Domination by second countable spaces

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Starting point

A characterization of metrizability

Exercise. . . from Engelking's book

4.2.B (Šneĭder [1945]). Show that a compact space X is metrizable if and only if the diagonal Δ is a G_δ-set in the Cartesian product $X \times X$ (see Problem 3.12.22(e); cf. Problem 4.5.15 and Exercise 5.1.I).

Kind of problems studied

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ subsets of $(K \times K) \setminus \Delta$. We write:

(A) each A_α is compact;

(B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;

(C) $(K \times K) \setminus \Delta = \bigcup \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.

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Theorem (Orihuela, B.C. 1987)

(A) + **(B)** + **(C)** + **(D)** $\Rightarrow$ *K is metrizable.*

(D) *For each compact set $F \subset (K \times K) \setminus \Delta$, there is A_α such that $F \subset A_\alpha$.*

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Open question

(A) + **(B)** + **(C)** $\overset{?}{\Rightarrow}$ K is metrizable.

The references

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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **126**, 1–10 (1987)

On a Theorem of Choquet and Dolecki

IWO LABUDA*

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Submitted by Ky Fan

Received June 6, 1985

The results of this paper clarify and extend slightly the previous work of Dolecki and Lechicki (*C. R. Acad. Sci. Paris* **293** (1981), 219–221; *J. Math. Anal. Appl.* **88** (1982), 547–584) and Hansell, Jayne, Rogers and the author (*Math. Z.* **189** (1985), 297–318). Let X, Y be Hausdorff spaces and $F\colon X \to Y$ an upper semicontinuous set-valued map. A subset K of $F(x)$ is said to be a peak of F at x, if, for every open set V containing K, there exists a neighbourhood U of x such that $F(U) \setminus F(x) \subset V$. Criteria ("Choquet–Dolecki Theorems") are given in order that F has the smallest possible peak. It turns out that in unexpectedly general situations an upper semicontinuous map F has, for every x in X, a peak which is the smallest possible at x and moreover compact.

1. THE RESULTS

Let X, Y be Hausdorff spaces, and let $F\colon X \to Y$ be a set-valued map. In what follows the term "map" means always such a set-valued map.

A *cap* (of upper semicontinuity) of F at x_0 is a set K in Y such that the

The references

ROCKY MOUNTAIN
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INHERENT COMPACTNESS OF UPPER CONTINUOUS SET VALUED MAPS

BRIAN L. DAVIS AND IWO LABUDA

The following theorem is due to Cascales and Orihuela [**8**], see also [**7**].

3.4. Theorem. *The following are equivalent for $\mathcal{B}$ with a countable base.*

(i) *For each $(y_n) \geq \mathcal{B}$, the closure $\overline{\{y_n : n \in \mathbf{N}\}}$ is countably compact.*

(ii) *$\mathcal{B}$ is countably compact at $\operatorname{adh}_\omega \mathcal{B}$ which itself is countably compact.*

The proof given here follows [**26**] rather than [**8**]. What is really interesting though, is the link between this result and the Vaĭnšteĭn-Choquet-Dolecki theorem. Cascales and Orihuela knew [**22**], but their applications do not go beyond points of countable character treated in [**25**]. On the other hand, a part of the arguments in [**26**] could have been skipped using the above theorem.

Notation

- $L, M, \ldots X, Y, \ldots$ topological spaces; E, F Banach or sometimes lcs;
- K compact Hausdorff space;
- 2^X subsets; $\mathscr{K}(X)$ family of compact sets;
- $C(X)$ continuous functions; $C_p(X)$ continuous functions endowed with the pointwise convergence topology τ_p;

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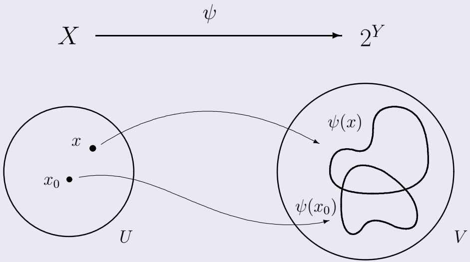
- $C(X)$ continuous functions; $C_p(X)$ continuous functions endowed with the pointwise convergence topology τ_p;
- $\Omega \subset \mathbb{C}$ open set; $\mathscr{H}(\Omega)$ space of holomorphic functions with the topology of uniform convergence on compact sets;
- $\Omega \subset \mathbb{R}^n$ open set; $\mathscr{D}'(\Omega)$ space of distributions;
- $\varinjlim E_n$ inductive limit of a sequence of Fréchet spaces.

