Definition 3.1.6. *Let $K \in \mathcal{K}_0^n$. The (relative)* form body *of K with respect to E, denoted by K^*, is defined to be the intersection of the supporting half-spaces to E with outer normals vectors in $\mathcal{U}_0(K)$, i.e.,*

$$K^* = \bigcap_{u \in \mathcal{U}_0(K)} H^-(E,u) = \bigcap_{u \in \mathcal{U}_0(K)} \bigl\{x : \langle x,u\rangle \leq h(E,u)\bigr\}. \tag{3.2}$$

For the sake of brevity, the relative form body of K with respect to E will be called just *form body* of K, and will be denoted by K^*, without any reference to the convex body E. When E is a particular convex body or some property of E holds, it will be pointed out.

The form body of K with respect to E is the body constructed by considering the 0-extreme supporting halfspaces, i.e., halfspaces corresponding to 0-extreme hyperplanes, of K and intersecting them as supporting halfspaces of E (see Figure 3.1).

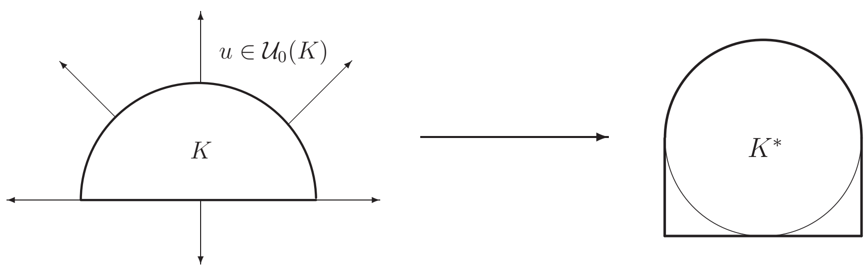

Figure 3.1: The form body of a half-circle with respect to the circle.

Remark 3.4. Just from the definition it follows that any form body is a p-tangential body for some $p \in \{0, \dots, n-1\}$. Moreover, it can be proved (see [49, p. 321, Theorem 2.2.8]) that a convex body $K \in \mathcal{K}_0^n$ is homothetic to its form body K^* if and only if K is homothetic to a tangential body of E. $\diamondsuit$

Remark 3.5. When K is *regular*, i.e., if every $x \in \operatorname{bd} K$ is a regular point, Remark 3.1 ensures that the outer unit normal vector at every $x \in \operatorname{bd} K$ is a 0-extreme normal vector. Thus, the form body of a regular convex body (with respect to E) is E itself (see Figure 3.2).

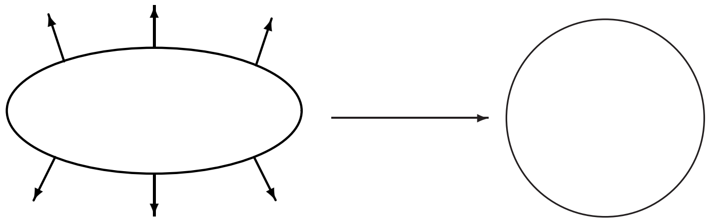

Figure 3.2: The form body of an ellipse with respect to the circle.

In the case of a polytope P (with interior points), the 0-extreme normal vectors of P are the outer normal vectors to its facets (i.e., $(n-1)$-dimensional faces). Hence, the form body of a polytope is always a polytope. In particular, the form body of P with respect to a regular convex body $E \in \mathcal{K}_0^n$ is another polytope circumscribed about E (i.e., all the facets touch E) whose facets are parallel to the ones of P (see Figure 3.3). $\diamondsuit$

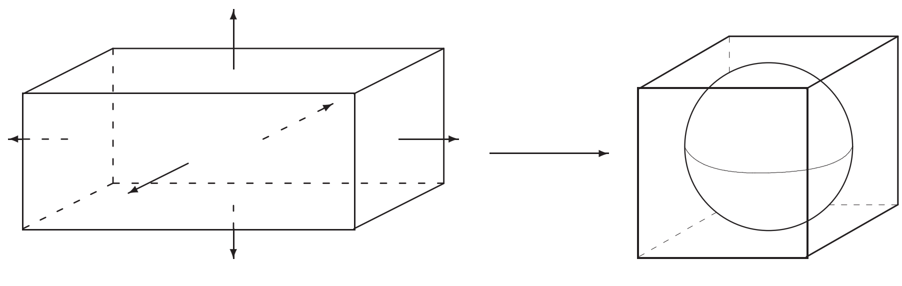

Figure 3.3: The form body of an orthogonal box with respect to the ball.

Remark 3.6. i) In the planar case, all form bodies are cap-bodies.

ii) For $n \geq 3$, any polytope circumscribed about the unit ball B^n is a form body of some convex body K which is not a cap-body.

iii) When $\mathcal{U}_0(E) \subseteq \mathcal{U}_0(K)$, then $K^* = E$. For example, in the plane, the form body of a ball with respect to a square is the square itself. $\diamondsuit$

3.1.2 0-extreme vectors of inner parallel bodies and the form body

Since we are interested in the behavior of the quermassintegrals with respect to the full system of parallel bodies of $K \in \mathcal{K}^n$, it is natural to look first at the relations among the 0-extreme normal vectors of the inner and outer parallel bodies (and consequently at their form bodies). If we deal with outer parallel bodies, the mentioned relation will be trivial in the case we will work on: when E is regular; indeed, since for the Minkowski sum of two convex bodies $K, L \in \mathcal{K}^n$ it holds that

$$\mathcal{U}_0(K) \cup \mathcal{U}_0(L) \subseteq \mathcal{U}_0(K+L), \tag{3.3}$$

(this property was proved in [41, Lemma 2.4]), when E is regular then Remark 3.5 ensures that $\mathbb{S}^{n-1} = \mathcal{U}_0(K) \cup \mathcal{U}_0(E) \subseteq \mathcal{U}_0(K+\lambda E) = \mathbb{S}^{n-1}$.

The following results state a relation between the 0-extreme normal vectors of $K \in \mathcal{K}^n$ and the ones of its inner parallel bodies. They can be found in [41, Lemma 4.4 and Lemma 4.5].

Lemma 3.1.7. *Let $K \in \mathcal{K}^n$, $-\mathrm{r}(K;E) < \lambda \le 0$ and $u \in \mathcal{U}_0(K_\lambda)$. Then*

$$h(K_\lambda, u) = h(K,u) - |\lambda|\, h(E,u) = h(K,u) + \lambda h(E,u). \tag{3.4}$$

Lemma 3.1.8. *Let $K \in \mathcal{K}^n$ and $-\mathrm{r}(K;E) < \lambda \le 0$. Then*

$$\mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_0(K). \tag{3.5}$$

A first question arising from the above lemma is for which convex bodies the equality holds in (3.5). With regard to it, we prove the following results.

Lemma 3.1.9 ([28]). *Let $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$ be regular. We write $\mathrm{r} = \mathrm{r}(K;E)$. If K is a tangential body of the outer parallel body $(K_{-\mathrm{r}})_{\mathrm{r}} = K_{-\mathrm{r}} + \mathrm{r}E$, then for any $-\mathrm{r} < \lambda \le 0$,*

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda). \tag{3.6}$$

Proof. First we show that for every $u \in \mathcal{U}_0(K)$, $h(K_\lambda,u) = h\big((K_{-\mathrm{r}})_{\mathrm{r}+\lambda}, u\big)$. Notice, on one hand, that since K is a tangential body of $(K_{-\mathrm{r}})_{\mathrm{r}}$ then $h(K,u) = h\big((K_{-\mathrm{r}})_{\mathrm{r}}, u\big)$ for any $u \in \mathcal{U}_0(K)$ and thus

$$h(K_\lambda,u) \le h(K,u) - |\lambda|\, h(E,u) = h(K_{-\mathrm{r}} + \mathrm{r}E, u) - |\lambda|\, h(E,u) = h(K_{-\mathrm{r}},u) + (\mathrm{r}+\lambda) h(E,u).$$

On the other hand, it is clear from the definition of inner parallel body that since

$$(K_{-\mathrm{r}})_{\mathrm{r}+\lambda} + |\lambda|\, E = K_{-\mathrm{r}} + \big(\mathrm{r} - |\lambda|\big)E + |\lambda|\, E = K_{-\mathrm{r}} + \mathrm{r}E = (K_{-\mathrm{r}})_{\mathrm{r}} \subseteq K,$$

then $(K_{-\mathrm{r}})_{\mathrm{r}+\lambda} \subseteq K_\lambda$ and hence we have $h(K_\lambda,u) \ge h(K_{-\mathrm{r}},u) + (\mathrm{r}+\lambda)h(E,u)$. Thus we obtain the equality $h(K_\lambda,u) = h\big((K_{-\mathrm{r}})_{\mathrm{r}+\lambda}, u\big)$ for any $u \in \mathcal{U}_0(K)$, as required.

Now, in order to prove that $\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K)$ for every $-\mathrm{r} < \lambda \le 0$, let $u \in \mathcal{U}_0(K)$. Since it holds that $h(K_\lambda,u) = h\big((K_{-\mathrm{r}})_{\mathrm{r}+\lambda}, u\big)$, the supporting hyperplanes $H(K_\lambda,u)$ and $H\big((K_{-\mathrm{r}})_{\mathrm{r}+\lambda}, u\big)$

coincide. Moreover, we know that $(K_{-\mathrm{r}})_{\mathrm{r}+\lambda} \subseteq K_\lambda$. So, at any common point x in the (non-empty) intersection of the support sets $K_\lambda \cap H(K_\lambda, u)$ and $(K_{-\mathrm{r}})_{\mathrm{r}+\lambda} \cap H\big((K_{-\mathrm{r}})_{\mathrm{r}+\lambda}, u\big)$, the corresponding normal cones verify that $N(K_\lambda, x) \subseteq N\big((K_{-\mathrm{r}})_{\mathrm{r}+\lambda}, x\big)$. On the other hand, since clearly $(K_{-\mathrm{r}})_{\mathrm{r}+\lambda}$ is regular then $\dim N\big((K_{-\mathrm{r}})_{\mathrm{r}+\lambda}, x\big) = 1$ (see Remark 3.1); hence $\dim N(K_\lambda, x) = 1$ which proves that $u \in \mathcal{U}_0(K_\lambda)$ (see Remark 3.5). Thus $\mathcal{U}_0(K) \subseteq \mathcal{U}_0(K_\lambda)$ and with (3.5) we get the result. $\square$

Lemma 3.1.10 ([28]). *Let $K \in \mathcal{K}^n$ be a regular convex body and write $\mathrm{r} = \mathrm{r}(K; E)$. Then $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda)$ for any $-\mathrm{r} < \lambda \leq 0$ if and only if $K = K_{-\mathrm{r}} + \mathrm{r}E$.*

Proof. If $K = K_{-\mathrm{r}} + \mathrm{r}E$ then Lemma 3.1.9 gives the result. So, we assume that $\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K)$ for all $\lambda \in (-\mathrm{r}, 0]$. Since K in regular, $\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K) = \mathbb{S}^{n-1}$ and hence, for all $u \in \mathbb{S}^{n-1} = \mathcal{U}_0(K_\lambda)$ Lemma 3.1.7 ensures that $h(K_\lambda, u) = h(K, u) + \lambda h(E, u)$. Therefore (see (1.2))

$$\begin{aligned}
\mathrm{W}_{n-1}(K_\lambda; E) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K_\lambda, u)\, d\mathrm{S}_{n-1}(E; u) \\
&= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K, u)\, d\mathrm{S}_{n-1}(E; u) + \frac{1}{n}\int_{\mathbb{S}^{n-1}} \lambda h(E, u)\, d\mathrm{S}_{n-1}(E; u) \\
&= \mathrm{W}_{n-1}(K; E) + \lambda \mathrm{W}_{n-1}(E; E) = \mathrm{W}_{n-1}(K; E) - |\lambda|\, \mathrm{W}_{n-1}(E; E).
\end{aligned}$$

Thus we get $\mathrm{W}_{n-1}(K; E) = \mathrm{W}_{n-1}(K_\lambda; E) + |\lambda|\, \mathrm{W}_{n-1}(E; E) = \mathrm{W}_{n-1}\big(K_\lambda + |\lambda|\, E\big)$ (linearity of mixed volumes; see Proposition 1.2.5 part v)), and since it always holds $K_\lambda + |\lambda|\, E \subseteq K$, we can conclude that $K = K_\lambda + |\lambda|\, E$. Notice that we have proved $K = K_\lambda + |\lambda|\, E$ for all $\lambda \in (-\mathrm{r}, 0]$, which implies that $K = K_{-\mathrm{r}} + \mathrm{r}E$, as required. $\square$

From (3.5) it follows that $\mathrm{cl}\,\mathcal{U}_0(K_\lambda) \subseteq \mathrm{cl}\,\mathcal{U}_0(K)$ for $-\mathrm{r}(K; E) < \lambda \leq 0$, and in general we get

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) \subseteq \mathrm{cl}\,\mathcal{U}_0(K_\rho) \tag{3.7}$$

for $-\mathrm{r}(K; E) < \lambda < \rho \leq 0$.

In the introduction of Chapter 2 it is given a *geometrical definition* of the inner parallel body of a convex body $K \in \mathcal{K}^n$ at distance $\lambda > 0$, with respect to the unit ball B^n, namely, that it is the intersection of the closed supporting half-spaces of K moved in a distance λ. This notion can be stated in a more general way; in fact, it follows from (3.4) and (3.5) that for a convex body K with inradius $\mathrm{r}(K; E)$, the (relative) inner parallel bodies of K, for $-\mathrm{r}(K; E) < \lambda \leq 0$, are given by

$$K_\lambda = \bigcap_{u \in \mathcal{U}_0(K)} \big\{x : \langle x, u\rangle \leq h(K, u) + \lambda h(E, u)\big\}. \tag{3.8}$$

Remark 3.7. From the above relation and (3.5) it can be easily seen that the inner parallel body of a polytope is always a polytope. $\diamondsuit$

Lemma 3.1.8 allows to show the following useful property regarding 0-extreme vectors and the Minkowski sum, namely, that the set of 0-extreme vectors of the Minkowski sum of two convex bodies does not depend on the size of the summands involved in it.

Lemma 3.1.11 ([28]). *Let $K, L \in \mathcal{K}^n$ and $\lambda > 0$. Then*

$$\mathcal{U}_0(K + L) = \mathcal{U}_0(K + \lambda L).$$

Proof. First we assume that $0 < \lambda \le 1$. Then, $K + L = K + \lambda L + (1 - \lambda)L$, i.e., $K + \lambda L$ is the inner parallel body of $K + L$ *with respect to* L at distance $1 - \lambda$. So, by Lemma 3.1.8 we get

$$\mathcal{U}_0(K + \lambda L) \subseteq \mathcal{U}_0(K + L).$$

On the other hand, it is clear that $\mathcal{U}_0(\lambda K) = \mathcal{U}_0(K)$ and hence $\mathcal{U}_0(\lambda K + \lambda L) = \mathcal{U}_0(K + L)$. Since $K + \lambda L = \lambda K + \lambda L + (1 - \lambda)K$, we get that $\lambda K + \lambda L$ is the inner parallel body of $K + L$ *with respect to* K at distance $1 - \lambda$ and thus, using again Lemma 3.1.8, it follows that

$$\mathcal{U}_0(K + L) = \mathcal{U}_0(\lambda K + \lambda L) \subseteq \mathcal{U}_0(K + \lambda L).$$

Together with the previous inclusion we get the result, i.e., $\mathcal{U}_0(K + L) = \mathcal{U}_0(K + \lambda L)$.

Finally, if $\lambda \ge 1$ (and hence $1/\lambda \le 1$) it is enough to consider $(1/\lambda)(K + \lambda L) = (1/\lambda)K + L$. Since $\mathcal{U}_0(K + \lambda L) = \mathcal{U}_0\big((1/\lambda)(K + \lambda L)\big) = \mathcal{U}_0\big((1/\lambda)K + L\big)$ we have just to apply the previous case, interchanging the roles of K and L, to obtain the result. $\square$

In the following lemma we prove a relation between the 0-extreme normal vectors of a convex body K and the ones of its form body K^* with respect to E. In [41, Lemma 4.6] it is shown that

$$\mathcal{U}_0(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_0(K). \tag{3.9}$$

Here we prove that equality holds under certain restrictions.

Lemma 3.1.12 ([29]). *Let $E \in \mathcal{K}^n_0$ be regular. Then for any $K \in \mathcal{K}^n_0$ it holds*

$$\mathcal{U}_0(K^*) = \mathrm{cl}\,\mathcal{U}_0(K). \tag{3.10}$$

Proof. First we prove that $\mathcal{U}_0(K) \subseteq \mathcal{U}_0(K^*)$. By using the characterization of 0-extreme normal vectors stated in Lemma 3.1.2, let $u \in \mathcal{U}_0(K)$ and let $u_1, u_2 \in \mathbb{S}^{n-1}$ and $\alpha, \beta > 0$ as in (3.1). Since $u \in \mathcal{U}_0(K)$, by the definition of form body (with respect to E) it holds $h(E, u) = h(K^*, u)$. On the other hand, since E is regular, $\mathcal{U}_0(E) = \mathbb{S}^{n-1}$, and then u is also a 0-extreme normal vector of E. Hence

$$h(K^*, u) = h(E, u) < \alpha h(E, u_1) + \beta h(E, u_2) \le \alpha h(K^*, u_1) + \beta h(K^*, u_2),$$

where the last inequality follows from $E \subset K^*$. Using again Lemma 3.1.2 we get $u \in \mathcal{U}_0(K^*)$.

Now we prove (3.10). By (3.9) we just have to see the inclusion $\mathcal{U}_0(K^*) \supseteq \mathrm{cl}\,\mathcal{U}_0(K)$. Thus, let $u \in \mathrm{cl}\,\mathcal{U}_0(K)$ and suppose that $u \notin \mathcal{U}_0(K^*)$.

We take a sequence $(u_k)_{k \in \mathbb{N}} \subset \mathcal{U}_0(K)$ with $u_k \to u$ for $k \to \infty$. Since we already know that $\mathcal{U}_0(K) \subset \mathcal{U}_0(K^*)$ then $u_k \in \mathcal{U}_0(K^*)$ for all $k \in \mathbb{N}$ and hence, by definition of form body, we get

$h(E,u_k)=h(K^*,u_k)$ for all $k\in\mathbb{N}$. Therefore $h(E,u)=h(K^*,u)$ by the continuity of the support function. It ensures that there exists $x\in\operatorname{bd}K^*\cap\operatorname{bd}E$ such that $u\in N(K^*,x)\cap N(E,x)$.

Since we suppose that $u\notin\mathcal{U}_0(K^*)$ then by definition of 0-extreme normal vector, u can be written as $u=u_1+u_2$ with $u_1,u_2\neq u$ linearly independent normal vectors at the same boundary point $x\in\operatorname{bd}K^*$, which implies that $\dim N(K^*,x)\geq 2$. Notice however that $\dim N(E,x)=1$ since $u\in\mathcal{U}_0(E)$ (by the regularity of E, $\mathcal{U}_0(E)=\mathbb{S}^{n-1}$ and then $u\in\mathcal{U}_0(E)$). On the other hand it is clear that $\dim N(E,x)\geq\dim N(K^*,x)$ because $E\subset K^*$. Hence we get $\dim N(E,x)\geq 2$, a contradiction. It shows that $u\in\mathcal{U}_0(K^*)$. □

Remark 3.8. Notice that there are examples for which $\mathcal{U}_0(K^*)\subset\operatorname{cl}\mathcal{U}_0(K)$ strictly if E is not a regular convex body. In fact, taking E as the unit square in the plane and $K=B^2$ then $(B^2)^*=E$ (see Remark 3.6), and we get $\mathcal{U}_0(B^2)=\mathbb{S}^1$ whereas $\mathcal{U}_0\big((B^2)^*\big)=\mathcal{U}_0(E)=\big\{\pm(1,0),\pm(0,1)\big\}$. ♢

Remark 3.9. Using an analogous argument as in the proof of Lemma 3.1.12 it is shown that any $u\in\mathbb{S}^{n-1}$ such that $h(K^*,u)=h(E,u)$ is a 0-extreme normal vector of K^*, for E regular. ♢

3.1.3 Some relations involving inner parallel bodies and the form body

Besides the relations between the 0-extreme normal vectors of K, K^* and the inner parallel bodies of K, in order to prove the main results in this chapter, we will need some relations between (some of) these bodies themselves. In [41, Lemma 4.7] it is proved that, for $K\in\mathcal{K}^n_0$ and for every $-\mathrm{r}(K;E)\leq\lambda\leq 0$, the following holds:

$$K_\lambda\supseteq\frac{\mathrm{r}(K_\lambda;E)}{\mathrm{r}(K;E)}K,\tag{3.11}$$

with equality if and only if K and K^* are homothetic; notice that in the limit case $\lambda=-\mathrm{r}(K;E)$ the set $0\cdot K$ is defined just as a point contained in $\ker(K;E)$. The following result provides a key relation between a convex body K, its inner parallel bodies and its form body (see [41, Lema 4.8]).

Lemma 3.1.13. *Let $K\in\mathcal{K}^n_0$. For every $-\mathrm{r}(K;E)\leq\lambda\leq 0$ it holds*

$$K_\lambda+|\lambda|\,K^*\subseteq K.\tag{3.12}$$

In [28] we characterize the convex bodies verifying the equality in (3.12). Before stating and proving this result, we make the following observation, which will be needed later.

Since quermassintegrals are particular cases of mixed volumes, as noticed in Section 1.2, and provided (1.2), the following integral expression for the quermassintegrals of $K\in\mathcal{K}^n$ holds:

$$\mathrm{W}_i(K;E)=\frac{1}{n}\int_{\mathbb{S}^{n-1}}h(K,u)\,d\mathrm{S}\big(K[n-i-1],E[i];u\big).\tag{3.13}$$

Then, it is natural to think that the differentiability of quermassintegrals is related, in a certain way, with the differentiability of the support function of $K \in \mathcal{K}^n$, both with respect to the parameter defining the full system of parallel bodies of K. The following result can be found in [41, Lemma 4.9], and provide a first approach to this relation. We will write $K_\lambda^* = (K_\lambda)^*$ to denote the form body of the inner parallel body of K at distance $|\lambda|$.

Lemma 3.1.14. *Let $K \in \mathcal{K}^n$. For every $u \in \mathbb{S}^{n-1}$, the derivative of $h(\lambda, u) := h(K_\lambda, u)$ with respect to λ exists almost everywhere for $-\mathrm{r}(K;E) < \lambda < \infty$ and*

$$\frac{d}{d\lambda} h(\lambda, u) \geq h(K_\lambda^*, u).$$

Equality holds if, for every $-\mathrm{r}(K;E) < \lambda < \infty$,

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K_\lambda + K_\lambda^*). \tag{3.14}$$

Since $E \subseteq K_\lambda^*$, it is clear (see Proposition 1.1.10) that

$$\frac{d}{d\lambda} h(\lambda, u) \geq h(E, u).$$

Moreover (see [41, p. 81]), equality holds for all $u \in \mathbb{S}^{n-1}$ if and only if $K = K_{-\mathrm{r}(K;E)} + \mathrm{r}(K;E)E$.

Remark 3.10. Notice that since $\mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_0(K)$ (see Lemma 3.1.8) then it always holds $K_\lambda^* \supseteq K^*$ for any $-\mathrm{r}(K;E) < \lambda \leq 0$. $\diamondsuit$

Now we prove the announced characterization of the equality case in (3.12).

Theorem 3.1.15 ([28]). *Let $K \in \mathcal{K}_0^n$ and let $E \in \mathcal{K}_0^n$ be regular. We write $\mathrm{r} = \mathrm{r}(K;E)$. Then $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \leq \lambda \leq 0$ if and only if K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ verifying that for all $-\mathrm{r} \leq \lambda \leq 0$,*

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*). \tag{3.15}$$

Proof. We start by assuming that $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \leq \lambda \leq 0$, in particular, that $K = K_{-\mathrm{r}} + \mathrm{r}K^*$, and we prove that K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$. In order to do that, we first show that $K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*$ is the inner parallel body of K at distance $|\lambda|$, i.e., $K_\lambda = K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*$, for any $-\mathrm{r} \leq \lambda \leq 0$.

From (3.12) and using Remark 3.10 we get that, for any convex body K and for $-\mathrm{r} < \lambda \leq 0$,

$$K_\lambda \supseteq (K_\lambda)_{-\mathrm{r}(K_\lambda;E)} + \mathrm{r}(K_\lambda;E)K_\lambda^* = K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K_\lambda^* \supseteq K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*.$$

Since when $\lambda = -\mathrm{r}$ we get trivially an identity, we obtain the inclusion $K_\lambda \supseteq K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*$ for the full interval $[-\mathrm{r}, 0]$. So it remains to be proved the reverse inclusion in the particular case when $K = K_{-\mathrm{r}} + \mathrm{r}K^*$. Notice that

$$K_\lambda = (K_{-\mathrm{r}} + \mathrm{r}K^*)_\lambda = \bigcap_{u \in \mathcal{U}_0(K)} \Big\{ x : \langle x, u \rangle \leq h(K_{-\mathrm{r}} + \mathrm{r}K^*, u) - |\lambda|\, h(E, u) \Big\} \tag{3.16}$$

(cf. (3.8)). On the other hand, by Lemma 3.1.11, $\mathcal{U}_0(K)=\mathcal{U}_0(K_{-\mathrm{r}}+\mathrm{r}K^*)=\mathcal{U}_0\bigl(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*\bigr)$ for $-\mathrm{r}<\lambda\leq 0$. Hence we can write

$$\begin{aligned} K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^* &= \bigcap_{u\in\mathcal{U}_0\bigl(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*\bigr)} \Bigl\{x:\langle x,u\rangle\leq h\bigl(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*,u\bigr)\Bigr\} \\ &= \bigcap_{u\in\mathcal{U}_0(K)} \Bigl\{x:\langle x,u\rangle\leq h\bigl(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*,u\bigr)\Bigr\} \end{aligned} \tag{3.17}$$

Thus if $x\in K_\lambda$, $-\mathrm{r}<\lambda\leq 0$, it lies in the intersection given in (3.16), and in order to show that $x\in K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*$ we have to prove that it lies also in (3.17). So for any $u\in\mathcal{U}_0(K)$ it follows

$$\begin{aligned} \langle x,u\rangle &\leq h(K_{-\mathrm{r}}+\mathrm{r}K^*,u)-|\lambda|\,h(E,u)=h(K_{-\mathrm{r}},u)+\mathrm{r}h(K^*,u)-|\lambda|\,h(K^*,u) \\ &= h(K_{-\mathrm{r}},u)+(\mathrm{r}+\lambda)h(K^*,u)=h\bigl(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*,u\bigr), \end{aligned}$$

i.e., $x\in K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*$, which shows that $K_\lambda\subseteq K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*$ for $-\mathrm{r}<\lambda\leq 0$. The case $\lambda=-\mathrm{r}$ holds trivially.

Thus we have shown that

$$K_\lambda=(K_{-\mathrm{r}}+\mathrm{r}K^*)_\lambda=K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^* \tag{3.18}$$

in the full range $-\mathrm{r}\leq\lambda\leq 0$ and on account of Lemma 3.1.11, it follows that

$$\mathcal{U}_0(K_\lambda)=\mathcal{U}_0\bigl(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*\bigr)=\mathcal{U}_0(K_{-\mathrm{r}}+\mathrm{r}K^*)=\mathcal{U}_0(K).$$

We assume now that K is not a tangential body of $(K_{-\mathrm{r}})_\mathrm{r}=K_{-\mathrm{r}}+\mathrm{r}E$. Then by the definition of tangential body (see Definition 3.1.3) there exists $u_0\in\mathcal{U}_0(K)$ such that

$$H\bigl((K_{-\mathrm{r}})_\mathrm{r},u_0\bigr)\cap H(K,u_0)=\emptyset;$$

in particular, we have that the distance, say μ, between the above two hyperplanes is strictly positive. On the other hand, since $u_0\in\mathcal{U}_0(K)=\mathcal{U}_0(K_\lambda)$ then $h(K_\lambda,u_0)=h(K,u_0)-|\lambda|\,h(E,u_0)$ for every $-\mathrm{r}\leq\lambda\leq 0$ (see Lemma 3.1.7), and hence the distance between the parallel hyperplanes $H(K,u_0)$ and $H(K_\lambda,u_0)$ is $|\lambda|\,h(E,u_0)$. Moreover, the distance between the (parallel) hyperplanes $H\bigl((K_{-\mathrm{r}})_{\mathrm{r}+\lambda},u_0\bigr)$ and $H\bigl((K_{-\mathrm{r}})_\mathrm{r},u_0\bigr)$ is also $|\lambda|\,h(E,u_0)$ since the body $(K_{-\mathrm{r}})_\mathrm{r}=K_{-\mathrm{r}}+\mathrm{r}E$ is just the outer parallel body of $(K_{-\mathrm{r}})_{\mathrm{r}+\lambda}=K_{-\mathrm{r}}+(\mathrm{r}+\lambda)E=K_{-\mathrm{r}}+\bigl(\mathrm{r}-|\lambda|\bigr)E$ at distance $|\lambda|$, for every λ. Thus, the distance between $H(K_\lambda,u_0)$ and $H\bigl((K_{-\mathrm{r}})_{\mathrm{r}+\lambda},u_0\bigr)$ is $\mu>0$, *for every* λ. But this leads to a contradiction, since when $|\lambda|\to\mathrm{r}$, the distance between the hyperplanes $H(K_\lambda,u_0)$ and $H\bigl((K_{-\mathrm{r}})_{\mathrm{r}+\lambda},u_0\bigr)$ goes to zero.

Thus we already know that K is a tangential body of $(K_{-\mathrm{r}})_\mathrm{r}$, and it remains to be proved that K verifies condition (3.15). But it is a direct consequence of Lemma 3.1.11: for all $-\mathrm{r}\leq\lambda\leq 0$,

$$\mathcal{U}_0(K)=\mathcal{U}_0\bigl(K_\lambda+|\lambda|\,K^*\bigr)=\mathcal{U}_0(K_\lambda+K^*).$$

Reciprocally, now we assume that K is a tangential body of $(K_{-\mathrm{r}})_{\mathrm{r}} = K_{-\mathrm{r}} + \mathrm{r}E$ verifying (3.15) for all $-\mathrm{r} \leq \lambda \leq 0$. Since K is a tangential body of $(K_{-\mathrm{r}})_{\mathrm{r}}$, Lemma 3.1.9 ensures that $\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K)$ for every $-\mathrm{r} < \lambda \leq 0$ and hence $K^* = K^*_\lambda$ for $-\mathrm{r} < \lambda \leq 0$. We work first in the semi-opened interval $(-\mathrm{r}, 0]$. Notice that, since E is regular, we can apply Lemma 3.1.12 to get

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*) \supseteq \mathcal{U}_0(K_\lambda) \cup \mathcal{U}_0(K^*) = \mathcal{U}_0(K) \cup \mathrm{cl}\,\mathcal{U}_0(K) = \mathrm{cl}\,\mathcal{U}_0(K) \supseteq \mathcal{U}_0(K).$$

Therefore, in particular, $\mathcal{U}_0(K)$ is closed and $\mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*)$. Thus, the above properties allow to conclude that

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*) = \mathcal{U}_0(K_\lambda + K^*_\lambda),$$

i.e., we get condition (3.14) in Lemma 3.1.14. Then it follows that for all $\lambda \in (-\mathrm{r}, 0]$ and $u \in \mathbb{S}^{n-1}$,

$$\frac{d}{d\lambda} h(\lambda, u) = h(K^*_\lambda, u) \equiv h(K^*, u).$$

Now we fix $u \in \mathbb{S}^{n-1}$ and define the function

$$f(\lambda) = h(K, u) - h(\lambda, u) + \lambda h(K^*, u),$$

which is absolutely continuous (since $h(\lambda, u)$ is concave in λ, see [21, Theorem 1.1]), almost everywhere differentiable by Lemma 3.1.14 and clearly verifies that $f'(\lambda) = 0$. Thus f is a constant function and since $f(0) = 0$ we obtain $f \equiv 0$, i.e.,

$$h(K, u) = h(\lambda, u) - \lambda h(K^*, u) = h(K_\lambda, u) + |\lambda|\, h(K^*, u) = h\big(K_\lambda + |\lambda|\, K^*, u\big),$$

for all $u \in \mathbb{S}^{n-1}$. Theorem 1.1.11 allows to conclude that $K = K_\lambda + |\lambda|\, K^*$ for every $\lambda \in (-\mathrm{r}, 0]$. It remains to be proved the result for $\lambda = -\mathrm{r}$. Condition (3.15) for $\lambda = -\mathrm{r}$ can be written as

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_{-\mathrm{r}} + K^*) = \mathcal{U}_0(K_{-\mathrm{r}} + \mathrm{r}K^*),$$

where the last equality comes again from Lemma 3.1.11. The above identity together with the fact that $K = K_\lambda + |\lambda|\, K^*$ for every $\lambda \in (-\mathrm{r}, 0]$ allow to express the sets K and $K_{-\mathrm{r}} + \mathrm{r}K^*$ in the following way:

$$\begin{aligned}
K &= \bigcap_{u \in \mathcal{U}_0(K)} \Big\{x : \langle x, u\rangle \leq h\big(K_\lambda, u\big) + |\lambda|\, h(K^*, u)\Big\}, \quad \text{for any } \lambda \in (-\mathrm{r}, 0],\\
K_{-\mathrm{r}} + \mathrm{r}K^* &= \bigcap_{u \in \mathcal{U}_0(K_{-\mathrm{r}} + \mathrm{r}K^*)} \Big\{x : \langle x, u\rangle \leq h\big(K_{-\mathrm{r}}, u\big) + \mathrm{r}h(K^*, u)\Big\}\\
&= \bigcap_{u \in \mathcal{U}_0(K)} \Big\{x : \langle x, u\rangle \leq h\big(K_{-\mathrm{r}}, u\big) + \mathrm{r}h(K^*, u)\Big\}.
\end{aligned}$$

By Lemma 3.1.13 we have $K \supseteq K_{-\mathrm{r}} + \mathrm{r}K^*$. In order to show the reverse inclusion, let $x \in K$. Then $\langle x, u\rangle \leq h\big(K_\lambda, u\big) + |\lambda|\, h(K^*, u)$ for all $u \in \mathcal{U}_0(K)$ and all $\lambda \in (-\mathrm{r}, 0]$. Taking limits when λ tends to $-\mathrm{r}$, and taking into account that the support function is continuous with respect to the Hausdorff metric, we get that $\langle x, u\rangle \leq h\big(K_{-\mathrm{r}}, u\big) + \mathrm{r}h(K^*, u)$ for all $u \in \mathcal{U}_0(K)$, i.e., $x \in K_{-\mathrm{r}} + \mathrm{r}K^*$ due to the above description of this set. It concludes the proof. □

Remark 3.11. It is enough to assume $K = K_{-\mathrm{r}} + \mathrm{r}K^*$ in the statement of Theorem 3.1.15, since it is equivalent to the condition $K = K_\lambda + |\lambda|\, K^*$ for all $\lambda \in [-\mathrm{r}, 0]$: clearly one direction is trivial; for the converse just notice that if $K = K_{-\mathrm{r}} + \mathrm{r}K^*$ then $K_\lambda = K_{-\mathrm{r}} + (\mathrm{r} + \lambda)K^*$ (see (3.18)), and hence, for all $\lambda \in [-\mathrm{r}, 0]$,

$$K_\lambda + |\lambda|\, K^* = K_{-\mathrm{r}} + \big(\mathrm{r} - |\lambda|\big)K^* + |\lambda|\, K^* = K_{-\mathrm{r}} + \mathrm{r}K^* = K.$$

We have settled the theorem in this way in order to establish the precise characterization of the equality case in (3.12). $\diamondsuit$

Remark 3.12. Condition (3.15) can not be omitted: if we write σ to denote a line segment of length $\ell \geq 2$ in $\mathbb{R}^3$ and we take a point x lying outside the solid cylinder with circular cross section of radius 1 and axis the line aff σ, the convex body K obtained as the convex hull $K = \mathrm{conv}\{\sigma + B^3, x\}$ (see Figure 3.4) verifies:

- $\ker K = \sigma$ and $\mathrm{r}(K; B^3) = 1$;
- K is a 1-tangential body of $\sigma + B^3 = K_{-1} + B^3$;
- K^* is just the convex hull of B^3 and a suitable point;
- condition (3.15) does not hold for $\lambda = -1$ (the inradius);

hence $K \neq K_{-1} + K^*$. $\diamondsuit$

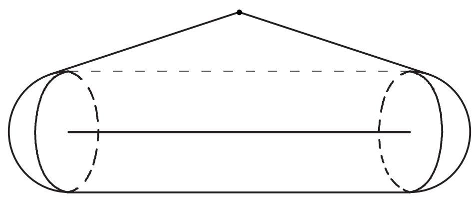

Figure 3.4: A tangential body of $K_{-\mathrm{r}} + \mathrm{r}B^3$ not verifying (3.15).

Remark 3.13. Notice that for E regular, if $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r}(K; E) < \lambda \leq 0$ then

$$\frac{d}{d\lambda} h(\lambda, u) = h(K^*_\lambda, u) = h(K^*, u)$$

for all $u \in \mathbb{S}^{n-1}$. $\diamondsuit$

3.1.4 A glance at some conjectures on Aleksandrov-Fenchel equality

Despite the increasing interest in Aleksandrov-Fenchel inequality (see Theorem 1.2.10), as well as the several new approaches to it, the problem of the complete characterization of the equality case remains still open. For a deeper study of this problem we refer to [47, 48, 49, 50]. It turns out that the study of the differentiability of the quermassintegrals of $K \in \mathcal{K}^n$ has a connection with this problem. To be more precise, the notion of r-extreme normal vector admits a generalization, which we will introduce next, and it turns out to be related with both, the equality case in Aleksandrov-Fenchel inequality and the problem of the differentiability of the quermassintegrals. In order to introduce this more general concept, we will need the notion of touching cone, which was already explained at the beginning of this chapter (see page 37).

Definition 3.1.16. *For $K_1, \dots, K_{n-1} \in \mathcal{K}^n$, a vector $u \in \mathbb{S}^{n-1}$ is a $(K_1, \dots, K_{n-1})$*-extreme normal vector *if there exist $(n-1)$-dimensional linear subspaces $H_1, \dots, H_{n-1} \subset \mathbb{R}^n$ such that $T(K_i, u) \subset H_i$ for $i = 1, \dots, n-1$ and $\dim(H_1 \cap \dots \cap H_{n-1}) = 1$.*

In particular, for $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$ regular, $u \in \mathbb{S}^{n-1}$ is an r-extreme normal vector of K if and only if u is $\big(K[n-1-r], E[r]\big)$-extreme. For $K_1, \dots, K_{n-1} \in \mathcal{K}^n$, the closure of the set of $(K_1, \dots, K_{n-1})$-extreme unit normal vectors plays a role in the classification of the equality case in Aleksandrov-Fenchel inequality (1.7). We deal with it in the following. Let $K_2, \dots, K_{n-1} \in \mathcal{K}^{n-1}$ be a fixed $(n-2)$-tuple of convex bodies and let $K, L \in \mathcal{K}^n$. In [47, Lemma 2.5] it is proved that if

$$\mathrm{V}(K, K, K_2, \dots, K_{n-1}) > 0, \quad \text{and} \quad \mathrm{V}(L, L, K_2, \dots, K_{n-1}) > 0$$

then equality holds in Aleksandrov-Fenchel inequality (1.7) if and only if the mixed surface area measures $\mathrm{S}(K, K_2, \dots, K_{n-1}; \cdot)$ and $\mathrm{S}(L, K_2, \dots, K_{n-1}; \cdot)$ are proportional. So it is natural to wonder what the equality

$$\mathrm{S}(K, K_2, \dots, K_{n-1}; \cdot) = \mathrm{S}(L, K_2, \dots, K_{n-1}; \cdot)$$

means for the convex bodies K and L. It is known (see [47, Lemma 3.4]) that if L is regular and strictly convex then, for every $K \in \mathcal{K}^n$,

$$\operatorname{supp} \mathrm{S}(K, K_2, \dots, K_{n-1}; \cdot) \subseteq \operatorname{supp} \mathrm{S}(L, K_2, \dots, K_{n-1}; \cdot).$$

Here $\operatorname{supp} \mu$ denotes the *support* of a Borel measure μ, i.e., the complement of the largest open set on which the measure vanishes. This shows, in particular, the following lemma, which can be found in [47, pp.134-135].

Lemma 3.1.17. *Let $K, L \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$ be regular and strictly convex. If $h(K, u) = h(L, u)$ for every $u \in \operatorname{supp} \mathrm{S}(E, K_2, \dots, K_{n-1}; \cdot)$, then $\mathrm{S}(K, K_2, \dots, K_{n-1}; \cdot) = \mathrm{S}(L, K_2, \dots, K_{n-1}; \cdot)$.*

The reciprocal of this statement is still an open problem, see [47, Conjecture 3.2]. On the other hand, the following conjecture (see [47, Conjecture 3.5]) establishes a relation between extreme normal vectors and mixed surface area measures. In a certain way it is related also with the equality case in the Aleksandrov-Fenchel inequality, although we will not deal with it here (for further explanations about it we refer to [47]).

Conjecture 3.1.18. *Let $K_1,\dots,K_{n-1}\in\mathcal{K}^n$. The closure of the set of $(K_1,\dots,K_{n-1})$-extreme unit normal vectors is* $\operatorname{supp}\mathrm{S}(K_1,\dots,K_{n-1};\cdot)$.

There are several cases which are known to be true (see [47, pp. 134–13]); we detail just the ones we will use in the next section of this chapter.

Proposition 3.1.19. *Conjecture 3.1.18 is true in the following cases:*

i) *if $K_1,\dots,K_{n-1}$ are polytopes;*
ii) *if $K_1=\dots=K_p$ and the bodies $K_{p+1},\dots,K_{n-1}$ are regular and strictly convex for some $p\in 0,\dots,n-1$.*

There exist some known cases in which equality in Aleksandrov-Fenchel inequality (1.7) holds. For a deep study of this topic we refer to [49].

3.2 Setting Hadwiger's problem in dimension n

The problem of studying the differentiability of the quermassintegrals of a convex body K with respect to the parameter of definition of the full system of parallel bodies of K, in the 3-dimensional case and with respect to the ball B^3, goes back to Hadwiger [23]. In Chapter 2 the development of this problem can be found, as well as some connections of it with the roots of Steiner polynomial. In this chapter we state and study the problem in dimension n, with respect to an arbitrary (fixed) $E\in\mathcal{K}_0^n$ (with some additional hypothesis in certain cases). This will lead to n classes of convex bodies which will be introduced next, in analogy with the 3 classes $\mathcal{R}_\alpha$, $\mathcal{R}_\beta$ and $\mathcal{R}_\gamma$ appearing in the classical 3-dimensional case.

We start by stating the problem. From the concavity of the full system of parallel bodies (Lemma 2.1.3), the concavity of the real functions $\mathrm{W}_i(\lambda):=\mathrm{W}_i(K_\lambda;E)$ (Theorem 2.1.4) and the polynomial expression for the quermassintegrals of the outer parallel bodies (Theorem 1.2.8), it is easy to see (see also Proposition 1.1.8) that the analogous result to Lemma 2.2.1 holds in arbitrary dimension n, namely:

Lemma 3.2.1. *For any $\mathcal{K}\in\mathcal{K}^n$, the functionals $\mathrm{W}_i(\lambda)=\mathrm{W}_i(K_\lambda;E)$ have left and right derivatives at each point $-\mathrm{r}(K;E)\leq\lambda<\infty$ and they satisfy*

$$'\mathrm{W}_i(\lambda)\geq\mathrm{W}_i'(\lambda)\geq(n-i)\mathrm{W}_{i+1}(\lambda),\tag{3.19}$$

for $i=0,\dots,n-1$.

Again it is well-known (see [6, 34]) that the volume is always differentiable and $\mathrm{V}'(\lambda)=n\mathrm{W}_1(\lambda)$ for $-\mathrm{r}(K;E)\leq\lambda<\infty$ (cf. Theorem 2.2.2); notice that in the case $\lambda=-\mathrm{r}(K;E)$ we refer to differentiability from the right. Moreover, if $\lambda\geq 0$ then all quermassintegrals are differentiable at λ

and $\mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda)$. The question arises for which convex bodies equalities hold in (3.19) for the full range $-\mathrm{r}(K;E) \leq \lambda < \infty$. With this notation we introduce the following definition.

Definition 3.2.2. *A convex body $K \in \mathcal{K}^n$ belongs to the class $\mathcal{R}_p$, $0 \leq p \leq n-1$, if for all $0 \leq i \leq p$ and for $-\mathrm{r}(K;E) \leq \lambda < \infty$ it holds*

$$'\mathrm{W}_i(\lambda) = \mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda). \tag{3.20}$$

Since $\mathrm{V}'(\lambda) = n\mathrm{W}_1(\lambda)$ the class $\mathcal{R}_0 = \mathcal{K}^n$ is the family of all convex bodies in $\mathbb{R}^n$. Moreover $\mathcal{R}_{i+1} \subset \mathcal{R}_i$, $i = 0, \dots, n-2$, and all these inclusions are strict, as will follow from Theorem 3.4.1 (see Remark 3.17).

We finish this section stating a result which establishes a lower bound for the derivative of the quermassintegrals in terms of a mixed volume involving the form body of the original body K. This result was proved in [41, Lemma 3.5], and will be needed later on.

Lemma 3.2.3. *Let $K \in \mathcal{K}^n$. For each $1 \leq i \leq n-1$, the derivative of $\mathrm{W}_i(\lambda)$ exists almost everywhere in $-\mathrm{r}(K;E) < \lambda \leq 0$ and*

$$\mathrm{W}_i'(\lambda) \geq (n-i)\mathrm{V}\big(K_\lambda[n-i-1], K_\lambda^*, E[i]\big). \tag{3.21}$$

Remark 3.14. From (3.21), the inequalities in the right-hand side of (3.19) follow immediately provided that $E \subseteq K^*$, for $i = 0, \dots, n-1$, just taking $\lambda = 0$. $\diamondsuit$

3.3 Convex bodies lying in the classes $\mathcal{R}_p$

In this section we will determine first the convex bodies belonging to the smallest class, i.e., $\mathcal{R}_{n-1}$. Then, for each of the remaining classes $\mathcal{R}_p$, $p = 1, \dots, n-2$, necessary conditions for a convex body to lie in it will be stated in terms of the support function of the form body of K_λ, the mixed area measures and the set of p-extreme normal vectors of K_λ, $\mathcal{U}_p(K_\lambda)$.

3.3.1 Characterizing $\mathcal{R}_{n-1}$

We start characterizing the smallest class, i.e., $\mathcal{R}_{n-1}$. Before stating and proving the result, we mention that in [34] it was pointed out that if E is a *summand* of K, i.e., if there exists $L \in \mathcal{K}^n$ such that $K = E + L$, then

$$\mathrm{W}_i(K_\lambda; E) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K;E)\big(-|\lambda|\big)^k, \tag{3.22}$$

for $-1 \leq \lambda \leq 0$ and $i = 0, \dots, n$. We present this result here since we will need it for the proof of the following theorem, although we will deal again with this fact in the next chapter.

Theorem 3.3.1 ([29]). *The only sets in $\mathcal{R}_{n-1}$ are the outer parallel bodies of k-dimensional convex bodies, for $0 \leq k \leq n-1$, i.e.,*

$$\mathcal{R}_{n-1} = \bigl\{K = L + \lambda E : L \in \mathcal{K}^n,\ \dim L \leq n-1,\ \lambda \geq 0\bigr\}.$$

Proof. For the sake of brevity we write $\mathrm{r} = \mathrm{r}(K;E)$. If K is a k-dimensional convex body, $k \leq n-1$, then $\mathrm{r} = 0$ and the full system of parallel bodies is reduced to the family of outer parallel sets. Hence the equalities in (3.20) trivially hold for all $i = 0, \dots, n-1$ and we have that $K \in \mathcal{R}_{n-1}$. Thus we suppose that $K \in \mathcal{K}^n_0$, which implies that $\mathrm{r} > 0$.

If $K = L + \lambda_0 E$ with $L \in \mathcal{K}^n$, $\dim L \leq n-1$, and $\lambda_0 > 0$, then clearly $\mathrm{r} = \lambda_0$ and the inner parallel body $K_\lambda = L + \bigl(\lambda_0 - |\lambda|\bigr)E$ for $-\lambda_0 \leq \lambda \leq 0$. Moreover $K_{-\mathrm{r}} = L$. Then from (3.22) we get that

$$\mathrm{W}_i(\lambda) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K;E)\bigl(-|\lambda|\bigr)^k = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K;E)\lambda^k$$

for all $i = 0, \dots, n$ and $-\lambda_0 \leq \lambda \leq 0$, and clearly all the quermassintegrals are differentiable and $\mathrm{W}'_i(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda)$, for $i = 0, \dots, n-1$. Hence $K \in \mathcal{R}_{n-1}$.

Conversely, if $K \in \mathcal{R}_{n-1}$ we have in particular that the last quermassintegral W_{n-1} is differentiable and $\mathrm{W}'_{n-1}(\lambda) = \mathrm{W}_n(K;E) = \mathrm{V}(E)$, for all $\lambda \in [-\mathrm{r}, 0]$. Then integration with respect to λ yields

$$\mathrm{W}_{n-1}(\lambda) - \mathrm{W}_{n-1}(-\mathrm{r}) = \int_{-\mathrm{r}}^{\lambda} \mathrm{W}'_{n-1}(s)\,ds = \int_{-\mathrm{r}}^{\lambda} \mathrm{V}(E)\,ds = \mathrm{V}(E)(\lambda + \mathrm{r}),$$

for all $\lambda \in [-\mathrm{r}, 0]$. In particular, when $\lambda = 0$ we get $\mathrm{W}_{n-1}(0) - \mathrm{W}_{n-1}(-\mathrm{r}) = \mathrm{r}\mathrm{V}(E)$, i.e.,

$$\mathrm{W}_{n-1}(K;E) = \mathrm{W}_{n-1}(K_{-\mathrm{r}};E) + \mathrm{r}\mathrm{W}_{n-1}(E;E) = \mathrm{W}_{n-1}(K_{-\mathrm{r}} + \mathrm{r}E;E), \tag{3.23}$$

where the last equality follows from the linearity of $\mathrm{W}_{n-1}(K;E) = \mathrm{V}\bigl(K, E[n-1]\bigr)$ in its first variable (see Proposition 1.2.5, part v)). Since $K_{-\mathrm{r}} + \mathrm{r}E \subset K$ we get from (3.23) that $K_{-\mathrm{r}} + \mathrm{r}E = K$, which proves the required statement. □

In the case of the original Hadwiger problem, i.e., when $n = 3$ and $E = B^3$, we obtain the characterization of the so called class $\mathcal{R}_\gamma$ (see Theorem 2.3.1).

As a direct consequence of Theorem 3.3.1 and the remark after Lemma 3.1.14 the following corollary can be stated:

Corollary 3.3.2. *A convex body $K \in \mathcal{K}^n$ lies in $\mathcal{R}_{n-1}$ if and only if the derivative*

$$\frac{d}{d\lambda} h(\lambda, u) \equiv h(E, u)$$

almost everywhere, for all $u \in \mathbb{S}^{n-1}$.

3.3.2 Necessary conditions for the remaining classes $\mathcal{R}_p$

Throughout all this subsection $E \in \mathcal{K}_0^n$ will be a regular and strictly convex body. With this assumption we state necessary conditions for $K \in \mathcal{K}^n$ to lie in the class $\mathcal{R}_p$, $0 \leq p \leq n-2$. The main theorem in this part will be the following one:

Theorem 3.3.3 ([29]). *Let $K \in \mathcal{K}^n$ and let $E \in \mathcal{K}_0^n$ be a regular and strictly convex body. If $K \in \mathcal{R}_p \backslash \mathcal{R}_{n-1}$, $0 \leq p \leq n-2$, then for all $\lambda \in \bigl(-\mathrm{r}(K;E), 0\bigr]$ the following holds:*

i.a) $h(K_\lambda^*, u) = h(E, u)$ *for all* $u \in \operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr)$ *and* $i = 0, \dots, p$.

i.b) $h(K_\lambda^*, u) = h(E, u)$ *for all* $u \in \operatorname{cl} \mathcal{U}_p(K_\lambda)$.

ii) *If* $p \neq 0$, *then* $\mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr) = \mathrm{S}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i-1]; \cdot\bigr)$ *for* $i = 1, \dots, p$.

iii) $\operatorname{supp} \mathrm{S}\bigl(K_\lambda^*[n-1]; \cdot\bigr) \cup \Bigl(\bigcup_{i=0}^p \operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr)\Bigr) \subseteq \operatorname{cl} \mathcal{U}_0(K_\lambda)$.

iv) $\operatorname{cl} \mathcal{U}_0(K_\lambda) = \operatorname{cl} \mathcal{U}_1(K_\lambda) = \cdots = \operatorname{cl} \mathcal{U}_p(K_\lambda)$.

For the proof of Theorem 3.3.3 we need the following lemma, in which it is proved that all the above conditions are equivalent for any convex body $K \in \mathcal{K}_0^n$.

Lemma 3.3.4 ([29]). *Let $K \in \mathcal{K}_0^n$ and let $E \in \mathcal{K}_0^n$ be a regular and strictly convex body. For $0 \leq p \leq n-2$, the following conditions are equivalent:*

i.a) $h(K_\lambda^*, u) = h(E, u)$ *for all* $u \in \operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr)$ *and* $i = 0, \dots, p$.

i.b) $h(K_\lambda^*, u) = h(E, u)$ *for all* $u \in \operatorname{cl} \mathcal{U}_p(K_\lambda)$.

ii) *If* $p \neq 0$, *then* $\mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr) = \mathrm{S}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i-1]; \cdot\bigr)$ *for* $i = 1, \dots, p$.

iii) $\operatorname{supp} \mathrm{S}\bigl(K_\lambda^*[n-1]; \cdot\bigr) \cup \Bigl(\bigcup_{i=0}^p \operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr)\Bigr) \subseteq \operatorname{cl} \mathcal{U}_0(K_\lambda)$.

iv) $\operatorname{cl} \mathcal{U}_0(K_\lambda) = \operatorname{cl} \mathcal{U}_1(K_\lambda) = \cdots = \operatorname{cl} \mathcal{U}_p(K_\lambda)$.

Proof. Property (i.b) is just a reformulation of (i.a). In fact, since E is regular and strictly convex, $\operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr)$ is the closure of the set of $\bigl(K_\lambda[n-i-1], E[i]\bigr)$-extreme normal vectors (see Proposition 3.1.19), which is the set $\operatorname{cl} \mathcal{U}_i(K_\lambda)$ since $u \in \mathbb{S}^{n-1}$ is an i-extreme normal vector of K_λ if and only if u is $\bigl(K_\lambda[n-1-i], E[i]\bigr)$-extreme (see page 49); i.e.,

$$\operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr) = \operatorname{cl} \mathcal{U}_i(K_\lambda), \tag{3.24}$$

$i = 0, \dots, p$. Since $\mathcal{U}_i(K_\lambda) \subseteq \mathcal{U}_p(K_\lambda)$ for all $i = 0, \dots, p-1$ we get the equivalence between properties (i.a) and (i.b).

Now we prove that (i.a) is equivalent to (ii). Since E is regular and strictly convex and $i \geq 1$, Lemma 3.1.17 shows that (i.a) implies that $\mathrm{S}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i-1]; \cdot\bigr) = \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr)$. This proves property (ii).

And conversely, we now assume that for all $i = 1, \dots, p$, the mixed surface area measures $\mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr) = \mathrm{S}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i-1]; \cdot\bigr)$. Then using the formula for the mixed volumes given in (1.2) we get

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} h(K_\lambda^*, u)\, d\mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; u\bigr) &= n\mathrm{V}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i]\bigr)\\ &= \int_{\mathbb{S}^{n-1}} h(E, u)\, d\mathrm{S}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i-1]; u\bigr)\\ &= \int_{\mathbb{S}^{n-1}} h(E, u)\, d\mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; u\bigr)\end{aligned}$$

and therefore

$$\int_{\mathbb{S}^{n-1}} \bigl[h(K_\lambda^*, u) - h(E, u)\bigr]\, d\mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; u\bigr) = 0. \tag{3.25}$$

Since $E \subseteq K_\lambda^*$ and hence $h(K_\lambda^*, u) \geq h(E, u)$ (see Proposition 1.1.10), we get that (3.25) is equivalent to $h(K_\lambda^*, u) = h(E, u)$ for all $u \in \operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr)$ and $i = 1, \dots, p$. In order to show (i.a) for $i = 0$ notice that $\operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-2], E; \cdot\bigr) = \operatorname{cl} \mathcal{U}_1(K_\lambda) \supseteq \operatorname{cl} \mathcal{U}_0(K_\lambda) = \operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-1]; \cdot\bigr)$. Hence we also obtain $h(K_\lambda^*, u) = h(E, u)$ for all $u \in \operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-1]; \cdot\bigr)$. This proves (i.a).

Now we show that (i.a) implies (iii). By Lemma 3.2.3 in the particular case when $i = 0$ we get

$$\mathrm{V}'(\lambda) \geq n\mathrm{V}\bigl(K_\lambda[n-1], K_\lambda^*\bigr) \geq n\mathrm{V}\bigl(K_\lambda[n-1], E\bigr), \tag{3.26}$$

where the second inequality follows from the monotonicity of the mixed volumes (Proposition 1.2.5, part vi)) and $E \subseteq K_\lambda^*$. Since the volume is differentiable and $\mathrm{V}'(\lambda) = n\mathrm{W}_1(\lambda) = n\mathrm{V}\bigl(K_\lambda[n-1], E\bigr)$, we have equalities in (3.26), and hence we can assure that any convex body $K \in \mathcal{K}_0^n$ satisfies

$$\mathrm{V}\bigl(K[n-1], E\bigr) = \mathrm{V}\bigl(K[n-1], K^*\bigr).$$

In particular, the above relation applied to the form body K_λ^* ensures that $\mathrm{V}\bigl(K_\lambda^*[n-1], E\bigr) = \mathrm{V}(K_\lambda^*)$. Using again the formula for the mixed volumes given by (1.2) we get that for any $K \in \mathcal{K}_0^n$ it holds that $h(K_\lambda^*, u) = h(E, u)$ for all $u \in \operatorname{supp} \mathrm{S}\bigl(K_\lambda^*[n-1]; \cdot\bigr)$. This condition joined to (i.a) gives

$$\begin{aligned} h(K_\lambda^*, u) = h(E, u) \quad &\text{for all}\\ &u \in \operatorname{supp} \mathrm{S}\bigl(K_\lambda^*[n-1]; \cdot\bigr) \cup \left(\bigcup_{i=0}^{p} \operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr)\right). \end{aligned} \tag{3.27}$$

On the other hand it is clear that (see Remark 3.9) since E is regular then $h(K_\lambda^*, u) = h(E, u)$ if and only if $u \in \mathcal{U}_0(K_\lambda^*)$. Moreover, by using Lemma 3.1.12 we know that $\mathcal{U}_0(K_\lambda^*) = \operatorname{cl} \mathcal{U}_0(K_\lambda)$ and hence we have $h(K_\lambda^*, u) = h(E, u)$ if and only if $u \in \operatorname{cl} \mathcal{U}_0(K_\lambda)$. From here using (3.27) we get the required property (iii).

In order to prove (iii) implies (iv) notice that $\operatorname{supp} \mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr) = \operatorname{cl} \mathcal{U}_i(K_\lambda)$, for all $i = 0, \dots, p$ (cf. (3.24)), since E is regular and strictly convex. Hence we get from (iii) that, in particular, $\operatorname{cl} \mathcal{U}_p(K_\lambda) \subseteq \operatorname{cl} \mathcal{U}_0(K_\lambda)$. Since it always holds $\mathcal{U}_0(K_\lambda) \subseteq \cdots \subseteq \mathcal{U}_p(K_\lambda)$, we obtain (iv).

It remains to be shown that (iv) implies (i.a). Using again the identity given in (3.24), i.e., that $\operatorname{supp}\mathrm{S}\bigl(K_\lambda[n-i-1],E[i];\cdot\bigr)=\operatorname{cl}\mathcal{U}_i(K_\lambda)$ for $i=0,\dots,p$, we get from (iv) that for all $i=1,\dots,p$

$$\operatorname{supp}\mathrm{S}\bigl(K_\lambda[n-i-1],E[i];\cdot\bigr)=\operatorname{supp}\mathrm{S}\bigl(K_\lambda[n-1];\cdot\bigr).$$

On the other hand, since E is regular and using Lemma 3.1.12, we know that $h(K_\lambda^*,u)=h(E,u)$ if and only if $u\in\mathcal{U}_0(K_\lambda^*)=\operatorname{cl}\mathcal{U}_0(K_\lambda)$. Thus if $u\in\operatorname{supp}\mathrm{S}\bigl(K_\lambda[n-i-1],E[i];\cdot\bigr)$ for $i\in\{0,\dots,p\}$, then $u\in\operatorname{supp}\mathrm{S}\bigl(K_\lambda[n-1];\cdot\bigr)=\operatorname{cl}\mathcal{U}_0(K_\lambda)$ (cf. (3.24)), which implies that $h(K_\lambda^*,u)=h(E,u)$ and shows (i.a). This concludes the proof of the lemma. □

Now we prove Theorem 3.3.3 by showing that $K\in\mathcal{R}_p$ implies property (i.a). From Lemma 3.3.4 we get the remaining statements.

Proof of Theorem 3.3.3. First notice that by hypothesis $K\notin\mathcal{R}_{n-1}$. Since all k-dimensional convex bodies, $k\le n-1$, are contained in $\mathcal{R}_{n-1}$, we have $K\in\mathcal{K}_0^n$. Hence we can apply Lemma 3.3.4 and prove just property (i.a).

Since $E\subseteq K_\lambda^*$, the monotonicity of the mixed volumes (Proposition 1.2.5, part vi)) implies that

$$\mathrm{V}\bigl(K_\lambda[n-p-1],K_\lambda^*,E[p]\bigr)\ge \mathrm{W}_{p+1}(\lambda) \tag{3.28}$$

and hence we get from (3.21) that

$$\mathrm{W}_p'(\lambda)\ge(n-p)\mathrm{W}_{p+1}(\lambda).$$

Thus, if $K\in\mathcal{R}_p$ then we have equality in the previous inequality and also in (3.28). Moreover since $K\in\mathcal{R}_p\subset\cdots\subset\mathcal{R}_0$ we get $\mathrm{V}\bigl(K_\lambda[n-i-1],K_\lambda^*,E[i]\bigr)=\mathrm{W}_{i+1}(\lambda)$ for all $i=0,\dots,p$. Using the formula for the mixed volumes given by (1.2) we can write

$$\begin{aligned}0&=n\mathrm{V}\bigl(K_\lambda[n-i-1],K_\lambda^*,E[i]\bigr)-n\mathrm{V}\bigl(K_\lambda[n-i-1],E[i+1]\bigr)\\&=\int_{\mathbb{S}^{n-1}}\bigl[h(K_\lambda^*,u)-h(E,u)\bigr]\,d\mathrm{S}\bigl(K_\lambda[n-i-1],E[i];u\bigr),\end{aligned}$$

which is equivalent to $h(K_\lambda^*,u)=h(E,u)$ for all $u\in\operatorname{supp}\mathrm{S}\bigl(K_\lambda[n-i-1],E[i];\cdot\bigr)$, for $i=0,\dots,p$. This proves (i.a) and the theorem. □

Remark 3.15. If $K\in\mathcal{R}_p\backslash\mathcal{R}_{n-1}$, $1\le p\le n-2$, then property (ii) of Theorem 3.3.3 ensures that the mixed surface area measures $\mathrm{S}\bigl(K_\lambda[n-i-1],E[i];\cdot\bigr)$ and $\mathrm{S}\bigl(K_\lambda^*,K_\lambda[n-i-1],E[i-1];\cdot\bigr)$ coincide. Hence we can rewrite (i.a) as

$$h(K_\lambda^*,u)=h(E,u)\quad\text{ for }\ u\in\operatorname{supp}\mathrm{S}\bigl(K_\lambda^*,K_\lambda[n-i-1],E[i-1];\cdot\bigr)$$

for $i=1,\dots,p$. We also can write that

$$\begin{aligned}\operatorname{cl}\mathcal{U}_i(K_\lambda)&=\operatorname{cl}\Bigl\{\bigl(K_\lambda[n-i-1],E[i]\bigr)\text{-extreme normal vectors}\Bigr\}\\&=\operatorname{supp}\mathrm{S}\bigl(K_\lambda[n-i-1],E[i];\cdot\bigr)=\operatorname{supp}\mathrm{S}\bigl(K_\lambda^*,K_\lambda[n-i-1],E[i-1];\cdot\bigr).\end{aligned}$$

They provide new conditions for a convex body to belong to the class $\mathcal{R}_p$, $1\le p\le n-2$. ◇

Using (1.2) the following corollary is an immediate consequence of ii) in Theorem 3.3.3.

Corollary 3.3.5 ([29]). *If $K \in \mathcal{R}_p \backslash \mathcal{R}_{n-1}$, $1 \le p \le n-2$, then for any convex body $L \in \mathcal{K}^n$ and for all $\lambda \in \bigl(-\mathrm{r}(K;E), 0\bigr]$ it holds*

$$\mathrm{V}\bigl(L, K_\lambda^*, K_\lambda[n-i-1], E[i-1]\bigr) = \mathrm{V}\bigl(L, K_\lambda[n-i-1], E[i]\bigr), \quad i = 1, \dots, p.$$

Proof. Since $K \in \mathcal{R}_p \backslash \mathcal{R}_{n-1}$, Theorem 3.3.3 part ii) ensures that, for all $i = 1, \dots, p$, it holds that $\mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr) = \mathrm{S}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i-1]; \cdot\bigr)$. Then using Formula (1.2),

$$\begin{aligned}
\mathrm{V}\bigl(L, K_\lambda^*, K_\lambda[n-i-1], E[i-1]\bigr) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L,u)\, d\mathrm{S}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i-1]; \cdot\bigr) \\
&= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L,u)\, d\mathrm{S}\bigl(K_\lambda[n-i-1], E[i]; \cdot\bigr) \\
&= \mathrm{V}\bigl(L, K_\lambda[n-i-1], E[i]\bigr).
\end{aligned}$$

□

Replacing L by K_λ, K_λ^* and E in the expression of Corollary 3.3.5 we get that the relations

$$\mathrm{W}_{i+1}(\lambda) = \mathrm{V}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i]\bigr) = \mathrm{V}\bigl(K_\lambda^*[2], K_\lambda[n-i-1], E[i-1]\bigr) \tag{3.29}$$

hold for all $i = 1, \dots, p$. In particular, we have equality in the Aleksandrov-Fenchel inequality for the convex bodies K_λ^* and E:

$$\mathrm{V}\bigl(K_\lambda^*, K_\lambda[n-i-1], E[i]\bigr)^2 = \mathrm{V}\bigl(K_\lambda^*[2], K_\lambda[n-i-1], E[i-1]\bigr)\mathrm{V}\bigl(K_\lambda[n-i-1], E[i+1]\bigr).$$

Theorem 3.3.3 allows to exclude convex sets from the classes $\mathcal{R}_p$. Thus we prove, for instance, that there are no polytopes lying in $\mathcal{R}_p$, $p = 1, \dots, n-1$, when $E \in \mathcal{K}_0^n$ is a regular and strictly convex body.

Corollary 3.3.6 ([29]). *There are no (full-dimensional) polytopes in $\mathcal{R}_p$, for all $1 \le p \le n-1$.*

Proof. Since $\mathcal{R}_{n-1} \subset \cdots \subset \mathcal{R}_1$ it is enough to show the assertion for the biggest class, $\mathcal{R}_1$. Let $P \in \mathcal{K}_0^n$ be a convex polytope lying in the class $\mathcal{R}_1$ and let $\lambda \in \bigl(-\mathrm{r}(P;E), 0\bigr]$. Theorem 3.3.3, item i.b), ensures that

$$h(P_\lambda^*, u) = h(E, u) \quad \text{for all } \ u \in \mathrm{cl}\,\mathcal{U}_1(P_\lambda). \tag{3.30}$$

On the other hand we know that P_λ is also a polytope (see Remark 3.7), and moreover P_λ^* is a polytope all whose $(n-1)$-faces touch E (see Remark 3.5). Then $h(P_\lambda^*, u) = h(E, u)$ if and only if u is a 0-extreme normal vector of P_λ^*. Hence from (3.30) we can assure that $\mathcal{U}_1(P_\lambda) \subset \mathcal{U}_0(P_\lambda^*)$.

Let $u \in \mathcal{U}_1(P_\lambda) \backslash \mathcal{U}_0(P_\lambda)$. Notice that such a vector u exists since P_λ is a polytope. By definition of 0-extreme normal vector, u can be written as $u = u_1 + u_2$ with $u_1, u_2 \neq u$ linearly independent normal vectors at the same boundary point of P. Then the 2-dimensional cone determined by u_1

and u_2 contains u in its relative interior and it provides a 1-dimensional neighborhood $\mathcal{V} \subset \mathbb{S}^{n-1}$ of the vector u. Moreover $\mathcal{V} \subset \mathcal{U}_1(P_\lambda) \subset \mathcal{U}_0(P_\lambda^*)$. This leads to a contradiction, since we have shown that there exists a relative 1-dimensional open set $\mathcal{V} \subset \mathbb{S}^{n-1}$ of 0-extreme normal vectors of the polytope P_λ^*. □

Remark 3.16. The only polytopes in $\mathcal{R}_p$, $1 \leq p \leq n-1$, have empty interior, and they lie in $\mathcal{R}_{n-1}$. Notice also that if we remove the hypothesis of regularity and strict convexity for E then Corollary 3.3.6 is not true, since trivially there are polytopes in the classes $\mathcal{R}_p$; indeed, just taking E a polytope then $E \in \mathcal{R}_{n-1}$. ♢

3.4 Tangential bodies in $\mathcal{R}_p$

This section is devoted to study the tangential bodies in connection with the problem of the differentiability of the quermassintegrals. First we determine the tangential bodies lying in each class. Then we get a new necessary condition for a convex body K to lie in $\mathcal{R}_p$, now in terms of the quermassintegrals of K and its form body.

Theorem 3.4.1 ([29]). *A tangential body $K \in \mathcal{K}^n$ of E lies in the class $\mathcal{R}_p$ if and only if K is an $(n-p-1)$-tangential body of E.*

Proof. Since K is a tangential body of E we have $\mathrm{r}(K;E) = 1$ and we know from Remark 3.2 that $K_\lambda = \big(1-|\lambda|\big)K$. Hence

$$\mathrm{W}_i(\lambda) = \big(1-|\lambda|\big)^{n-i}\mathrm{W}_i(K;E) = (1+\lambda)^{n-i}\mathrm{W}_i(K;E) \tag{3.31}$$

for all $i = 0, \ldots, n$ and then

$$\mathrm{W}_i'(\lambda) = (n-i)(1+\lambda)^{n-i-1}\mathrm{W}_i(K;E). \tag{3.32}$$

We suppose first that $K \in \mathcal{R}_p$. Then $\mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda)$, for $i = 0, \ldots, p$ and thus

$$\mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(K_\lambda;E) = (n-i)\mathrm{W}_{i+1}\big((1+\lambda)K;E\big) = (n-i)(1+\lambda)^{n-i-1}\mathrm{W}_{i+1}(K;E)$$

for $i = 0, \ldots, p$. The last two expressions for the derivative $\mathrm{W}_i'(\lambda)$ together allow to conclude that $\mathrm{W}_i(K;E) = \mathrm{W}_{i+1}(K;E)$ for all $i = 0, \ldots, p$. Then, Favard's Theorem 3.1.5 proves that K is a $(n-p-1)$-tangential body of E.

Conversely, if K is an $(n-p-1)$-tangential body of E, we have $\mathrm{W}_i(K;E) = \mathrm{W}_{i+1}(K;E)$ for $i = 0, \ldots, p$. Then we get from (3.31) and (3.32) that

$$\mathrm{W}_i'(\lambda) = (n-i)(1+\lambda)^{n-i-1}\mathrm{W}_{i+1}(K;E) = (n-i)\mathrm{W}_{i+1}(\lambda)$$

for $i = 0, \ldots, p$, which shows that $K \in \mathcal{R}_p$. □

Remark 3.17. Notice that all the inclusions $\mathcal{R}_{i+1}\subset\mathcal{R}_i$, $i=0,\dots,n-2$, are strict, as follows from Theorem 3.4.1 and the fact that there exist $(i+1)$-tangential bodies of E which are not i-tangential bodies of E; indeed, it can be easily constructed a centrally symmetric (i.e., such that $K=-K$) $(i+1)$-tangential body of E which is not an i-tangential body just taking the convex hull of E and $2(i+1)$ suitable chosen points outside E (see also the proof of Theorem 4.2.2, where a –not centrally symmetric– 2-tangential body which is not a 1-tangential body is constructed). $\diamondsuit$

We finish this section by proving the following theorem. We remark that this result was already proved for the class $\mathcal{R}_0$ in [42, Theorem 9].

Theorem 3.4.2 ([29]). *Let $K\in\mathcal{K}_0^n$ and write $\mathrm{r}=\mathrm{r}(K;E)$. If $K\in\mathcal{R}_p$, for any $0\leq p\leq n-1$, then*

$$\begin{aligned}\mathrm{W}_p(K;E)-\mathrm{W}_p(K_{-\mathrm{r}};E)\leq{}&\mathrm{W}_{p+1}(K^*;E)^{\frac{-1}{n-p-1}}\\&\left(\mathrm{W}_{p+1}(K;E)^{\frac{n-p}{n-p-1}}-\left[\mathrm{W}_{p+1}(K;E)^{\frac{1}{n-p-1}}-\mathrm{r}\mathrm{W}_{p+1}(K^*;E)^{\frac{1}{n-p-1}}\right]^{n-p}\right).\end{aligned}\tag{3.33}$$

Equality holds if and only if K is homothetic to an $(n-p-1)$-tangential body of E.

Proof. Let $\lambda\in[-\mathrm{r},0]$ with $\mathrm{r}=\mathrm{r}(K;E)$. By Lemma 3.1.13 we know that $K_\lambda+|\lambda|K^*\subseteq K$. Then for all $0\leq i\leq n$,

$$\begin{aligned}\mathrm{W}_i(K;E)^{\frac{1}{n-i}}\geq\mathrm{W}_i\big(K_\lambda+|\lambda|K^*;E\big)^{\frac{1}{n-i}}&\geq\mathrm{W}_i(K_\lambda;E)^{\frac{1}{n-i}}+|\lambda|\mathrm{W}_i(K^*;E)^{\frac{1}{n-i}}\\&=\mathrm{W}_i(\lambda)^{\frac{1}{n-i}}-\lambda\mathrm{W}_i(K^*;E)^{\frac{1}{n-i}},\end{aligned}$$

where the last inequality comes from Brunn-Minkowski's inequality for relative quermassintegrals (see Theorem 1.2.12). Since $K\in\mathcal{R}_p$ we have $\mathrm{W}_p'(\lambda)=(n-p)\mathrm{W}_{p+1}(\lambda)$, and taking $i=p+1$ in the previous inequality we can integrate from $-\mathrm{r}$ to 0 with respect to λ:

$$\begin{aligned}\frac{1}{n-p}\big[\mathrm{W}_p(0)-\mathrm{W}_p(-\mathrm{r})\big]&=\int_{-\mathrm{r}}^0\mathrm{W}_{p+1}(\lambda)\,d\lambda\\&\leq\int_{-\mathrm{r}}^0\left(\mathrm{W}_{p+1}(K;E)^{\frac{1}{n-p-1}}+\lambda\mathrm{W}_{p+1}(K^*;E)^{\frac{1}{n-p-1}}\right)^{n-p-1}\\&=\frac{1}{n-p}\left.\frac{\left(\mathrm{W}_{p+1}(K;E)^{\frac{1}{n-p-1}}+\lambda\mathrm{W}_{p+1}(K^*;E)^{\frac{1}{n-p-1}}\right)^{n-p}}{\mathrm{W}_{p+1}(K^*;E)^{\frac{1}{n-p-1}}}\right|_{-\mathrm{r}}^{0},\end{aligned}$$

from which we get directly the required inequality (3.33). Equality holds in this inequality if and only if both $K=K_\lambda+|\lambda|K^*$ and equality holds in Brunn-Minkowski's inequality for every $0\leq i\leq p+1$ and $-\mathrm{r}\leq\lambda\leq0$.

We suppose first that equality holds in (3.33). From the Brunn-Minkowski equality case we know that K and K^* are homothetic (see Theorem 1.2.12), and this is the case if and only if K is homothetic to a tangential body of E (see Remark 3.4). Since $K \in \mathcal{R}_p$ Theorem 3.4.1 ensures that then K is homothetic to an $(n-p-1)$-tangential body of E. It just remains to be proved that any homothetical copy of an $(n-p-1)$-tangential body K of E satisfies $K = K_\lambda + |\lambda| K^*$. In fact, if K is such a set then $K_\lambda = \bigl(1 - |\lambda|/\mathrm{r}\bigr)K$ (see Remark 3.2) and clearly $K^* = (1/\mathrm{r})K$. Therefore

$$K = \left(1 - \frac{|\lambda|}{\mathrm{r}}\right) K + \frac{|\lambda|}{\mathrm{r}} K = K_\lambda + |\lambda| K^*.$$

Conversely, if K is homothetic to an $(n-p-1)$-tangential body of E then we already know that $K = K_\lambda + |\lambda| K^*$ and that K, K^* are homothetic, which implies equality in Brunn-Minkowski's inequality. So we get equality in (3.33). $\square$

Chapter 4

Bounding quermassintegrals of inner parallel bodies

There is an essential *geometrically intuitive* difference between outer and inner parallel bodies of a convex body $K \in \mathcal{K}^n$. On the one hand, the difference lies in the fact that outer parallel bodies are built just by using a vectorial operation in the Euclidean space, while inner parallel bodies do not correspond to any such operation. On the other hand it is precisely this difference what makes the study of inner parallel bodies not only interesting but useful since, as we have seen in the previous chapter, it is connected with other nice problems for convex bodies. We will provide several examples of the *unruly* behavior of inner parallel bodies, for instance with respect to the boundary structure or the volume, showing so their *uncontrollable* comportment (see Figure 2.1 in Chapter 2 or Figure 4.1).

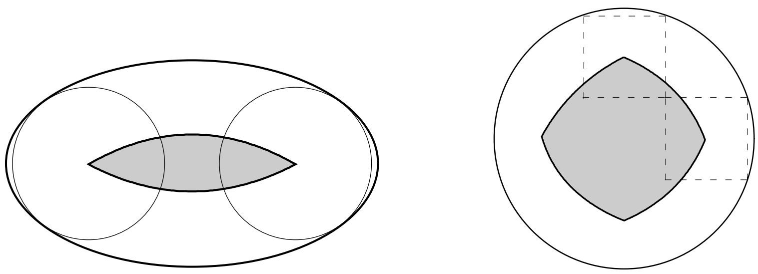

Figure 4.1: Inner parallel body of an ellipse (relative to B^2) and a circle (relative to the square).

In the previous chapters it was pointed out that both, the boundary structure and the volume of the outer parallel bodies of K can be *controlled*; however in Chapter 3 we have seen that the boundary structure of the inner parallel bodies is rather more difficult to control. Motivated by a conjecture of Matheron, in this chapter we provide bounds for the volume of the inner parallel body of a convex body K involving the alternating Steiner polynomial of K. As a consequence we get that this conjecture is not true since, in fact, we prove it is not possible to bound the volume of the inner parallel body in terms of just the alternating Steiner polynomial itself. Finally we get also upper and lower bounds for the quermassintegrals of the inner parallel body of K.

The original work we collect in this chapter can be found in [28] and in [30].

4.1 Using classes $\mathcal{R}_p$ to bound the volume of the inner parallel body

The volume of the outer parallel body of K with respect to $E \in \mathcal{K}_0^n$ at distance $\lambda \geq 0$, K_λ, is a polynomial of degree n in λ, as already noticed in Chapter 1. Steiner formulae for quermassintegrals (1.4) provide analogous expressions for the relative quermassintegrals of K_λ, $\lambda \geq 0$. There is however no explicit formula for the volume nor for the quermassintegrals of the inner parallel bodies of a convex body K. This leads, in a natural way, to consider the problem of studying whether it is possible to give lower/upper bounds for the volume of the inner parallel bodies of $K \in \mathcal{K}^n$ in terms of the quermassintegrals of the original body.

We recall (see Section 3.3) that when E is a summand of K then

$$\mathrm{W}_i(K_{-\lambda}; E) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K; E)(-\lambda)^k, \tag{4.1}$$

for $0 \leq \lambda \leq 1$ and $i = 0, \ldots, n$. In fact, Matheron proved in [34] that for $0 \leq \lambda \leq 1$ and $i = 0, \ldots, n$ identity (4.1) holds if and only if E is a summand of K, and he conjectured that it was enough to assume (4.1) just for $i = 0$, i.e., the case of the volume, and even more:

Conjecture 4.1.1 (Matheron, [34]). *Let $K \in \mathcal{K}^n$. Then for $0 \leq \lambda \leq 1$,*

$$\mathrm{V}(K_{-\lambda}) \geq \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)(-\lambda)^i. \tag{4.2}$$

The equality holds if and only if E is a summand of K.

The right-hand side in (4.2) is usually called the *alternating Steiner polynomial* of K. Matheron proved Conjecture 4.1.1 for $n = 2$.

We will see that the classes $\mathcal{R}_p$ of convex bodies studied in Chapter 3 allow to give bounds (upper or lower, depending on the class) for the volume of the inner parallel bodies involving the alternating Steiner polynomial. As a consequence of these bounds we prove that there exist many sets for which inequality (4.2) does not hold, proving so the non-validity of Matheron's conjecture.

We remark that the non-validity of the conjecture in the 3-dimensional space and for $E = B^3$ was already mentioned in [43].

In order to prove the main result in this section we need the following lemma which provides an upper bound for the (left) derivative of the i-th quermassintegral. From now on, $E \in \mathcal{K}_0^n$ will be again a fixed convex body with interior points and we will write, following the notation in the previous chapter, $\mathrm{W}_i(\lambda) = \mathrm{W}_i(K_\lambda; E)$ for $\lambda \leq 0$ and $\mathrm{r} = \mathrm{r}(K;E)$ for the sake of brevity, unless it is not clear from context.

Lemma 4.1.2 ([30]). *Let $K \in \mathcal{K}_0^n$ and $-\mathrm{r} < \lambda \leq 0$. Then for $i = 0, \dots, n$,*

$$'\mathrm{W}_i(\lambda) \leq \frac{n-i}{\mathrm{r}+\lambda}\,\mathrm{W}_i(\lambda).$$

Equality holds if and only if K is homothetic to a tangential body of E.

Proof. The inradius of K_λ, $-\mathrm{r} \leq \lambda \leq 0$, is given by $\mathrm{r}(K_\lambda; E) = \mathrm{r} - |\lambda| = \mathrm{r} + \lambda$ (see (2.1)). Then, by (3.11) it holds that

$$\frac{\mathrm{r}+\lambda}{\mathrm{r}}\,K \subseteq K_\lambda. \tag{4.3}$$

Let now $t \in [0, \mathrm{r})$ such that $-\mathrm{r} < \lambda - t \leq \lambda \leq 0$, using again (3.11) we have

$$\frac{\mathrm{r}+\lambda-t}{\mathrm{r}+\lambda}\,K_\lambda = \frac{\mathrm{r}(K_{\lambda-t};E)}{\mathrm{r}(K_\lambda;E)}\,K_\lambda \subseteq K_{\lambda-t}$$

and therefore

$$\left(1 - \frac{t}{\mathrm{r}+\lambda}\right)^{n-i} \mathrm{W}_i(\lambda) \leq \mathrm{W}_i(\lambda - t).$$

Using this inequality we can bound the (left) derivative of W_i:

$$'\mathrm{W}_i(\lambda) = \lim_{t\to 0} \frac{\mathrm{W}_i(\lambda) - \mathrm{W}_i(\lambda - t)}{t} \leq \lim_{t\to 0} \frac{\left[1 - \left(1 - \frac{t}{\mathrm{r}+\lambda}\right)^{n-i}\right]\mathrm{W}_i(\lambda)}{t} = \frac{n-i}{\mathrm{r}+\lambda}\,\mathrm{W}_i(\lambda),$$

which shows the required inequality. In order to prove the equality case, we suppose first that $'\mathrm{W}_i(\lambda)(\mathrm{r}+\lambda) = (n-i)\mathrm{W}_i(\lambda)$, for $i \in \{0, \dots, n\}$. Since $K \in \mathcal{K}_0^n$, $\mathrm{W}_i(\lambda) > 0$ and we can write

$$\int_\lambda^0 \frac{'\mathrm{W}_i(t)}{\mathrm{W}_i(t)}\,dt = \int_\lambda^0 \frac{n-i}{\mathrm{r}+t}\,dt.$$

Hence, $\log \mathrm{W}_i(0) - \log \mathrm{W}_i(\lambda) = (n-i)\big[\log \mathrm{r} - \log(\mathrm{r}+\lambda)\big]$, from which we obtain

$$\frac{\mathrm{W}_i(K;E)}{\mathrm{W}_i(K_\lambda;E)} = \frac{\mathrm{W}_i(0)}{\mathrm{W}_i(\lambda)} = \left(\frac{\mathrm{r}}{\mathrm{r}+\lambda}\right)^{n-i},$$

or equivalently,

$$\mathrm{W}_i(K_\lambda;E) = \mathrm{W}_i\left(\frac{\mathrm{r}+\lambda}{\mathrm{r}}\,K;E\right). \tag{4.4}$$

From (4.3) and (4.4) we get an equality,

$$K_\lambda = \frac{\mathrm{r}+\lambda}{\mathrm{r}} K$$

for all $\lambda \in (-\mathrm{r}, 0]$. Then Theorem 3.1.4 ensures that K is homothetic to a tangential body of E.

Conversely, if K is homothetic to a tangential body of E, we know from Remark 3.2 that $K_\lambda = (1+\lambda/\mathrm{r})K$. Hence $\mathrm{W}_i(\lambda) = (1+\lambda/\mathrm{r})^{n-i}\mathrm{W}_i(K;E)$ for all $i = 0, \dots, n$ and then

$$'\mathrm{W}_i(\lambda) = \mathrm{W}_i'(\lambda) = (n-i)\frac{(\mathrm{r}+\lambda)^{n-i-1}}{\mathrm{r}^{n-i}}\mathrm{W}_i(K;E) = \frac{n-i}{\mathrm{r}+\lambda}\left(\frac{\mathrm{r}+\lambda}{\mathrm{r}}\right)^{n-i}\mathrm{W}_i(K;E) = \frac{n-i}{\mathrm{r}+\lambda}\mathrm{W}_i(\lambda).$$

It concludes the proof of the lemma. □

Thus (cf. (3.19)) we have obtained an upper bound of the derivative of W_i in terms of W_i. Throughout the rest of this section λ will always be non-negative, $\lambda \geq 0$. Now we prove the main result in this part.

Theorem 4.1.3 ([30]). *Let $K \in \mathcal{K}^n$ be a convex body lying in the class $\mathcal{R}_p$, $0 \leq p \leq n-1$. For every $0 \leq \lambda < \mathrm{r}$ it holds:*

i) If $p = n-1$ then

$$\mathrm{V}(K_{-\lambda}) = \sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)(-\lambda)^i. \tag{4.5}$$

ii) If p is even, $0 \leq p \leq n-2$ then

$$\mathrm{V}(K_{-\lambda}) \geq \sum_{i=0}^{p+2}\binom{n}{i}\mathrm{W}_i(K;E)(-\lambda)^i - \binom{n}{p+2}(n-p-2)\int_0^\lambda \frac{(\lambda-s)^{p+2}}{\mathrm{r}-s}\mathrm{W}_{p+2}(-s)\,ds. \tag{4.6}$$

iii) If p is odd, $1 \leq p \leq n-2$ then

$$\mathrm{V}(K_{-\lambda}) \leq \sum_{i=0}^{p+2}\binom{n}{i}\mathrm{W}_i(K;E)(-\lambda)^i + \binom{n}{p+2}(n-p-2)\int_0^\lambda \frac{(\lambda-s)^{p+2}}{\mathrm{r}-s}\mathrm{W}_{p+2}(-s)\,ds. \tag{4.7}$$

Equality holds in both inequalities if and only if K is homothetic to an $(n-p-2)$-tangential body of the convex body E.

This theorem is a direct consequence of the following more general result.

Theorem 4.1.4 ([30]). *Let $K \in \mathcal{K}^n_0$ be a convex body lying in the class $\mathcal{R}_p$, $0 \leq p \leq n-2$. For every $0 \leq t \leq \lambda < \mathrm{r}$ it holds:*

i) If p is even, $0 \leq p \leq n-2$ then

$$\mathrm{V}(K_{-\lambda}) \geq \sum_{i=0}^{p+2}\binom{n}{i}\mathrm{W}_i(-t)(t-\lambda)^i - \binom{n}{p+2}(n-p-2)\int_t^\lambda \frac{(\lambda-s)^{p+2}}{\mathrm{r}-s}\mathrm{W}_{p+2}(-s)\,ds.$$

ii) If p is odd, $1 \le p \le n-2$ then

$$\mathrm{V}(K_{-\lambda}) \le \sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(-t)(t-\lambda)^i + \binom{n}{p+2}(n-p-2)\int_t^\lambda \frac{(\lambda-s)^{p+2}}{\mathrm{r}-s}\mathrm{W}_{p+2}(-s)\,ds.$$

Equality holds in both inequalities if and only if K is homothetic to an $(n-p-2)$-tangential body of the convex body E.

Notice that (4.6) and (4.7) in Theorem 4.1.3 are obtained by replacing $t=0$ in Theorem 4.1.4, and equality (4.5) is a direct consequence of Theorem 3.3.1. Notice also that if $\dim K \le n-1$ then $\mathrm{r} = \mathrm{r}(K;E) = 0$ and hence the result in Theorem 4.1.3 is trivial.

Proof of Theorem 4.1.4. We fix $0 \le \lambda < \mathrm{r}$. For $0 \le t \le \lambda$ we define the function

$$F(t) = \mathrm{V}(-\lambda) - \sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(-t)(t-\lambda)^i + (-1)^p\binom{n}{p+2}(n-p-2)\int_t^\lambda \frac{(\lambda-s)^{p+2}}{\mathrm{r}-s}\mathrm{W}_{p+2}(-s)\,ds.$$

Since $K \in \mathcal{R}_p$, by definition W_i is differentiable and $\mathrm{W}_i'(s) = (n-i)\mathrm{W}_{i+1}(s)$, for $i = 0, \dots, p$. Then it is an easy computation to check that the first derivative of F is

$$\begin{aligned} F'(t) = (-1)^p\binom{n}{p+1}(\lambda-t)^{p+1} \Bigg[&(n-p-1)\mathrm{W}_{p+2}(-t) - \mathrm{W}_{p+1}'(-t) \\ &+ \frac{n-p-1}{p+2}(\lambda-t)\left(\mathrm{W}_{p+2}'(-t) - \frac{n-p-2}{\mathrm{r}-t}\mathrm{W}_{p+2}(-t)\right)\Bigg]. \end{aligned}$$

From (3.19) we know that

$$(n-p-1)\mathrm{W}_{p+2}(-t) - \mathrm{W}_{p+1}'(-t) \le 0. \tag{4.8}$$

Since $\mathrm{W}_{p+2}'(-t) \le {}'\mathrm{W}_{p+2}(-t)$, we can apply Lemma 4.1.2 which ensures that

$$\mathrm{W}_{p+2}'(-t) - \frac{n-p-2}{\mathrm{r}-t}\mathrm{W}_{p+2}(-t) \le 0. \tag{4.9}$$

Thus if p is even then $F'(t) \le 0$, whereas for p odd we have $F'(t) \ge 0$. Since clearly $F(\lambda) = 0$ and $t \le \lambda$, we conclude that:

- for p even it holds $F(t) \ge F(\lambda) = 0$, which proves i);
- for p odd it holds $F(t) \le F(\lambda) = 0$, which proves ii).

Now we deal with the equality case. We have to show that $F(t)$ is identically zero for every fixed $0 \le \lambda < \mathrm{r}$ if and only if K is homothetic to an $(n-p-2)$-tangential body of E. If $F(t) \equiv 0$ then equality must hold in (4.8) and (4.9) for all t and λ, $0 \le t \le \lambda < \mathrm{r}$. Since $K \in \mathcal{R}_p$, by definition equality holds in (4.8) if and only if $K \in \mathcal{R}_{p+1}$. On the other hand, by Lemma 4.1.2 equality holds in (4.9) if and only if K is homothetic to a tangential body of E. Since, by Theorem 3.4.1, the only tangential bodies in $\mathcal{R}_{p+1}$ are the $(n-p-2)$-tangential bodies, we get the result.

Conversely, if K is homothetic to an $(n-p-2)$-tangential body of E then by Lemma 4.1.2 equality holds in (4.9). On the other hand, we have again that $K_{-t} = (1-t/\mathrm{r})K$, which implies $\mathrm{W}_{p+1}(-t) = (1-t/\mathrm{r})^{n-p-1}\mathrm{W}_{p+1}(K;E)$. Hence

$$\begin{aligned}\mathrm{W}'_{p+1}(-t) &= (n-p-1)\frac{1}{\mathrm{r}}\left(1-\frac{t}{\mathrm{r}}\right)^{n-p-2}\mathrm{W}_{p+1}(K;E)\\ &= (n-p-1)\left(1-\frac{t}{\mathrm{r}}\right)^{n-p-2}\mathrm{W}_{p+2}(K;E) = (n-p-1)\mathrm{W}_{p+2}(-t)\end{aligned}$$

since if K is homothetic to an $(n-p-2)$-tangential body of E, then the homothecy factor is $\mathrm{r} = \mathrm{r}(K;E)$ and $\mathrm{W}_{p+1}(K;E) = \mathrm{r}\mathrm{W}_{p+2}(K;E)$ (see Theorem 3.1.5 and the remark afterwards). It concludes the equality case and the proof of the theorem. □

Notice that Theorem 4.1.3 provides both upper and lower bounds for the volume of the inner parallel body of a convex body K lying in the class $\mathcal{R}_p$, $p = 1,\dots,n-2$. Since $K \in \mathcal{R}_p \subset \mathcal{R}_{p-1}$, if p is even, and hence $p-1$ is odd, then

$$\begin{aligned}&\sum_{i=0}^{p+2}\binom{n}{i}\mathrm{W}_i(K;E)(-\lambda)^i - \binom{n}{p+2}(n-p-2)\int_0^\lambda \frac{(\lambda-s)^{p+2}}{\mathrm{r}-s}\mathrm{W}_{p+2}(-s)\,ds\\ &\qquad \le V(-\lambda) \le\\ &\sum_{i=0}^{p+1}\binom{n}{i}\mathrm{W}_i(K;E)(-\lambda)^i + \binom{n}{p+1}(n-p-1)\int_0^\lambda \frac{(\lambda-s)^{p+1}}{\mathrm{r}-s}\mathrm{W}_{p+1}(-s)\,ds\end{aligned}$$

and similarly for the case when p is odd.

As a consequence of this theorem we get the following corollary, which proves the non-validity of Matheron's conjecture by showing that there are many sets not verifying inequality (4.2).

Corollary 4.1.5 ([30]). *Let $K \in \mathcal{K}^n$, n odd, be a convex body lying in $\mathcal{R}_{n-2}$. Then for $0 \le \lambda < \mathrm{r}$,*

$$\mathrm{V}(K_{-\lambda}) \le \sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)(-\lambda)^i. \tag{4.10}$$

Equality holds if and only if $K \in \mathcal{R}_{n-1}$.

Corollary 4.1.5 is a direct consequence of Theorem 4.1.3, since if the dimension n is odd then $p = n-2$ is so, and hence we get (4.10) from (4.7).

Remark 4.1. Notice that the equality case in Corollary 4.1.5 supports the second conjectured property in Conjecture 4.1.1; namely that the volume of the inner parallel body $K_{-\lambda}$ verifies that $\mathrm{V}(K_{-\lambda}) = \sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)(-\lambda)^i$ if and only if E is a summand of K (cf. Theorem 3.3.1). ♢

The continuity of the functionals involved in Theorem 4.1.4 allows to assure that this result as well as Theorem 4.1.3 are true for the limit case when $\lambda = \mathrm{r}$. Thus we get the following corollary.

Corollary 4.1.6 ([30]). *Let $K \in \mathcal{K}^n$ be a convex body lying in $\mathcal{R}_p$, $0 \le p \le n-1$.*

i) If $p = n-1$ then

$$\sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)(-\mathrm{r})^i = 0.$$

ii) If p is even, $0 \le p \le n-2$ then

$$\sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(K;E)(-\mathrm{r})^i - \binom{n}{p+2}(n-p-2)\int_0^{\mathrm{r}} (\mathrm{r}-s)^{p+1}\mathrm{W}_{p+2}(-s)\,ds \le 0.$$

iii) If p is odd, $1 \le p \le n-2$ then

$$\sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(K;E)(-\mathrm{r})^i + \binom{n}{p+2}(n-p-2)\int_0^{\mathrm{r}} (\mathrm{r}-s)^{p+1}\mathrm{W}_{p+2}(-s)\,ds \ge 0.$$

Equality holds in both inequalities if and only if K is homothetic to an $(n-p-2)$-tangential body of the convex body E.

Remark 4.2. Notice that in the case when $K \in \mathcal{R}_{n-1}$, i.e., when K is an outer parallel body of some convex body, then the inradius $\mathrm{r} = \mathrm{r}(K;E)$ is a root of the alternating Steiner polynomial.$\diamondsuit$

The following two corollaries, which are particular cases of Corollary 4.1.6, allow to establish new inequalities for an arbitrary convex body $K \in \mathcal{K}^n = \mathcal{R}_0$.

Corollary 4.1.7. *For any convex body $K \in \mathcal{K}^n$ it holds*

$$\mathrm{V}(K) \le n\mathrm{W}_1(K;E)\,\mathrm{r} - \frac{n(n-1)}{2}\mathrm{W}_2(K;E)\,\mathrm{r}^2 + \frac{n(n-1)(n-2)}{2}\int_0^{\mathrm{r}} (\mathrm{r}-s)\mathrm{W}_2(-s)\,ds.$$

Equality holds if and only if K is homothetic to an $(n-2)$-tangential body of E.

In [42] Sangwine-Yager proves an inequality in terms of the volume, the first and second quermassintegrals and the inradius of K, namely

$$0 \ge \mathrm{V}(K) - n\mathrm{r}\mathrm{W}_1(K;E) + (n-1)\mathrm{r}^2\mathrm{W}_2(K;E),$$

where equality holds if K is an $(n-2)$-tangential body of E.

However a characterization of the equality is not given. The following corollary establishes an inequality involving the volume, the first and second quermassintegrals and the inradius of K, in which the equality case is completely characterized.

Corollary 4.1.8 ([30]). *For any convex body $K \in \mathcal{K}^n$ it holds*

$$(n-3)\mathrm{V}(K) + 2\mathrm{W}_1(K;E)\,\mathrm{r} - (n-1)\mathrm{W}_2(K;E)\,\mathrm{r}^2 \geq 0.$$

Equality holds if and only if K is homothetic to an $(n-2)$-tangential body of E.

Proof. Since, up to translations, $\mathrm{r}(K_{-s};E)E \subseteq K_{-s}$, the monotonicity of the mixed volumes (see Proposition 1.2.5, part vi)) implies that $(\mathrm{r}-s)\mathrm{W}_2(-s) \leq \mathrm{W}_1(-s)$. Therefore

$$\int_0^{\mathrm{r}} (\mathrm{r}-s)\mathrm{W}_2(-s)\,ds \leq \int_0^{\mathrm{r}} \mathrm{W}_1(-s)\,ds = \frac{1}{n}\mathrm{V}(K),$$

since the volume is differentiable and $\mathrm{V}'(-s) = -n\mathrm{W}_1(-s)$. Hence, using Corollary 4.1.7 we get

$$\mathrm{V}(K) \leq n\mathrm{W}_1(K;E)\,\mathrm{r} - \frac{n(n-1)}{2}\mathrm{W}_2(K;E)\,\mathrm{r}^2 + \frac{(n-1)(n-2)}{2}\mathrm{V}(K).$$

Simplifying we get the required inequality. Notice that $(\mathrm{r}-s)\mathrm{W}_2(-s) = \mathrm{W}_1(-s)$ if and only if K_{-s} (and hence K) is homothetic to an $(n-2)$-tangential body of E (see Theorem 3.1.5 and remark afterwards). It concludes the proof. □

Remark 4.3. We know that the relative in- and circumradius are related by $\mathrm{r}(K;E)\,\mathrm{R}(E;K) = 1$ (see (1.5)). Then, using also that $\mathrm{W}_i(K;E) = \mathrm{W}_{n-i}(E;K)$, all the previous inequalities can be rewritten in terms of the relative circumradius. For instance, the inequality in Corollary 4.1.8 can be expressed, interchanging E by K in order to write it with the usual notation, as

$$(n-3)\mathrm{R}(K;E)^2\mathrm{V}(E) + 2\mathrm{R}(K;E)\mathrm{W}_{n-1}(K;E) - (n-1)\mathrm{W}_{n-2}(K;E) \geq 0.$$ ◊

4.2 The volume of the inner parallel body and the alternating Steiner polynomial

In the previous section we have proved that when dealing with convex bodies lying in some of the classes $\mathcal{R}_p$, it depends on the parity of p that we get an upper or a lower bound in terms of a closely related function to the alternating Steiner polynomial; just when $p = n-2$ the precise polynomial is obtained. In this section we will show that, except in very particular cases, it is not possible to bound the volume of the inner parallel body in terms of the alternating Steiner polynomial.

Throughout this section λ will be again non negative, $\lambda \geq 0$. Notice that the alternating Steiner polynomial of K in the variable λ is just the Steiner polynomial of K in $-\lambda$. Following the notation used for the Steiner polynomial in Chapter 1, we have that $f_E(K,-\lambda) = \sum_{i=0}^n \binom{n}{i}\mathrm{W}_i(K;E)(-\lambda)^i$ is the alternating Steiner polynomial of $K \in \mathcal{K}^n$ with respect to the fixed convex body $E \in \mathcal{K}_0^n$.

A first approximation to the above mentioned problem using 1-tangential bodies can lead to the idea that, depending on the parity of the dimension, upper or lower bounds for $\mathrm{V}(K_{-\lambda})$ in terms of $f_E(K,-\lambda)$ can be obtained:

Theorem 4.2.1 ([30]). *Let $K \in \mathcal{K}_0^n$ be a 1-tangential body of E. If n is odd then inequality* (4.10) *holds. If n is even then inequality* (4.2) *holds. In either case equality holds if and only if $K = E$.*

Proof. If $K \in \mathcal{K}_0^n$ is a 1-tangential body of E, Theorem 3.1.5 asserts that $\mathrm{V}(K) = \mathrm{W}_i(K;E)$, for all $i = 1, \dots, n-1$, and so we can rewrite the alternating Steiner polynomial $f_E(K, -\lambda)$, $\lambda \geq 0$, in the following way

$$f_E(K,-\lambda) = \mathrm{V}(K)\left[\sum_{i=0}^{n-1}\binom{n}{i}(-\lambda)^i + \frac{\mathrm{V}(E)}{\mathrm{V}(K)}(-\lambda)^n\right] = \mathrm{V}(K)\Big[(1-\lambda)^n - \big(1-\alpha(K)\big)(-\lambda)^n\Big],$$

where $\alpha(K) = \mathrm{V}(E)/\mathrm{V}(K)$. Observe that $0 < \alpha(K) \leq 1$ since $E \subseteq K$. On the other hand, since K is a tangential body of E we have that $\mathrm{r}(K;E) = 1$ and thus $K_{-\lambda} = (1-\lambda)K$. Hence, the volume $\mathrm{V}(K_{-\lambda}) = (1-\lambda)^n \mathrm{V}(K)$. Therefore

$$\mathrm{V}(K_{-\lambda}) - f_E(K,-\lambda) = \big(1-\alpha(K)\big)(-\lambda)^n.$$

Clearly if the dimension n is odd (even) the above difference is negative (positive) and inequality (4.10) (inequality (4.2)) holds. Equality holds if and only if $\alpha(K) = \mathrm{V}(E)/\mathrm{V}(K) = 1$, i.e., only when $K = E$. □

This first impression is however wrong, since it is also possible to find examples in odd (even) dimension for which inequality (4.2) (inequality (4.10)) holds, as the following result shows.

Theorem 4.2.2 ([30]). *There exist convex bodies in odd (even) dimension for which inequality* (4.2) *(inequality* (4.10)*) holds.*

Proof. Let $K \in \mathcal{K}_0^n$ be a 2-tangential body of B^n. Then on account of Theorem 3.1.5 we can rewrite the alternating Steiner polynomial of K as

$$\begin{aligned} f_{B^n}(K,-\lambda) &= \mathrm{V}(K)\left[\sum_{i=0}^{n-2}\binom{n}{i}(-\lambda)^i + n\frac{\mathrm{W}_{n-1}(K)}{\mathrm{V}(K)}(-\lambda)^{n-1} + \frac{\mathrm{V}(B^n)}{\mathrm{V}(K)}(-\lambda)^n\right] \\ &= \mathrm{V}(K)\Big[(1-\lambda)^n - n\big(1-\beta(K)\big)(-\lambda)^{n-1} - \big(1-\alpha(K)\big)(-\lambda)^n\Big], \end{aligned}$$

where $\beta(K) = \mathrm{W}_{n-1}(K)/\mathrm{V}(K)$ and $\alpha(K) = \mathrm{V}(B^n)/\mathrm{V}(K)$. Notice that since K is a 2-tangential body of B^n, its inradius $\mathrm{r}(K) = 1$, and hence $\alpha(K) \leq \beta(K) \leq 1$ (see (1.6)). Then, using again that $K_\lambda = (1-\lambda)K$, we can write the difference $\mathrm{V}(K_{-\lambda}) - f_{B^n}(K,-\lambda)$ as

$$\mathrm{V}(K_{-\lambda}) - f_{B^n}(K,-\lambda) = \mathrm{V}(K)(-\lambda)^{n-1}\Big[n\big(1-\beta(K)\big) - \big(1-\alpha(K)\big)\lambda\Big]. \tag{4.11}$$

For the sake of brevity we write $G(\lambda, K) = n\big(1-\beta(K)\big) - \big(1-\alpha(K)\big)\lambda$. Since $0 \leq \lambda \leq 1 = \mathrm{r}(K)$ then $G(\lambda, K) \geq G(1, K)$. Hence, if we construct a 2-tangential body $K \in \mathcal{K}_0^n$ of B^n such that $G(1, K) \geq 0$

for any value of the dimension, we get the desired example: when n is odd (even) the above difference in (4.11) is positive (negative) and therefore inequality (4.2) (inequality (4.10)) holds.

In order to get such a body it is enough to consider the 2-tangential body K^t constructed in [25, Proof of Theorem 1.2] which we reproduce in the next: let $P^t \in \mathcal{K}_0^3$, $t \geq 2$, be the pyramid over a square basis with vertices

$$(\pm t, \pm t, -1)^\intercal, \quad \bigl(0, 0, 1 + 2/(t^2 - 1)\bigr)^\intercal.$$

The coordinates are chosen such that the largest ball contained in P^t is B^3 and that all 2-faces (facets) of P^t touch B^3. Next P^t is embedded in the canonical way into $\mathbb{R}^n$ for $n \geq 3$ and let $K^t = \operatorname{conv}\{P^t, B^n\} \in \mathcal{K}_0^n$. If H is a support plane of K^t which is not a support plane of B^n it must be a support plane of P^t containing no 2-face of P^t, since they are, by construction, already tangent to the ball. So, H may contain only vertices or edges of P^t. Thus H is not an $(n-3)$-extreme support hyperplane of K^t, because the outer normal vectors to vertices or edges of a polytope are, respectively, $(n-1)$- and $(n-2)$-extreme normal vectors which are not $(n-3)$-extreme.

This shows that K^t is a 2-tangential body of B^n. It is easy to see that for the pyramid P^t there exists a constant c such that its 3-dimensional volume is not smaller than $c\,t^2$. Hence there exists a constant c_n depending only on the dimension such that $\mathrm{V}(K^t) \geq c_n\, t^2$. On the other hand, the circumradius of K^t is certainly less than $2t$ and so by (1.6) we have the bound $\mathrm{W}_{n-1}(K^t) \leq 2t\mathrm{V}(B^n)$. Finally, notice that the inequality $G(1, K^t) \geq 0$ is equivalent to the relation $(n-1)\mathrm{V}(K^t) \geq n\mathrm{W}_{n-1}(K^t) - \mathrm{V}(B^n)$, which clearly holds if t is large enough. $\square$

Thus, as mentioned before, Corollary 4.1.5, Theorem 4.2.1 and Theorem4.2.2 show that in general it is hopeless to give upper or lower bounds for the volume of the inner parallel body of a convex body in terms of exactly the alternating Steiner polynomial. It is necessary to deal with particular families of sets (cf. Corollary 4.1.5 and Theorem 4.2.1).

4.3 Bounding the quermassintegrals of the inner parallel body

In this section we get some bounds for the relative quermassintegrals of the inner parallel body in terms of the ones of the original body K. These results improve previous bounds which were obtained in [9]. For instance, the following lower bound for the relative quermassintegrals of the inner parallel body of K at distance λ was proved in [9, Theorem 2].

Theorem 4.3.1. *Let $K \in \mathcal{K}^n$. For $-\mathrm{r} \leq \lambda \leq 0$ and $i = 0, \ldots, n-1$ it holds*

$$\mathrm{W}_i(K_\lambda; E) \leq \mathrm{W}_i(K; E) - |\lambda| \sum_{k=0}^{n-i-1} \mathrm{V}\bigl(K_\lambda[k], K[n-i-k-1], E[i+1]\bigr).$$

The key point in the proof of this result is the fact that $K_{-\mathrm{r}}+\mathrm{r}E \subseteq K$; thus, it should be possible to get sharper inequalities using the fact that $K_{-\mathrm{r}} + \mathrm{r}E \subseteq K_{-\mathrm{r}} + \mathrm{r}K^* \subseteq K$ (see Lemma 3.1.13). Indeed, and by Theorem 3.1.15, we can get better bounds than the ones in Theorem 4.3.1 for the quermassintegrals of the inner parallel body, providing also conditions for the equality case.

Theorem 4.3.2 ([28]). *Let $K \in \mathcal{K}_0^n$ and $E \in \mathcal{K}_0^n$ be regular and strictly convex. For $-\mathrm{r} \leq \lambda \leq 0$ and $i = 0, \dots, n-1$,*

$$\mathrm{W}_i(K_\lambda; E) \leq \mathrm{W}_i(K; E) - |\lambda| \sum_{k=0}^{n-i-1} \mathrm{V}\bigl(K_\lambda[k], K[n-i-k-1], K^*, E[i]\bigr). \tag{4.12}$$

If K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ verifying condition (3.15) *then equality holds in all the inequalities. Conversely, if equality holds in* (4.12) *for some $i \in \{0, \dots, n-1\}$ then K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$.*

Proof. Using (3.12) and the monotonicity and linearity of mixed volumes (see Proposition 1.2.5) we get that for $-\mathrm{r} \leq \lambda \leq 0$,

$$\begin{aligned}
\mathrm{W}_i(K; E) &= \mathrm{V}\bigl(K[n-i], E[i]\bigr) \geq \mathrm{V}\bigl(K_\lambda + |\lambda|\, K^*, K[n-i-1], E[i]\bigr) \\
&= \mathrm{V}\bigl(K_\lambda, K[n-i-1], E[i]\bigr) + |\lambda|\, \mathrm{V}\bigl(K^*, K[n-i-1], E[i]\bigr) \\
&\geq \mathrm{V}\bigl(K_\lambda, K_\lambda + |\lambda|\, K^*, K[n-i-2], E[i]\bigr) + |\lambda|\, \mathrm{V}\bigl(K^*, K[n-i-1], E[i]\bigr) \\
&\geq \mathrm{V}\bigl(K_\lambda, K_\lambda, K[n-i-2], E[i]\bigr) + |\lambda|\, \mathrm{V}\bigl(K_\lambda, K^*, K[n-i-2], E[i]\bigr) \\
&\quad + |\lambda|\, \mathrm{V}\bigl(K^*, K[n-i-1], E[i]\bigr) \geq \dots \\
&\geq \mathrm{W}_i(K_\lambda; E) + |\lambda| \sum_{k=0}^{n-i-1} \mathrm{V}\bigl(K_\lambda[k], K^*, K[n-i-k-1], E[i]\bigr).
\end{aligned}$$

Now we deal with the equality case. If K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ verifying condition (3.15), Theorem 3.1.15 ensures that $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \leq \lambda \leq 0$, and hence equality holds in (4.12). Conversely, now we assume that equality holds in (4.12) for some $i \in \{0, \dots, n-1\}$. Then we have, in particular, that

$$\mathrm{V}\bigl(K[n-i], E[i]\bigr) = \mathrm{V}\bigl(K_\lambda + |\lambda|\, K^*, K[n-i-1], E[i]\bigr),$$

or equivalently, using the formula for the mixed volumes given in (1.2) we get that

$$\int_{\mathbb{S}^{n-1}} h(K,u)\, d\mathrm{S}\bigl(K[n-i-1], E[i]; u\bigr) = \int_{\mathbb{S}^{n-1}} h\bigl(K_\lambda + |\lambda|\, K^*, u\bigr)\, d\mathrm{S}\bigl(K[n-i-1], E[i]; u\bigr).$$

Since $K_\lambda + |\lambda|\, K^* \subseteq K$ (see Lemma 3.1.13), we get that the above identity for the integrals is equivalent to $h\bigl(K_\lambda + |\lambda|\, K^*, u\bigr) = h(K,u)$ for all $u \in \operatorname{supp} \mathrm{S}\bigl(K[n-i-1], E[i]; u\bigr)$. On the other hand, since E is regular and strictly convex, $\operatorname{supp} \mathrm{S}\bigl(K[n-i-1], E[i]; u\bigr) = \mathrm{cl}\,\mathcal{U}_i(K) \supseteq \mathrm{cl}\,\mathcal{U}_0(K)$ (see Proposition 3.1.19). So we get $h\bigl(K_\lambda + |\lambda|\, K^*, u\bigr) = h(K,u)$ for all $u \in \mathrm{cl}\,\mathcal{U}_0(K)$. Notice that it implies, in particular, that K is a tangential body of $K_\lambda + |\lambda|\, K^*$.

Now observe that, for every $u \in \operatorname{cl}\mathcal{U}_0(K) = \mathcal{U}_0(K^*)$ (see Lemma 3.1.12) it holds

$$h(K,u) = h\bigl(K_\lambda + |\lambda|\, K^*, u\bigr) = h(K_\lambda, u) + |\lambda|\, h(K^*, u) = h(K_\lambda, u) + |\lambda|\, h(E,u) = h\bigl(K_\lambda + |\lambda|\, E, u\bigr),$$

which shows that K is a tangential body of $K_\lambda + |\lambda|\, E$, for all $\lambda \in [-\mathrm{r}, 0]$; in particular, K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ as required. □

Notice that the assumptions of regularity and strict convexity for E are needed just for the equality case; the inequalities hold for any $E \in \mathcal{K}_0^n$.

The particular case $i = 0$ provides a new upper bound for the volume of the inner parallel body.

Corollary 4.3.3 ([28]). *Let $K \in \mathcal{K}_0^n$ and let $E \in \mathcal{K}_0^n$ be a regular and strictly convex body. For $-\mathrm{r} \le \lambda \le 0$ it holds*

$$\mathrm{V}(K_\lambda) \le \mathrm{V}(K) - |\lambda| \sum_{k=0}^{n-1} \mathrm{V}\bigl(K_\lambda[k], K[n-k-1], K^*\bigr).$$

If K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ verifying condition (3.15) *then equality holds. Conversely, if equality holds then K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$.*

Notice that in the case $\lambda = -\mathrm{r}$ we get the following lower bound for the volume of K:

$$\mathrm{V}(K) \ge \mathrm{r} \sum_{k=0}^{n-1} \mathrm{V}\bigl(K_{-\mathrm{r}}[k], K[n-k-1], K^*\bigr).$$

Remark 4.4. Inequalities in Theorem 4.3.2 allow to give an alternative proof to the fact that the left derivative of the i-th quermassintegral with respect to λ, $-\mathrm{r} < \lambda \le 0$ is bounded from below by $(n-i)\mathrm{V}\bigl(K_\lambda[n-i-1], K_\lambda^*, E[i]\bigr)$, which was proved in [41, Lemma 3.5] (see Lemma 3.2.3): for $h \ge 0$, using (4.12) and considering that $K_{\lambda-h}$ is an inner parallel body of K_λ if $\lambda - h > -\mathrm{r}$, we get

$$\begin{aligned}
{}'\mathrm{W}_i(\lambda) &= \lim_{h\to 0} \frac{\mathrm{W}_i(K_\lambda; E) - \mathrm{W}_i(K_{\lambda-h}; E)}{h} \\
&\ge \lim_{h\to 0} \frac{h \sum_{k=0}^{n-i-1} \mathrm{V}\bigl(K_{\lambda-h}[k], K_\lambda[n-i-k-1], K_\lambda^*, E[i]\bigr)}{h} \\
&= \sum_{k=0}^{n-i-1} \mathrm{V}\bigl(K_\lambda[n-i-1], K_\lambda^*, E[i]\bigr) = (n-i)\mathrm{V}\bigl(K_\lambda[n-i-1], K_\lambda^*, E[i]\bigr).
\end{aligned}$$

Moreover, since $E \subseteq K_\lambda^*$ for all $-\mathrm{r} < \lambda \le 0$, then we also get ${}'\mathrm{W}_i(\lambda) \ge (n-i)\mathrm{W}_{i+1}(K_\lambda; E)$. ◇

4.4 Inequalities for convex bodies lying in the class $\mathcal{R}_p$

Under the assumption that the convex body $K \in \mathcal{K}_0^n$ lies in the class $\mathcal{R}_p$, we can improve the previous inequalities.

Proposition 4.4.1 ([28]). *Let $K \in \mathcal{K}_0^n$ be a convex body lying in the class $\mathcal{R}_p$ and let $E \in \mathcal{K}_0^n$ be regular and strictly convex. For $0 \le i \le p$ and for $-\mathrm{r} \le \lambda \le 0$ it holds*

$$\begin{aligned} \mathrm{W}_i(K_\lambda; E) \ge \mathrm{W}_i(K;E) - (n-i)\,|\lambda|\,\mathrm{W}_{i+1}(K;E) + (n-i)\frac{\lambda^2}{2}\mathrm{V}\big(K[n-i-2], K^*, E[i+1]\big) \\ - (n-i)\sum_{k=1}^{n-i-2}\int_\lambda^0 t\mathrm{V}\big(K_t[k], K[n-i-k-2], K^*, E[i+1]\big)\,dt. \end{aligned} \tag{4.13}$$

If K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ verifying condition (3.15) *then equality holds in all the inequalities. Conversely, if equality holds in* (4.13) *for some $i \in \{0, \ldots, p\}$ then K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$.*

Proof. If we consider inequality (4.12) in the case of the $(i+1)$-th quermassintegral, $i = 0, \ldots, p$,

$$\begin{aligned} \mathrm{W}_{i+1}(K_\lambda; E) \le \mathrm{W}_{i+1}(K;E) - |\lambda|\,\mathrm{V}\big(K[n-i-2], K^*, E[i+1]\big) \\ - |\lambda| \sum_{k=1}^{n-i-2}\mathrm{V}\big(K_\lambda[k], K[n-i-k-2], K^*, E[i+1]\big), \end{aligned}$$

integrating from λ to 0 we get

$$\begin{aligned} \int_\lambda^0 \mathrm{W}_{i+1}(K_t;E)\,dt \le \int_\lambda^0 \Big[\mathrm{W}_{i+1}(K;E) + t\mathrm{V}\big(K[n-i-2], K^*, E[i+1]\big)\Big]\,dt \\ + \int_\lambda^0 t \sum_{k=1}^{n-i-2}\mathrm{V}\big(K_t[k], K[n-i-k-2], K^*, E[i+1]\big)\,dt. \end{aligned}$$

Since $K \in \mathcal{R}_p$, we have that $\mathrm{W}_i'(K_t) = (n-i)\mathrm{W}_{i+1}(K_t)$ for $i = 0, \ldots, p$, and hence

$$\begin{aligned} \frac{1}{n-i}\big[\mathrm{W}_i(K;E) - \mathrm{W}_i(K_\lambda;E)\big] \le -\lambda \mathrm{W}_{i+1}(K;E) - \frac{\lambda^2}{2}\mathrm{V}\big(K[n-i-2], K^*, E[i+1]\big) \\ + \sum_{k=1}^{n-i-2}\int_\lambda^0 t\mathrm{V}\big(K_t[k], K[n-i-k-2], K^*, E[i+1]\big)\,dt, \end{aligned}$$

which concludes the proof of the inequality. The conditions for the equality case follow directly from Theorem 4.3.2. □

Proposition 4.4.1 for the class $\mathcal{R}_0$ leads to the following corollary.

Corollary 4.4.2 ([28]). *Let $K \in \mathcal{K}_0^n$ and let $E \in \mathcal{K}_0^n$ be a regular and strictly convex body. For $-\mathrm{r} \le \lambda \le 0$ we have*

$$\mathrm{V}(K_\lambda) \ge \mathrm{V}(K) - n\,|\lambda|\,\mathrm{W}_1(K;E) + n\frac{\lambda^2}{2}\mathrm{V}\big(K[n-2], K^*, E\big) - n\sum_{k=1}^{n-2}\int_\lambda^0 t\mathrm{V}\big(K_t[k], K[n-k-2], K^*, E\big)\,dt.$$

If K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ verifying condition (3.15) *then equality holds. Conversely, if equality holds then K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$.*

This inequality strengths the one obtained by Brannen in [9, Corollary 2], namely,

$$\mathrm{V}(K_\lambda) \geq \mathrm{V}(K) - n\,|\lambda|\,\mathrm{W}_1(K;E) + n\frac{\lambda^2}{2}\mathrm{W}_2(K;E) + n\sum_{k=1}^{n-2}\int_0^{|\lambda|} t\mathrm{V}\big(K_t[k], K[n-k-2], E[2]\big)\,dt. \quad (4.14)$$

Notice that when $\lambda = -\mathrm{r}$, Corollary 4.4.2 provides an upper bound for the volume of K:

$$\frac{1}{n}\mathrm{V}(K) \leq \mathrm{r}\mathrm{W}_1(K;E) - \frac{\mathrm{r}^2}{2}\mathrm{V}\big(K[n-2], K^*, E\big) + \sum_{k=1}^{n-2}\int_{-\mathrm{r}}^{0} t\mathrm{V}\big(K_t[k], K[n-k-2], K^*, E\big)\,dt.$$

Moreover, in the case $n = 3$, the above inequality is written as

$$\frac{1}{3}\mathrm{V}(K) \leq \mathrm{r}\mathrm{W}_1(K;E) - \frac{\mathrm{r}^2}{2}\mathrm{V}\big(K, K^*, E\big) + \int_{-\mathrm{r}}^{0} t\mathrm{V}\big(K_t, K^*, E\big)\,dt. \quad (4.15)$$

Since it improves the corresponding inequality by Brannen (4.14) for $n = 3$ which, in turn, is sharper than the so called Osserman inequality in the particular case of $n = 3$ and $E = B^3$ (for a proof of this assertion see [9, p. 3982]), namely

$$\mathrm{V}(K) \leq 3\mathrm{r}(K)\mathrm{W}_1(K) - 2\mathrm{r}(K)^2\big(\kappa_3\mathrm{W}_1(K)\big)^{1/2}$$

(see [37]), we get thus that (4.15) is a strengthening of Osserman inequality.

Remark 4.5. Before finishing this chapter we would like to point out that in [43, p. 175] it is proved that for any convex body $K \in \mathcal{K}_0^n$ with inradius $\mathrm{r} = \mathrm{r}(K)$ and for all $-\mathrm{r} \leq \lambda \leq 0$,

$$\mathrm{V}(K_\lambda) \geq \mathrm{V}(K) - 3\,|\lambda|\,\mathrm{W}_1(K) + 2\lambda^2\mathrm{W}_2(K) + \lambda^2\mathrm{W}_2(K_\lambda), \quad (4.16)$$

where equality holds for all $-\mathrm{r} \leq \lambda \leq 0$ if and only if K is a 1-tangential body of $K_{-\mathrm{r}} + \mathrm{r}B^3$. The proof of the equality case is not correct since, in fact, if equality holds in (4.16) then K is a 1-tangential body of $K_{-\mathrm{r}} + \mathrm{r}B^3$, but not every 1-tangential body of $K_{-\mathrm{r}} + \mathrm{r}B^3$ satisfies (4.16); condition (3.15) is needed (see Figure 3.4). Inequality in Corollary 4.4.2 for $n = 3$ and $E = B^3$,

$$\mathrm{V}(K_\lambda) \geq \mathrm{V}(K) - 3\,|\lambda|\,\mathrm{W}_1(K) + 3\frac{\lambda^2}{2}\mathrm{V}(K, K^*, E) - 3\int_{\lambda}^{0} t\mathrm{V}(K_t, K^*, E)\,dt,$$

strengths (4.16) (the proof of this fact is analogous to the one in [9, p. 3982]). ♢

Chapter 5

From Brunn-Minkowski to Poincaré type inequalities

The functional $\Phi : \mathcal{K}^n \longrightarrow \mathcal{C}(\mathbb{S}^{n-1})$ assigning to every convex body K its support function $h_K : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ maps $\mathcal{K}^n$ into the abstract cone $\Phi(\mathcal{K}^n)$ of functions in $\mathcal{C}(\mathbb{S}^{n-1})$ which are the support function of some convex body (see Theorem 1.1.11). Thus, in a first step, we can consider any functional $F : \mathcal{K}^n \longrightarrow \mathbb{R}$ as defined on $\Phi(\mathcal{K}^n)$.

This map Φ allows also to *translate* Minkowski sum, and so outer parallel bodies, into $\mathcal{C}(\mathbb{S}^{n-1})$. From this point of view, if K is a convex body, $K+\lambda B^n$, $\lambda \geq 0$, can be identified with $h_K+\lambda$ (since $h_{B^3}(u) = 1$ in every direction $u \in \mathbb{S}^{n-1}$). Notice that $h_K - \lambda$ is not, in general, the support function of the inner parallel body $K_{-\lambda}$; however, if K is of class $\mathcal{C}^2_+$, for $\lambda \geq 0$ small enough it is easy to check that $h_K - \lambda$ is the support function of some convex body L. In fact, for any $\psi \in \mathcal{C}^2(\mathbb{S}^{n-1})$ and λ small enough in absolute value, $h_K + \lambda\psi$ is a support function.

Following the spirit of the previous chapters (definition of functionals on convex bodies with respect to the full system of parallel bodies), for $\psi : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^2$ and $K \in \mathcal{K}^n$ of class $\mathcal{C}^2_+$, let $\varepsilon > 0$ be small enough such that $h_K + \lambda\psi \in \Phi(\mathcal{K}^n)$ for any $\lambda \in (-\varepsilon, \varepsilon)$, i.e., $h_K + \lambda\psi$ is the support function of some convex body. Then, in a second step, a functional $F : \mathcal{K}^n \longrightarrow \mathbb{R}$ can be seen as a function on the variable λ, $F : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$, just taking $F(\lambda) = F\bigl(h_K + \lambda\psi\bigr)$.

In this chapter we will be mainly interested in the case of the quermassintegrals W_i. As we have already seen in Chapter 2, the general Brunn-Minkowski inequality ensures that if K and L are convex bodies in $\mathbb{R}^n$ and $t \in [0,1]$ then

$$\mathrm{W}_i\bigl((1-t)K + tL\bigr)^{1/(n-i)} \geq (1-t)\mathrm{W}_i(K)^{1/(n-i)} + t\mathrm{W}_i(L)^{1/(n-i)}$$

for $i = 0, \dots, n$ (see Theorem 1.2.12), i.e., the functional $\mathrm{W}_i^{1/(n-i)} : \mathcal{K}^n \longrightarrow \mathbb{R}$ is concave, for $i = 0, \dots, n$. The above remarks allow now to consider $\mathrm{W}_i^{1/(n-i)}$ as functions of the real variable $\lambda \in (-\varepsilon, \varepsilon)$, for $\varepsilon > 0$ small enough. The concavity property of these functions will lead to certain *Poincaré type inequalities*, which are the main aim of this chapter.

As mentioned in Chapter 1, the classical version of Brunn-Minkowski inequality (1.9) is one of the fundamental results in the Theory of Convex Bodies, and it is the starting point of many other similar inequalities involving mixed volumes of convex bodies. Nevertheless, as it is pointed out in [19], its role goes beyond the limits of the Theory of Convex Bodies, having connections with several important inequalities, for instance, in Analysis. Its equivalent functional formulation, the Prékopa-Leindler inequality, is related to Young's convolution inequality. Bobkov and Ledoux [5] gave a proof of the Sobolev and Gagliardo-Nirenberg inequalities with optimal constant, based on the Brunn-Minkowski inequality.

In [12] an argument leading from the Brunn-Minkowski inequality to a Poincaré-type inequality on the boundary of smooth convex bodies with positive Gauss curvature is given. The main idea is based on the fact that the concavity of the functional involved in Brunn-Minkowski inequality implies that its *second variation* must be negative semi-definite. At this point it is precisely the idea of defining the volume as a function on a real variable which provides the key-tool to use the concavity of the functional. In this chapter the natural continuation of the aforementioned work [12] is developed and all the results we present here can be found in [13].

5.1 Preliminaries

Let $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^3$ on the sphere. Setting $\{E_1, \dots, E_{n-1}\}$ a local orthonormal frame of vector fields on $\mathbb{S}^{n-1}$, we denote by f_i, f_{ij} and f_{ijk}, respectively, the first, second and third covariant derivatives of f with respect to $\{E_1, \dots, E_{n-1}\}$, for $i, j, k \in \{1, \dots, n-1\}$. Thus, writing $\operatorname{grad} f$ and $\operatorname{Hess} f$ for the *gradient* and the *Hessian* of f, respectively, we have $f_i = E_i(f) = \langle \operatorname{grad} f, E_i \rangle = df(E_i)$, $f_{ij} = \operatorname{Hess} f(E_i, E_j)$ and $f_{ijk} = \big\langle \operatorname{grad}\big(\operatorname{Hess} f(E_i, E_j)\big), E_k \big\rangle$. For further definitions and properties on this subject we refer for instance to [57, Chapter 1, §2 and §3] or [32].

The following properties on the high order covariant derivatives hold (we write δ_{ij} to denote the standard Kronecker symbols). They can be found in [57, Chapter 1, §3].

Lemma 5.1.1. *Let $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^3$. Then*

$$\begin{aligned}
f_{ij} &= f_{ji}, \\
f_{ijk} &= f_{jik}, \\
f_{ijk} &= f_{ikj} + f_j \delta_{ik} - f_k \delta_{ij}.
\end{aligned}$$

On the other hand, we denote as usual by div the *divergence* on the sphere, i.e., for a vector field $X = \sum_{i=1}^{n-1} X^i E_i$ on $\mathbb{S}^{n-1}$, the divergence X is given by

$$\operatorname{div} X = \sum_{i=1}^{n-1} E_i(X^i).$$

It is clear from the definition of divergence that for $f \in \mathcal{C}^2(\mathbb{S}^{n-1})$ and a vector field X on the sphere, it holds that

$$\operatorname{div}(fX) = \langle \operatorname{grad} f, X \rangle + f \operatorname{div} X. \tag{5.1}$$

The well-known *divergence theorem* can be stated in the following way (it can be checked, for instance, in [57, Chapter 1, Theorem 2.1]):

Theorem 5.1.2 (Divergence theorem). *For any orientable compact Riemannian manifold Σ without boundary and for any vector field X on Σ it holds*

$$\int_\Sigma \operatorname{div} X = 0.$$

5.1.1 Elementary symmetric functions

Now we introduce some notions on elementary symmetric functions of the eigenvalues of square matrices that we will use to prove the main results in this chapter. We refer to [40, Chapter 1] for a further study of this topic.

Let $m \in \mathbb{N}$ be a positive integer. For an $m \times m$ symmetric matrix $A = (a_{ij})_{ij}$ having eigenvalues $\lambda_1, \ldots, \lambda_m$ and for $k \in \{0, \ldots, m\}$, the *k-th elementary symmetric function* of A is defined as the k-th elementary symmetric function of its eigenvalues, i.e.,

$$\begin{aligned} \mathrm{s}_k(A) &:= \mathrm{s}_k(\lambda_1, \ldots, \lambda_m) = \sum_{1 \le i_1 < \cdots < i_k \le m} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \text{if } k \ge 1, \\ \mathrm{s}_0(A) &:= 1. \end{aligned}$$

In particular, $\mathrm{s}_1(A) = \operatorname{tr} A$ and $\mathrm{s}_m(A) = \det A$ are the trace and the determinant of A, respectively. We also need to consider the following: for A and k as above and $i, j \in \{1, \ldots, m\}$, let

$$\mathrm{s}_k^{ij}(A) = \frac{\partial \mathrm{s}_k(A)}{\partial a_{ij}}.$$

The matrix consisting of these entries, i.e., $\left(\mathrm{s}_k^{ij}(A)\right)_{ij}$, is still symmetric and can be considered as a *k-th cofactor matrix of A*.

Remark 5.1. Note that in the case $k = m$, the matrix $\left(\mathrm{s}_m^{ij}(A)\right)_{ij}$ is the usual cofactor matrix, whereas for $k = 1$ we get $\mathrm{s}_1^{ij}(A) = \delta_{ij}$ since the trace does not depend on the form in which the matrix is expressed. Hence, $\left(\mathrm{s}_1^{ij}(A)\right)_{ij}$ is the $m \times m$ identity matrix I_m. ◊

In the sequel we will use some properties of elementary symmetric functions of matrices that, for convenience, we gather in the following statement. For their proofs and more detailed explanations we refer to [38] and [40].

Proposition 5.1.3. *Let $A=(a_{ij})_{ij}$ be an $m\times m$ symmetric and positive definite matrix and let $i,j,k\in\{1,\dots,m\}$. Then the following facts hold:*

i) *If A is diagonal then $\bigl(\mathrm{s}_k^{ij}(A)\bigr)_{ij}$ is diagonal.*

ii) *If $\lambda_1,\dots,\lambda_m$ denote the eigenvalues of A, then the eigenvalues of $\bigl(\mathrm{s}_k^{ij}(A)\bigr)_{ij}$ are given by*

$$\mathrm{s}_{k-1}\bigl(\mathrm{diag}(\lambda_1,\dots,\widehat{\lambda}_l,\dots,\lambda_m)\bigr),\quad l=1,\dots,m.$$

Here $\widehat{\lambda}$ means that we omit the value λ.

iii) $\mathrm{s}_k(A)=\mathrm{s}_{m-k}(A^{-1})\det A$.

iv) *The k-th elementary symmetric function*

$$\mathrm{s}_k(A)=\frac{1}{k}\sum_{i,j=1}^{m}\mathrm{s}_k^{ij}(A)a_{ij}. \tag{5.2}$$

v) *It holds $\mathrm{tr}\bigl(\mathrm{s}_k^{ij}(A)\bigr)_{ij}=(m-k)\mathrm{s}_{k-1}(A)$.*

5.1.2 Convex bodies of class $\mathcal{C}_+^2$

We already know that the functionals $\mathrm{W}_i^{1/(n-i)}:\mathcal{K}^n\longrightarrow\mathbb{R}$ are concave in the class of convex bodies. Throughout this chapter we develop some of the consequences of considering the "heuristically natural" negativity of the second variation of the functional provided by the concavity.

We denote by $N_K:\mathrm{bd}\,K\longrightarrow\mathbb{S}^{n-1}$ the *Gauss map* of K, i.e., for $x\in\mathrm{bd}\,K$, $N_K(x)$ is the outer unit normal vector to $\mathrm{bd}\,K$ at x. If K is of class $\mathcal{C}_+^2$ then its Gauss map N_K is differentiable on $\mathrm{bd}\,K$ and its differential dN_K is the *Weingarten map* of $\mathrm{bd}\,K$. Moreover N_K is invertible and its inverse $N_K^{-1}:\mathbb{S}^{n-1}\longrightarrow\mathrm{bd}\,K$ is also differentiable (hence N_K is a diffeomorphism) and the matrix associated to the linear map dN_K^{-1} is $\bigl((h_K)_{ij}+h_K\delta_{ij}\bigr)_{ij}$ (see [22, Section 2]). Moreover, it is known (see [49, Section 2.5]) that a convex body K is of class $\mathcal{C}_+^2$ if and only if its support function $h_K\in\mathcal{C}^2(\mathbb{S}^{n-1})$ and the $(n-1)\times(n-1)$ matrix

$$M_K^{-1}:=\bigl((h_K)_{ij}+h_K\delta_{ij}\bigr)_{ij} \tag{5.3}$$

is positive definite at each point of $\mathbb{S}^{n-1}$. This implies, in particular, that M_K is also positive definite. We will write $M>0$ to mean that a matrix M is positive definite. The eigenvalues of M_K are the *principal curvatures* of K and so, the eigenvalues of M_K^{-1} are the *principal radii* of K (see [49, Section 2.5] for precise definitions and properties).

Then it follows that the set

$$\mathfrak{S} = \Big\{ h \in \mathcal{C}^2(\mathbb{S}^{n-1}) : \big(h_{ij} + h\delta_{ij}\big)_{ij} > 0 \ \text{ on } \mathbb{S}^{n-1} \Big\}$$

consists of support functions of convex bodies of class $\mathcal{C}^2_+$; i.e., if $h \in \mathfrak{S}$, then there exists $K \in \mathcal{K}^n$ of class $\mathcal{C}^2_+$ such that $h = h_K$, and conversely, for any $K \in \mathcal{K}^n$ of class $\mathcal{C}^2_+$, $h_K \in \mathfrak{S}$.

On the other hand, in Chapter 1 we have seen that the quermassintegrals of $K \in \mathcal{K}^n$ can be expressed as an integral, up to a constant, of the support function with respect to the corresponding mixed surface area measure (cf. (1.2)), namely, for $i = 0, \dots, n-1$,

$$\mathrm{W}_i(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K \, d\mathrm{S}\big(K[n-i-1], B^n[i]; \cdot\big) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K \, d\mathrm{S}_{n-i-1}(K; \cdot).$$

In the particular case when K is a convex body of class $\mathcal{C}^2_+$, the mixed surface area measure involved in this integral expression can be written in terms of the $(n-i-1)$-st elementary symmetric function of M_K^{-1} (see [49, (5.3.11), p. 291]), namely, in general

$$\mathrm{S}_k(K; \cdot) = \binom{n-1}{k}^{-1} \mathrm{s}_k(M_K^{-1}) \, \mathcal{H}^{n-1}, \tag{5.4}$$

and then it holds

$$\mathrm{W}_i(K) = \frac{1}{n} \binom{n-1}{n-i-1}^{-1} \int_{\mathbb{S}^{n-1}} h_K \mathrm{s}_{n-i-1}(M_K^{-1}) \, d\mathcal{H}^{n-1}. \tag{5.5}$$

Here $\mathcal{H}^{n-1}$ denotes the usual $(n-1)$-dimensional *Hausdorff measure*.

From now on if $h \in \mathfrak{S}$ we will write K_h to denote the convex body (of class $\mathcal{C}^2_+$) whose support function is h. Then by (5.5) and Theorem 1.2.12 we get the following result.

Proposition 5.1.4. *For $0 \le i \le n-1$, let F_i denote the functional*

$$F_i : \mathfrak{S} \longrightarrow \mathbb{R}_+, \quad F_i(h) = \int_{\mathbb{S}^{n-1}} h \, \mathrm{s}_{n-i-1}(M_{K_h}^{-1}) \, d\mathcal{H}^{n-1}.$$

Then $F_i^{1/(n-i)}$ is concave in $\mathfrak{S}$.

Proposition 5.1.4 is the first necessary tool we will use in order to prove the results we include in this chapter.

5.2 A lemma concerning Hessian operators on the sphere

This section is devoted to prove the following lemma, as well as a useful formula which is crucial for the calculations needed in the main results of the chapter. For any matrix A, we will write $\big[\mathrm{s}_k^{ij}(A)\big]_j$ to denote the vector field defined by the i-th row of the matrix $\big(\mathrm{s}_k^{ij}(A)\big)_{ij}$.

Lemma 5.2.1 ([13]). *Let $f \in \mathcal{C}^2(\mathbb{S}^{n-1})$ and $k \in \{1,\dots,n-1\}$. Let $\{E_1,\dots,E_{n-1}\}$ be a local orthonormal frame of vector fields on $\mathbb{S}^{n-1}$. Then, for every $i \in \{1,\dots,n-1\}$,*

$$\operatorname{div}\Big(\big[\mathrm{s}_k^{ij}(\operatorname{Hess} f + f\mathrm{I}_{n-1})\big]_j\Big) = \sum_{j=1}^{n-1} E_j\Big(\mathrm{s}_k^{ij}(\operatorname{Hess} f + f\mathrm{I}_{n-1})\Big) = 0.$$

The case $k = n-1$ of the preceding lemma was proved by Cheng and Yau in [11, p. 504]. We also note that an analogous result is valid in the Euclidean setting, with $\operatorname{Hess} f + f\mathrm{I}_{n-1}$ replaced by $\operatorname{Hess} f$ (see, for instance, [38, Proposition 2.1] and [40, Section 2.3]). In order to show it we follow a similar argument to the one in [40] for the Euclidean case, using some (standard) tools from Differential Geometry on the sphere $\mathbb{S}^{n-1}$.

Proof. For $k \in \{0,\dots,n-1\}$, the k-th elementary symmetric function of a symmetric $(n-1)\times(n-1)$ matrix $A = (a_{ij})_{ij}$ can be written in the following way (see, for instance, [38]):

$$\mathrm{s}_k(A) = \frac{1}{k}\sum_{\substack{i_r,j_r=1\\ r=1,\dots,k}}^{n-1} \delta\binom{i_1,\dots,i_k}{j_1,\dots,j_k} a_{i_1j_1}\cdots a_{i_kj_k}, \tag{5.6}$$

where the Kronecker symbol $\delta\binom{i_1,\dots,i_k}{j_1,\dots,j_k}$ equals 1 (respectively, -1) when $i_1,\dots,i_k$ are all distinct and $(j_1,\dots,j_k)$ is an even (respectively, odd) permutation of $(i_1,\dots,i_k)$; otherwise it is 0. Using the above equality we have

$$\mathrm{s}_k^{ij}(A) = \frac{1}{(k-1)!}\sum_{\substack{i,j,i_r,j_r=1\\ r=1,\dots,k-1}}^{n-1} \delta\binom{i,i_1,\dots,i_{k-1}}{j,j_1,\dots,j_{k-1}} a_{i_1j_1}\cdots a_{i_{k-1}j_{k-1}}.$$

Hence we can write

$$\begin{aligned}
&(k-1)!\sum_{j=1}^{n-1} E_j\Big(\mathrm{s}_k^{ij}(\operatorname{Hess} f + f\mathrm{I}_{n-1})\Big)\\
&= \sum_{j=1}^{n-1}\sum_{\substack{i,j,i_r,j_r=1\\ r=1,\dots,k-1}}^{n-1} \delta\binom{i,i_1,\dots,i_{k-1}}{j,j_1,\dots,j_{k-1}} E_j\Big((f_{i_1j_1}+f\delta_{i_1j_1})\cdots(f_{i_{k-1}j_{k-1}}+f\delta_{i_{k-1}j_{k-1}})\Big)\\
&= \sum_{j=1}^{n-1}\sum_{\substack{i,j,i_r,j_r=1\\ r=1,\dots,k-1}}^{n-1} \delta\binom{i,i_1,\dots,i_{k-1}}{j,j_1,\dots,j_{k-1}} \Big[(f_{i_1j_1j}+f_j\delta_{i_1j_1})(f_{i_2j_2}+f\delta_{i_2j_2})\cdots(f_{j_{k-1}i_{k-1}}+f\delta_{i_{k-1}j_{k-1}})\\
&\qquad + \cdots + (f_{i_1j_1}+f\delta_{i_1j_1})\cdots(f_{i_{k-2}j_{k-2}}+f\delta_{i_{k-2}j_{k-2}})(f_{i_{k-1}j_{k-1}j}+f_j\delta_{i_{k-1}j_{k-1}})\Big].
\end{aligned} \tag{5.7}$$

In the last sum, for fixed $i_1,\dots,i_{k-1},j_1,\dots,j_{k-1},j$, we consider the terms

$$A = \delta_1(f_{i_1j_1j}+f_j\delta_{i_1j_1})C \quad\text{and}\quad B = \delta_2(f_{i_1jj_1}+f_{j_1}\delta_{i_1j})C\,,$$

where

$$\delta_1 = \delta\begin{pmatrix} i, i_1, i_2, \dots, i_{k-1} \\ j, j_1, j_2, \dots, j_{k-1} \end{pmatrix}, \quad \delta_2 = \delta\begin{pmatrix} i, i_1, i_2, \dots, i_{k-1} \\ j_1, j, j_2, \dots, j_{k-1} \end{pmatrix},$$

and

$$C = (f_{i_2 j_2} + f\delta_{i_2 j_2}) \cdots (f_{j_{k-1} i_{k-1}} + f\delta_{i_{k-1} j_{k-1}}).$$

Clearly $\delta_2 = -\delta_1$. Moreover by using the relation concerning the third covariant derivatives on $\mathbb{S}^{n-1}$ given in Lemma 5.1.1 we get that

$$A + B = \delta_1 C\big(f_{i_1 j_1 j} + f_j \delta_{i_1 j_1} - f_{i_1 j j_1} - f_{j_1}\delta_{i_1 j}\big) = 0.$$

This argument can be repeated for any other term of the last sum in (5.7); in fact, for any term A in the above sum, there exists another term B, uniquely determined, which cancels out with A. This concludes the proof. $\square$

As a consequence of this lemma, we can prove the following result, which states that we can "move" the first covariant derivative from one function to another in the suitable context.

Lemma 5.2.2. *Let A be a matrix of the form $A = \operatorname{Hess} f + f\mathrm{I}_{n-1}$ for some $f \in \mathcal{C}^2(\mathbb{S}^{n-1})$. Let $g, \phi \in \mathcal{C}^1(\mathbb{S}^{n-1})$ and $k \in \{1, \dots, n-1\}$. Then, for every $i \in \{1, \dots, n-1\}$,*

$$\int_{\mathbb{S}^{n-1}} g \sum_{j=1}^{n-1} \phi_j\, \mathrm{s}_k^{ij}(A)\, d\mathcal{H}^{n-1} = -\int_{\mathbb{S}^{n-1}} \phi \sum_{j=1}^{n-1} g_j\, \mathrm{s}_k^{ij}(A)\, d\mathcal{H}^{n-1}.$$

Proof. Let $i, k \in \{1, \dots, n-1\}$ be fixed and consider the vector field $\left[\mathrm{s}_k^{ij}(A)\right]_j$, for $j = 1, \dots, n-1$, given by the i-th row of the matrix $\left(\mathrm{s}_k^{ij}(A)\right)_{ij}$. By (5.1) it is clear that the divergence of the product $\phi\left[\mathrm{s}_k^{ij}(A)\right]_j$ gives

$$\operatorname{div}\left(\phi\left[\mathrm{s}_k^{ij}(A)\right]_j\right) = \left\langle \operatorname{grad}\phi, \left[\mathrm{s}_k^{ij}(A)\right]_j\right\rangle + \phi \operatorname{div}\left(\left[\mathrm{s}_k^{ij}(A)\right]_j\right) = \left\langle \operatorname{grad}\phi, \left[\mathrm{s}_k^{ij}(A)\right]_j\right\rangle,$$

where the last equality follows from Lemma 5.2.1. Thus we get immediately that

$$\operatorname{div}\left(\phi\left[\mathrm{s}_k^{ij}(A)\right]_j\right) = \sum_{j=1}^{n-1} \phi_j\, \mathrm{s}_k^{ij}(A)$$

and we can rewrite the integral in the statement as

$$\int_{\mathbb{S}^{n-1}} g \sum_{j=1}^{n-1} \phi_j\, \mathrm{s}_k^{ij}(A)\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} g \operatorname{div}\left(\phi\left[\mathrm{s}_k^{ij}(A)\right]_j\right)\, d\mathcal{H}^{n-1}.$$

On the other hand, using again (5.1) we get that

$$g \operatorname{div}\left(\phi\left[\mathrm{s}_k^{ij}(A)\right]_j\right) = \operatorname{div}\left(g\phi\left[\mathrm{s}_k^{ij}(A)\right]_j\right) - \left\langle \operatorname{grad} g, \phi\left[\mathrm{s}_k^{ij}(A)\right]_j\right\rangle.$$

Substituting this expression in the above integral and applying divergence Theorem 5.1.2 we reach the required result:

$$\int_{\mathbb{S}^{n-1}} g \sum_{j=1}^{n-1} \phi_j \, \mathrm{s}_k^{ij}(A) \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \left[\operatorname{div} \left(g\phi \left[\mathrm{s}_k^{ij}(A) \right]_j \right) - \left\langle \operatorname{grad} g, \phi \left[\mathrm{s}_k^{ij}(A) \right]_j \right\rangle \right] d\mathcal{H}^{n-1}$$

$$= - \int_{\mathbb{S}^{n-1}} \left\langle \operatorname{grad} g, \phi \left[\mathrm{s}_k^{ij}(A) \right]_j \right\rangle d\mathcal{H}^{n-1} = - \int_{\mathbb{S}^{n-1}} \phi \sum_{j=1}^{n-1} g_j \, \mathrm{s}_k^{ij}(A) \, d\mathcal{H}^{n-1}. \quad \square$$

5.3 Poincaré type inequalities for convex bodies of class $\mathcal{C}^2_+$

The classical Poincaré inequality (with optimal constant) can be established in the following terms (see, for instance, [36, Section 3]):

Theorem 5.3.1 (Poincaré inequality). *Let $\phi \in \mathcal{C}^1(\mathbb{S}^{n-1})$ be such that*

$$\int_{\mathbb{S}^{n-1}} \phi \, d\mathcal{H}^{n-1} = 0. \tag{5.8}$$

Then

$$(n-1) \int_{\mathbb{S}^{n-1}} \phi^2 \, d\mathcal{H}^{n-1} \leq \int_{\mathbb{S}^{n-1}} |\operatorname{grad} \phi|^2 \, d\mathcal{H}^{n-1}. \tag{5.9}$$

In this section we will get two Poincaré type inequalities for convex bodies which are of class $\mathcal{C}^2_+$. Before stating those theorems, we collect in the following subsection some previous results which will be needed in the proofs of the main ones.

5.3.1 Some preliminary results

We recall that if K is of class $\mathcal{C}^2_+$ then its support function $h_K \in \mathfrak{S}$; and conversely, for any $h \in \mathfrak{S}$ there exists a (unique) convex body K_h of class $\mathcal{C}^2_+$ having h as its support function. We consider, for $k \in \{0, \dots, n-1\}$, the function F_k as defined in Proposition 5.1.4. It is clear that for $\phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $h \in \mathfrak{S}$ and $\varepsilon > 0$ small enough, $h_\lambda = h + \lambda\phi \in \mathfrak{S}$ for $|\lambda| \leq \varepsilon$. Following the analogy with the notation stated in (5.3), we write M_λ^{-1} to denote the matrix $M_\lambda^{-1} = \big((h_\lambda)_{ij} + h_\lambda \delta_{ij}\big)_{ij}$. Notice that for $\lambda = 0$ we get $M_0^{-1} = M_{K_h}^{-1}$. With this notation we prove the following result.

Proposition 5.3.2 ([13]). *Let $k \in \{0, \dots, n-1\}$, $h \in \mathfrak{S}$, $\phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\varepsilon > 0$ be such that $h + \lambda\phi \in \mathfrak{S}$ for every $\lambda \in (-\varepsilon, \varepsilon)$. Let $f(\lambda) = F_k(h_\lambda)$ for $\lambda \in (-\varepsilon, \varepsilon)$. Then*

$$f'(\lambda) = (n-k) \int_{\mathbb{S}^{n-1}} \phi \, \mathrm{s}_{n-k-1}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1}. \tag{5.10}$$

Proof. Clearly we have

$$f'(\lambda) = \int_{\mathbb{S}^{n-1}} \frac{d}{d\lambda}\left[h_\lambda \, \mathrm{s}_{n-k-1}(M_\lambda^{-1})\right] d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \left[\phi \, \mathrm{s}_{n-k-1}(M_\lambda^{-1}) + h_\lambda \frac{d}{d\lambda}\mathrm{s}_{n-k-1}(M_\lambda^{-1})\right] d\mathcal{H}^{n-1}.$$

For the sake of brevity we denote by $m_{ij}^\lambda = (h_\lambda)_{ij} + h_\lambda \delta_{ij}$ the entries of the matrix M_λ^{-1}. Since the elementary symmetric functions of M_λ^{-1} are functions of its entries m_{ij}^λ (see e.g. (5.6)), we can just apply the chain-rule to get that

$$\frac{d}{d\lambda}\mathrm{s}_{n-k-1}(M_\lambda^{-1}) = \sum_{i,j=1}^{n-1} \frac{d}{d\, m_{ij}^\lambda}\mathrm{s}_{n-k-1}(M_\lambda^{-1}) \frac{d\, m_{ij}^\lambda}{d\lambda} = \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1})(\phi_{ij} + \phi\delta_{ij}), \tag{5.11}$$

and thus

$$\begin{aligned} f'(\lambda) &= \int_{\mathbb{S}^{n-1}} \left[\phi \, \mathrm{s}_{n-k-1}(M_\lambda^{-1}) + h_\lambda \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1})(\phi_{ij} + \phi\delta_{ij})\right] d\mathcal{H}^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} \phi \, \mathrm{s}_{n-k-1}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} h_\lambda \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1})\phi_{ij} \, d\mathcal{H}^{n-1} \\ &\quad + \int_{\mathbb{S}^{n-1}} h_\lambda \phi \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1})\delta_{ij} \, d\mathcal{H}^{n-1}. \end{aligned} \tag{5.12}$$

Applying Lemma 5.2.2 twice to the second integral of the sum in (5.12) we obtain

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} h_\lambda \sum_{i,j=1}^{n-1} \phi_{ij} \, \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1} &= -\int_{\mathbb{S}^{n-1}} \phi_i \sum_{i,j=1}^{n-1} (h_\lambda)_j \, \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} (h_\lambda)_{ij} \, \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1}. \end{aligned} \tag{5.13}$$

On the other hand, using the expression of the elementary symmetric function of a matrix given by (5.2) we get

$$\sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1}) \, m_{ij}^\lambda = (n-k-1)\mathrm{s}_{n-k-1}(M_\lambda^{-1}). \tag{5.14}$$

Then using (5.13) and (5.14) in (5.12) we obtain finally

$$\begin{aligned} f'(\lambda) &= \int_{\mathbb{S}^{n-1}} \phi \, \mathrm{s}_{n-k-1}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1})(h_\lambda)_{ij} \, d\mathcal{H}^{n-1} \\ &\qquad + \int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1}) h_\lambda \delta_{ij} \, d\mathcal{H}^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} \phi \, \mathrm{s}_{n-k-1}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1})\left[(h_\lambda)_{ij} + h_\lambda\delta_{ij}\right] d\mathcal{H}^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} \phi \, \mathrm{s}_{n-k-1}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1} + (n-k-1)\int_{\mathbb{S}^{n-1}} \phi \, \mathrm{s}_{n-k-1}(M_\lambda^{-1}) \, d\mathcal{H}^{n-1}. \end{aligned}$$

□

The next result is a straightforward consequence of Proposition 5.3.2.

Proposition 5.3.3 ([13]). *Let $k \in \{0, \dots, n-1\}$, $h \in \mathfrak{S}$, $\phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\varepsilon > 0$ be such that $h + \lambda\phi \in \mathfrak{S}$ for every $\lambda \in (-\varepsilon, \varepsilon)$. Let $f(\lambda) = F_k(h_\lambda)$ for $\lambda \in (-\varepsilon, \varepsilon)$. Then*

$$f''(0) = (n-k) \int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_{K_h}^{-1})(\phi_{ij} + \phi\delta_{ij})\, d\mathcal{H}^{n-1}. \tag{5.15}$$

Proof. Differentiating (5.10) and using again (5.11) we get

$$f''(\lambda) = \int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \mathrm{s}_{n-k-1}^{ij}(M_\lambda^{-1})(\phi_{ij} + \phi\delta_{ij})\, d\mathcal{H}^{n-1}$$

and substituting by $\lambda = 0$ we obtain immediately the required value for $f''(0)$. □

Finally we establish a lemma which will be needed in the proof of the second main result of this section.

Lemma 5.3.4 ([13]). *Let $K \in \mathcal{K}^n$ be of class $\mathcal{C}^2_+$, $\phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\psi = \phi \circ N_K$ on $\operatorname{bd} K$. Let $k \in \{1, \dots, n-1\}$. For each $u \in \mathbb{S}^{n-1}$, if we write $x = N_K^{-1}(u)$ then*

$$\frac{1}{\det M_K^{-1}} \left\langle \big(\mathrm{s}_k^{ij}(M_K^{-1})\big)_{ij} \operatorname{grad}\phi, \operatorname{grad}\phi \right\rangle (u) = \left\langle M_K^{-1} \operatorname{grad}\psi, \big(\mathrm{s}_{n-k}^{ij}(M_K)\big)_{ij} \operatorname{grad}\psi \right\rangle (x).$$

Proof. We fixed $u \in \mathbb{S}^{n-1}$ and let $T_u\mathbb{S}^{n-1}$ be the tangent space of $\mathbb{S}^{n-1}$ at u, i.e., the $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$ that is orthogonal to u. Then we can choose an ortonormal basis $\{e_1, \dots, e_{n-1}\}$ of $T_u\mathbb{S}^{n-1}$ such that $M_K^{-1}(u)$ is diagonal: it is enough to consider the eigenvectors of dN_K^{-1}, with corresponding eigenvalues the principal radii of curvature. Throughout this proof everything will be computed with respect to the above fixed basis.

Thus we can suppose that $M_K^{-1}(u)$ is diagonal, say $M_K^{-1}(u) = \operatorname{diag}(\lambda_1, \dots, \lambda_{n-1})$, and then

$$M_K(x) = \operatorname{diag}\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_{n-1}}\right).$$

In particular

$$\operatorname{grad}\psi(x) = M_K(x)\operatorname{grad}\phi(u) = \left(\frac{1}{\lambda_1}\phi_1(u), \dots, \frac{1}{\lambda_{n-1}}\phi_{n-1}(u)\right). \tag{5.16}$$

Proposition 5.1.3, parts i) and ii), ensures that the matrix $\big(\mathrm{s}_k^{ij}(M_K^{-1})\big)_{ij}(u)$ is also diagonal and its eigenvalues are given by $\mathrm{s}_{k-1}\big(\operatorname{diag}(\lambda_1, \dots, \widehat{\lambda_l}, \dots, \lambda_{n-1})\big)$, for $l = 1, \dots, n-1$. Moreover, part iii) of Proposition 5.1.3 applied to the matrix $\operatorname{diag}(\lambda_1, \dots, \widehat{\lambda_l}, \dots, \lambda_{n-1})$ allows to write (notice that in this case the order of the matrix is $m = n-2$)

$$\begin{aligned}
\frac{\mathrm{s}_{k-1}\big(\operatorname{diag}(\lambda_1, \dots, \widehat{\lambda_l}, \dots, \lambda_{n-1})\big)}{\det M_K^{-1}(u)} &= \frac{1}{\lambda_l} \frac{\mathrm{s}_{k-1}\big(\operatorname{diag}(\lambda_1, \dots, \widehat{\lambda_l}, \dots, \lambda_{n-1})\big)}{\lambda_1 \cdots \widehat{\lambda_l} \cdots \lambda_{n-1}} \\
&= \frac{1}{\lambda_l} \mathrm{s}_{n-k-1}\left(\operatorname{diag}\left(\frac{1}{\lambda_1}, \dots, \widehat{\frac{1}{\lambda_l}}, \dots, \frac{1}{\lambda_{n-1}}\right)\right)
\end{aligned}$$

Thus we get

$$\begin{aligned}\frac{1}{\det M_K^{-1}}\Big\langle \big(\mathrm{s}_k^{ij}(M_K^{-1})\big)_{ij}\operatorname{grad}\phi,\operatorname{grad}\phi\Big\rangle(u) &= \frac{\sum_{i,j=1}^{n-1}\mathrm{s}_k^{ij}(M_K^{-1})\,\phi_i\phi_j}{\det M_K^{-1}}(u) = \frac{\sum_{i=1}^{n-1}\mathrm{s}_k^{ii}(M_K^{-1})\,\phi_i^2}{\det M_K^{-1}}(u)\\ &= \sum_{i=1}^{n-1}\frac{\mathrm{s}_{k-1}\big(\operatorname{diag}(\lambda_1,\dots,\widehat{\lambda}_i,\dots,\lambda_{n-1})\big)}{\det M_K^{-1}(u)}\,\phi_i^2(u)\\ &= \sum_{i=1}^{n-1}\frac{1}{\lambda_i}\mathrm{s}_{n-k-1}\left(\operatorname{diag}\left(\frac{1}{\lambda_1},\dots,\widehat{\frac{1}{\lambda_i}},\dots,\frac{1}{\lambda_{n-1}}\right)\right)\phi_i^2(u)\\ &= \sum_{i=1}^{n-1}\frac{1}{\lambda_i}\mathrm{s}_{n-k}^{ii}\big(M_K(x)\big)\phi_i^2(u) = \sum_{i,j=1}^{n-1}\frac{1}{\lambda_i}\mathrm{s}_{n-k}^{ij}\big(M_K(x)\big)\phi_i(u)\phi_j(u)\\ &= \Big\langle \operatorname{grad}\phi(u),\big(\mathrm{s}_{n-k}^{ij}\big(M_K(x)\big)\big)_{ij}\operatorname{grad}\psi(x)\Big\rangle = \Big\langle M_K^{-1}\operatorname{grad}\psi,\big(\mathrm{s}_{n-k}^{ij}(M_K)\big)_{ij}\operatorname{grad}\psi\Big\rangle(x),\end{aligned}$$

where the last two identities follow from (5.16). □

5.3.2 The main results

The following theorem is one of the main results in this chapter, providing a Poincaré type inequality on the sphere $\mathbb{S}^{n-1}$; notice that the classical Poincaré inequality (see Theorem 5.3.1) is obtained in the particular case when $K = B^n$.

Theorem 5.3.5 ([13]). *Let $K \in \mathcal{K}^n$ be a convex body of class $\mathcal{C}^2_+$ and let $k \in \{1,\dots,n-1\}$. For every $\phi \in \mathcal{C}^1(\mathbb{S}^{n-1})$, if*

$$\int_{\mathbb{S}^{n-1}}\phi\,\mathrm{s}_k(M_K^{-1})\,d\mathcal{H}^{n-1} = 0 \tag{5.17}$$

then

$$(n-k)\int_{\mathbb{S}^{n-1}}\phi^2\mathrm{s}_{k-1}(M_K^{-1})\,d\mathcal{H}^{n-1} \le \int_{\mathbb{S}^{n-1}}\Big\langle\big(\mathrm{s}_k^{ij}(M_K^{-1})\big)_{ij}\operatorname{grad}\phi,\operatorname{grad}\phi\Big\rangle\,d\mathcal{H}^{n-1}. \tag{5.18}$$

Proof. By standard approximation (see e.g. [55, p. 150]) we may assume that $\phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$. Let $\varepsilon > 0$ be such that $h_K + \lambda\phi \in \mathfrak{S}$ for every $\lambda \in (-\varepsilon,\varepsilon)$. Setting $f(\lambda) = F_{n-k-1}(h_K + \lambda\phi)$ and defining $g(\lambda) = f^{1/(k+1)}(\lambda)$, it follows from Proposition 5.1.4 that g is a concave function in the interval $(-\varepsilon,\varepsilon)$ and so

$$g''(0) = \frac{1}{k+1}\left[\left(\frac{1}{k+1}-1\right)f(0)^{\frac{1}{k+1}-2}f'(0)^2 + f(0)^{\frac{1}{k+1}-1}f''(0)\right] \le 0.$$

Notice that Proposition 5.3.2 ensures that the assumption (5.17) can be rewritten as $f'(0) = 0$ and hence the condition $g''(0) \le 0$ becomes $f(0)^{-k/(k+1)}f''(0) \le 0$. Since

$$f(0) = F_{n-k-1}(h_K) = \frac{1}{n}\binom{n-1}{k}^{-1}\mathrm{W}_{n-k-1}(K) > 0$$

($K \in \mathcal{C}^2_+$ and hence $K \in \mathcal{K}^n_0$), it follows that $f''(0) \leq 0$ and using (5.15) we get

$$\int_{\mathbb{S}^{n-1}} \phi^2 \sum_{i,j=1}^{n-1} \mathrm{s}_k^{ij}(M_K^{-1})\,\delta_{ij}\, d\mathcal{H}^{n-1} \leq -\int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \mathrm{s}_k^{ij}(M_K^{-1})\,\phi_{ij}\, d\mathcal{H}^{n-1}.$$

By Proposition 5.1.3, part v), we have

$$\sum_{i,j=1}^{n-1} \mathrm{s}_k^{ij}(M_K^{-1})\,\delta_{ij} = \sum_{i=1}^{n-1} \mathrm{s}_k^{ii}(M_K^{-1}) = \mathrm{tr}\big(\mathrm{s}_k^{ij}(M_K^{-1})\big)_{ij} = (n-k)\mathrm{s}_{k-1}(M_K^{-1}),$$

and thus the previous inequality can be written as

$$(n-k)\int_{\mathbb{S}^{n-1}} \phi^2 \mathrm{s}_{k-1}(M_K^{-1})\, d\mathcal{H}^{n-1} \leq -\int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \mathrm{s}_k^{ij}(M_K^{-1})\,\phi_{ij}\, d\mathcal{H}^{n-1}.$$

Now we just have to apply Lemma 5.2.2 in the right-hand side of the above inequality to obtain

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \phi_{ij}\, \mathrm{s}_k^{ij}(M_K^{-1})\, d\mathcal{H}^{n-1} &= -\int_{\mathbb{S}^{n-1}} \sum_{i,j=1}^{n-1} \phi_j\phi_i\, \mathrm{s}_k^{ij}(M_K^{-1})\, d\mathcal{H}^{n-1}\\ &= -\int_{\mathbb{S}^{n-1}} \Big\langle \big(\mathrm{s}_k^{ij}(M_K^{-1})\big)_{ij}\, \mathrm{grad}\,\phi, \mathrm{grad}\,\phi \Big\rangle\, d\mathcal{H}^{n-1},\end{aligned}$$

which concludes the proof. □

The type of hypothesis as the one given in (5.17) is usually called the *mean-zero condition* of the problem.

With the help of Lemma 5.2.1 we can establish a Poincaré type inequality, analogous to the one in Theorem 5.3.5, now on the border of a convex body K of class $\mathcal{C}^2_+$.

Theorem 5.3.6 ([13]). *Let $K \in \mathcal{K}^n$ be a convex body of class $\mathcal{C}^2_+$ and let $k \in \{1,\dots,n-1\}$. For every $\psi \in \mathcal{C}^1(\partial K)$, if*

$$\int_{\mathrm{bd}\,K} \psi\, \mathrm{s}_{k-1}(M_K)\, d\mathcal{H}^{n-1} = 0 \tag{5.19}$$

then

$$k\int_{\mathrm{bd}\,K} \psi^2 \mathrm{s}_k(M_K)\, d\mathcal{H}^{n-1} \leq \int_{\mathrm{bd}\,K} \Big\langle \big(\mathrm{s}_k^{ij}(M_K)\big)_{ij}\, \mathrm{grad}\,\psi, M_K^{-1}\, \mathrm{grad}\,\psi \Big\rangle\, d\mathcal{H}^{n-1}. \tag{5.20}$$

Proof. Let $\phi = \psi \circ N_K^{-1}$ on $\mathbb{S}^{n-1}$. Notice that the Jacobian of the inverse of Weingarten map dN_K^{-1} is given by $\det M_K^{-1}(u) > 0$, for all $u \in \mathbb{S}^{n-1}$. Moreover, by part iii) of Proposition 5.1.3 we have

$$\mathrm{s}_k\Big(M_K\big(N_K^{-1}(u)\big)\Big) = \frac{\mathrm{s}_{n-k-1}\big(M_K^{-1}(u)\big)}{\det M_K^{-1}(u)},$$

for all $u \in \mathbb{S}^{n-1}$ and for every $k \in \{1, \dots, n-1\}$. Hence applying the change of variable $\phi = \psi \circ N_K^{-1}$ we get that the integrals

$$\begin{aligned} \int_{\operatorname{bd} K} \psi \, \mathrm{s}_{k-1}(M_K) \, d\mathcal{H}^{n-1} &= \int_{\mathbb{S}^{n-1}} \phi \, \mathrm{s}_{n-k}(M_K^{-1}) \, d\mathcal{H}^{n-1}, \\ \int_{\operatorname{bd} K} \psi^2 \, \mathrm{s}_k(M_K) \, d\mathcal{H}^{n-1} &= \int_{\mathbb{S}^{n-1}} \phi^2 \, \mathrm{s}_{n-k-1}(M_K^{-1}) \, d\mathcal{H}^{n-1}. \end{aligned} \tag{5.21}$$

On the other hand, using the same change of variable, Lemma 5.3.4 ensures that

$$\int_{\operatorname{bd} K} \left\langle M_K^{-1} \operatorname{grad} \psi, \left(\mathrm{s}_k^{ij}(M_K)\right)_{ij} \operatorname{grad} \psi \right\rangle d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \left\langle \left(\mathrm{s}_{n-k}^{ij}(M_K^{-1})\right)_{ij} \operatorname{grad} \phi, \operatorname{grad} \phi \right\rangle d\mathcal{H}^{n-1}.$$

From the first identity in (5.21), our condition (5.19) gives the corresponding hypothesis (5.17) in Theorem 5.3.5. So we can apply this theorem, second equality in (5.21) and the above relation to get the final result:

$$\begin{aligned} k \int_{\operatorname{bd} K} \psi^2 \mathrm{s}_k(M_K) \, d\mathcal{H}^{n-1} &= k \int_{\mathbb{S}^{n-1}} \phi^2 \, \mathrm{s}_{n-k-1}(M_K^{-1}) \, d\mathcal{H}^{n-1} \\ &\leq \int_{\mathbb{S}^{n-1}} \left\langle \left(\mathrm{s}_{n-k}^{ij}(M_K^{-1})\right)_{ij} \operatorname{grad} \phi, \operatorname{grad} \phi \right\rangle d\mathcal{H}^{n-1} \\ &= \int_{\operatorname{bd} K} \left\langle M_K^{-1} \operatorname{grad} \psi, \left(\mathrm{s}_k^{ij}(M_K)\right)_{ij} \operatorname{grad} \psi \right\rangle d\mathcal{H}^{n-1}. \end{aligned}$$

□

5.4 Some Poincaré type inequalities for general convex bodies

In the case $k = 1$, the results of the previous section can be established for general convex bodies (not necessarily of class $\mathcal{C}^2_+$). We prove the following theorem.

Theorem 5.4.1 ([13]). *Let $K \in \mathcal{K}^n_0$. For every $\phi \in \mathcal{C}^1(\mathbb{S}^{n-1})$, if*

$$\int_{\mathbb{S}^{n-1}} \phi \, d\mathrm{S}_1(K; \cdot) = 0 \tag{5.22}$$

then

$$\int_{\mathbb{S}^{n-1}} \phi^2 \, d\mathcal{H}^{n-1} \leq \frac{1}{n-1} \int_{\mathbb{S}^{n-1}} |\operatorname{grad} \phi|^2 \, d\mathcal{H}^{n-1}.$$

Proof. Proposition 1.1.4 ensures that K can be approximated (in the Hausdorff metric) by a sequence $(K_m)_{m \in \mathbb{N}}$ of convex bodies of class $\mathcal{C}^2_+$. For fixed $m \in \mathbb{N}$ we write $h_m = h_{K_m}$ to denote the support function of K_m.

By standard approximation (see, for instance, [55, p. 150]) we may assume that the function $\phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$. Let $\varepsilon > 0$ be small enough such that $h_m + \lambda\phi \in \mathfrak{S}$ for $|\lambda| \leq \varepsilon$. As usual, we denote by $(M^m_\lambda)^{-1}$ the matrix

$$(M^m_\lambda)^{-1} = \left((h_m + \lambda\phi)_{ij} + (h_m + \lambda\phi)\delta_{ij}\right)_{ij}$$

and we consider the function

$$f_m(\lambda) = \int_{\mathbb{S}^{n-1}} (h_m + \lambda\phi)\mathrm{s}_1\big((M^m_\lambda)^{-1}\big)\, d\mathcal{H}^{n-1}.$$

By Proposition 5.1.4, $f_m^{1/2}$ is concave and hence

$$2f_m(0)f''_m(0) - f'_m(0)^2 \leq 0. \tag{5.23}$$

Since K_m is of class $\mathcal{C}^2_+$ we know by (5.5) and Proposition 5.3.2 that

$$\begin{aligned} f_m(0) &= \int_{\mathbb{S}^{n-1}} h_m\, \mathrm{s}_1(M^{-1}_{K_m})\, d\mathcal{H}^{n-1} = n(n-1)\mathrm{W}_{n-2}(K), \\ f'_m(0) &= 2\int_{\mathbb{S}^{n-1}} \phi\, \mathrm{s}_1(M^{-1}_{K_m})\, d\mathcal{H}^{n-1}. \end{aligned}$$

Moreover, using that $\mathrm{s}_1^{ij}(M^{-1}_{K_m}) = \delta_{ij}$ for all $i, j \in \{1, \dots, n-1\}$ (see Remark 5.1) and by Proposition 5.3.3, we get

$$\begin{aligned} f''_m(0) &= 2\int_{\mathbb{S}^{n-1}} \phi \sum_{i,j=1}^{n-1} \mathrm{s}_1^{ij}(M^{-1}_{K_m})(\phi_{ij} + \phi\delta_{ij})\, d\mathcal{H}^{n-1} = 2\int_{\mathbb{S}^{n-1}} \phi \sum_{i=1}^{n-1}(\phi_{ii} + \phi)\, d\mathcal{H}^{n-1} \\ &= 2\int_{\mathbb{S}^{n-1}} \phi \left((n-1)\phi + \sum_{i=1}^{n-1}\phi_{ii}\right) d\mathcal{H}^{n-1}. \end{aligned}$$

Substituting the above expressions for $f_m(0), f'_m(0), f''_m(0)$ in (5.23) we obtain

$$n(n-1)\mathrm{W}_{n-2}(K_m)\int_{\mathbb{S}^{n-1}} \phi \left((n-1)\phi + \sum_{i=1}^{n-1}\phi_{ii}\right) d\mathcal{H}^{n-1} \leq \left(\int_{\mathbb{S}^{n-1}} \phi\, \mathrm{s}_1(M^{-1}_{K_m})\, d\mathcal{H}^{n-1}\right)^2.$$

Now, according to (5.4),

$$\int_{\mathbb{S}^{n-1}} \phi\, \mathrm{s}_1(M^{-1}_{K_m})\, d\mathcal{H}^{n-1} = (n-1)\int_{\mathbb{S}^{n-1}} \phi\, d\mathrm{S}_1(K_m; \cdot),$$

and thus

$$n(n-1)\mathrm{W}_{n-2}(K_m)\int_{\mathbb{S}^{n-1}} \phi \left((n-1)\phi + \sum_{i=1}^{n-1}\phi_{ii}\right) d\mathcal{H}^{n-1} \leq (n-1)^2 \left(\int_{\mathbb{S}^{n-1}} \phi\, d\mathrm{S}_1(K_m; \cdot)\right)^2. \tag{5.24}$$

On the other hand, Proposition 1.2.5, part viii), ensures that $\big(\mathrm{W}_{n-2}(K_m)\big)_{m\in\mathbb{N}}$ converges to the value $\mathrm{W}_{n-2}(K)$ as m tends to infinity (continuity of the mixed volumes), as well as the sequence of measures $\big(\mathrm{S}_1(K_m; \cdot)\big)_{m\in\mathbb{N}}$ converges weakly to $\mathrm{S}_1(K; \cdot)$, which means that

$$\lim_{m\to\infty} \int_{\mathbb{S}^{n-1}} \phi\, d\mathrm{S}_1(K_m; \cdot) = \int_{\mathbb{S}^{n-1}} \phi\, d\mathrm{S}_1(K; \cdot).$$

Thus, letting $m \to \infty$ in (5.24) we get

$$n\mathrm{W}_{n-2}(K) \int_{\mathbb{S}^{n-1}} \phi \left((n-1)\phi + \sum_{i=1}^{n-1} \phi_{ii} \right) \, d\mathcal{H}^{n-1} \leq (n-1) \left(\int_{\mathbb{S}^{n-1}} \phi \, d\mathrm{S}_1(K;\cdot) \right)^2 = 0,$$

where the last integral vanishes because of our hypothesis (5.22). Moreover, since K has non-empty interior then $\mathrm{W}_{n-2}(K) > 0$ and we get

$$(n-1) \int_{\mathbb{S}^{n-1}} \phi^2 \, d\mathcal{H}^{n-1} \leq - \int_{\mathbb{S}^{n-1}} \phi \sum_{i=1}^{n-1} \phi_{ii} \, d\mathcal{H}^{n-1}. \tag{5.25}$$

On the other hand, by (5.1) we have

$$\langle \operatorname{grad} \phi, \operatorname{grad} \phi \rangle = \operatorname{div}(\phi \operatorname{grad} \phi) - \phi \operatorname{div}(\operatorname{grad} \phi) = \operatorname{div}(\phi \operatorname{grad} \phi) - \phi \sum_{i=1}^{n-1} \phi_{ii},$$

and taking integrals and using the divergence Theorem 5.1.2 we obtain

$$\int_{\mathbb{S}^{n-1}} |\operatorname{grad} \phi|^2 \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \left[\operatorname{div}(\phi \operatorname{grad} \phi) - \phi \sum_{i=1}^{n-1} \phi_{ii} \right] d\mathcal{H}^{n-1} = - \int_{\mathbb{S}^{n-1}} \phi \sum_{i=1}^{n-1} \phi_{ii} \, d\mathcal{H}^{n-1}.$$

Together with (5.25) it proves the result. □

In order to apply this kind of results it would be useful to understand when a Borel measure μ on $\mathbb{S}^{n-1}$ is the 1-st order surface area measure of some convex body. This problem is known as the *Christoffel problem*, and we will not deal here with this question, for which we refer, for instance, to [49, Section 4.3].

We just would like to mention that necessary and sufficient conditions for a measure μ to be the 1-st surface area measure of some convex body were obtained by Firey [17] and Berg [2] (see also [49, Section 4.3]), but they are not easy to use in practice. A considerable progress (in a larger class of problems) has been made by Guan and Ma in [22] and Sheng, Trudinger and Wang in [51], where a rather simple sufficient condition is found:

Theorem 5.4.2. *Let $f \in \mathcal{C}^1(\mathbb{S}^{n-1})$, $f > 0$, having Lipschitz first derivatives, and let $g = 1/f$. If*

$$\int_{\mathbb{S}^{n-1}} u f(u) \, d\mathcal{H}^{n-1}(u) = 0$$

and the matrix $(g_{ij} + g\delta_{ij})_{ij}$ is positive semi-definite on $\mathbb{S}^{n-1}$, then there exists a convex body K, uniquely determined up to translations, such that

$$\mathrm{S}_1(K;\cdot) = f\mathcal{H}^{n-1},$$

i.e., f is the density *of the measure $\mathrm{S}_1(K;\cdot)$ with respect to $\mathcal{H}^{n-1}$.*

In order to finish this chapter we intend to prove a new Poincaré type inequality for general convex bodies, but now with mean-zero condition in terms of their radial function (see Definition 1.1.9). It will be obtained as a consequence of our previous Theorem 5.4.1 and Theorem 5.4.2, which will play a crucial role in the proof.

Theorem 5.4.3 ([13]). *Let $K \in \mathcal{K}_0^n$ be a convex body with $0 \in \operatorname{int} K$. If*

$$\int_{\mathbb{S}^{n-1}} u\rho_K(u)\, d\mathcal{H}^{n-1}(u) = 0 \tag{5.26}$$

and for every $\phi \in \mathcal{C}^1(\mathbb{S}^{n-1})$ it holds that

$$\int_{\mathbb{S}^{n-1}} \phi\rho_K\, d\mathcal{H}^{n-1} = 0, \tag{5.27}$$

then

$$\int_{\mathbb{S}^{n-1}} \phi^2 d\mathcal{H}^{n-1} \leq \frac{1}{n-1}\int_{\mathbb{S}^{n-1}} |\operatorname{grad}\phi|^2\, d\mathcal{H}^{n-1}.$$

Proof. In order to prove this theorem it suffices to show that if K is a convex body with $0 \in \operatorname{int} K$ verifying (5.26), then there exists another convex body $\overline{K} \in \mathcal{K}^n$ such that ρ_K is the density of $\mathrm{S}_1(\overline{K};\cdot)$ with respect to $\mathcal{H}^{n-1}$ on $\mathbb{S}^{n-1}$, i.e., such that $\mathrm{S}_1(\overline{K};\cdot) = \rho_K\mathcal{H}^{n-1}$. In this case, condition (5.27) can be rewritten as

$$\int_{\mathbb{S}^{n-1}} \phi\, d\mathrm{S}_1(\overline{K};\cdot) = 0$$

for every $\phi \in \mathcal{C}^1(\mathbb{S}^{n-1})$, and hence, Theorem 5.4.1 ensures that

$$\int_{\mathbb{S}^{n-1}} \phi^2\, d\mathcal{H}^{n-1} \leq \frac{1}{n-1}\int_{\mathbb{S}^{n-1}} |\operatorname{grad}\phi|^2\, d\mathcal{H}^{n-1}.$$

Thus in order to conclude the proof we have to show the existence of such a set $\overline{K}$.

We know that the radial function of the convex body K is related with the support function of its dual K° by means of the identity (1.1), namely,

$$\rho_K = \frac{1}{h_{K^\circ}}, \qquad \text{on } \mathbb{S}^{n-1}.$$

Let $(K_m)_{m\in\mathbb{N}}$ be a sequence of convex bodies of class $\mathcal{C}^2_+$, with $0 \in \operatorname{int} K_m$ for all $m \in \mathbb{N}$, converging to K° in the Hausdorff metric. In particular, setting $h_m = h_{K_m}$, we have that $h_m \to h_{K^\circ}$ uniformly on $\mathbb{S}^{n-1}$. Moreover, we have the following properties:

i) since h_m is the support function of a convex body containing the origin in its interior, it is clear that $1/h_m > 0$ for all $m \in \mathbb{N}$;

ii) since K_m is of class $\mathcal{C}^2_+$ we can assure that for every $m \in \mathbb{N}$

$$\big((h_m)_{ij} + h_m\delta_{ij}\big)_{ij} > 0 \quad \text{on } \mathbb{S}^{n-1}; \tag{5.28}$$

iii) moreover, $1/h_m \in \mathcal{C}^2(\mathbb{S}^{n-1})$ and in particular, it has Lipschitz first derivatives;

iv) by (5.26) it is possible to construct the sequence K_m satisfying that

$$\int_{\mathbb{S}^{n-1}} u \frac{1}{h_m(u)}\, d\mathcal{H}^{n-1}(u) = 0 \quad \text{for all } m \in \mathbb{N}.$$

The above properties i), ii), iii), iv) ensure that we can apply Theorem 5.4.2 to the functions $f = 1/h_m$ and $g = h_m$ for every $m \in \mathbb{N}$, obtaining convex bodies $\overline{K}_m$ such that

$$\mathrm{S}_1(\overline{K}_m; \cdot) = \frac{1}{h_m}\mathcal{H}^{n-1}. \tag{5.29}$$

On the other hand, since K° is a convex body containing the origin in its interior, its support function is bounded up and below by two strictly positive constants; or analogously, we have that $c_1 < 1/h_{K^\circ} < c_2$ for suitable constants $c_1, c_2 > 0$. Then, since K_m are also convex bodies containing the origin in its interior for all $m \in \mathbb{N}$, and using the uniform convergence of $h_m \to h_{K^\circ}$, we can assure the existence of another two constants $\overline{c}_1, \overline{c}_2 > 0$ (not depending on m) such that

$$\overline{c}_1 \leq \frac{1}{h_m} \leq \overline{c}_2 \quad \text{on } \mathbb{S}^{n-1}, \ \text{ for all } m \in \mathbb{N}.$$

Thus we have obtained a sequence of convex bodies $(\overline{K}_m)_{m\in\mathbb{N}}$ such that, for each $m \in \mathbb{N}$, the associated 1-st order surface area measure of $\overline{K}_m$ is given by the function $1/h_m$ (cf. (5.29)) which is bounded up and below. Then by [22, Lemma 3.1] we get that the support function of $\overline{K}_m$, $h_{\overline{K}_m}$, is also bounded for all $m \in \mathbb{N}$. Hence, the sequence $\left(\overline{K}_m\right)_{m\in\mathbb{N}}$ is bounded, and Blaschke selection Theorem 1.1.12 ensures that, up to a subsequence, it converges to a convex body $\overline{K}$ in the Hausdorff metric. Hence, Proposition 1.2.5, part viii), ensures that the sequence of measures $\mathrm{S}_1(\overline{K}_m; \cdot)$ converges weakly to $\mathrm{S}_1(\overline{K}; \cdot)$ as m tends to infinity. Consequently, and by (5.29) we get

$$\mathrm{S}_1(\overline{K}; \cdot) = \frac{1}{h_{K^\circ}}\mathcal{H}^{n-1} = \rho_K \mathcal{H}^{n-1}.$$

Thus we have obtained a convex body $\overline{K}$ verifying the required properties, which concludes the proof of the theorem. □

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