Definitions

Upper semi-continuous set-valued map (multi-function)



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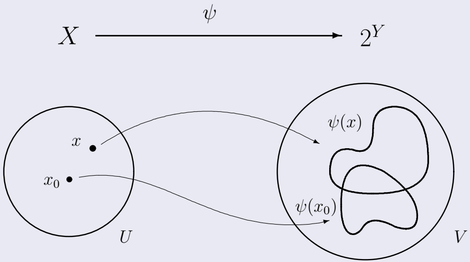

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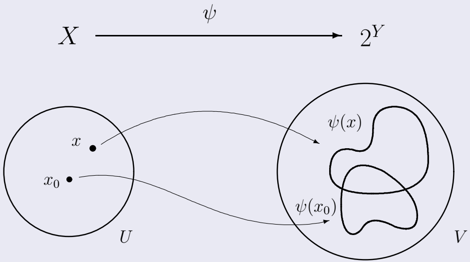

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2. Y is countably K-determined if there is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and $\psi : \Sigma \to 2^Y$ that is upper semi-continuous compact-valued and such that $Y = \bigcup_{\alpha \in \Sigma} \psi(\alpha)$.

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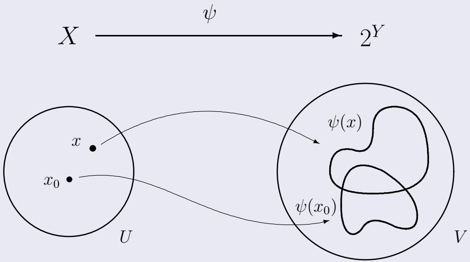

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$\mathbb{N}^{\mathbb{N}} \Leftrightarrow$ any Polish space P

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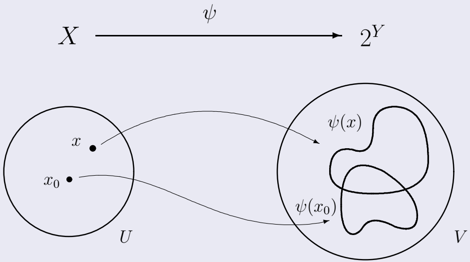

Σ ⇔ any second countable space M (Lindelöf Σ)

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Well known facts

1. If $\psi : X \to 2^Y$ that is upper semi-continuous compact-valued, then $K \subset X$ is compact $\Rightarrow$ $\psi(K)$ is compact;

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3. K-analytic $\Rightarrow$ countably K-determined $\Rightarrow$ Lindelöf;
4. countably K-determined + metrizable $\Rightarrow$ separable;
5. if X is K-analytic ($\psi : \mathbb{N}^{\mathbb{N}} \to 2^X$) and $A_\alpha := \psi(\{\beta : \beta \leq \alpha\})$ then:

   (A) each A_α is compact;
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6. ditto, if X is countably K-determined, there is a second countable space M and a family $\{A_K : K \in \mathscr{K}(M)\}$ such that:
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A topological space X is dominated by a Polish space, if there is a Polish space P and a family $\{A_K : K \in \mathscr{K}(P)\} \subset X$ such that:

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Proposition

For a topological space X the TFAE:

1. X is dominated by a Polish space;
2. There is a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of X with:
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K-analyticity implies domination by Polish spaces

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If X is K-analytic ($\psi : \mathbb{N}^{\mathbb{N}} \to 2^X$) and $A_\alpha := \psi(\{\beta : \beta \leq \alpha\})$ then:

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The question

- When domination by Polish spaces implies K-analyticity?
- How useful is a positive answer to the above?

A nice old case

Talagrand, Ann. of Math. 1979

Annals of Mathematics, 110 (1979), 407-438

Espaces de Banach faiblement $\mathcal{K}$-analytiques

Par Michel Talagrand

PROPOSITION 6.13. *Soit K un espace compact. Les assertions suivantes sont équivalentes:*

a). *K est de type $\mathcal{E}_1$.*

b). *Il existe une application croissante $\sigma \to A_\sigma$ de Σ (muni de l'ordre produit) dans l'ensemble des compacts de $\mathcal{C}_s(K)$ telle que $\bigcup_{\sigma\in\Sigma} A_\sigma$ sépare les points de K.*

Démonstration. Nous savons déjà que a) implique b) l'application

Another case with domination

Valdivia, J. London Math. Soc. 1987

QUASI-LB-SPACES

MANUEL VALDIVIA

We shall see later that properties (1) and (2) are important in order to obtain some results on the closed graph theorem. This is the reason for introducing the following definitions. *A quasi-*LB*-representation in a topological vector space* F *is a family* $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *of Banach discs satisfying the following conditions:*

1. $\bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = F$;
2. *if* $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ *and* $\alpha \leqslant \beta$ *then* $A_\alpha \subset A_\beta$.

Domination by Polish implies (many times) K-analyticity

- Let X be a topological space $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of X with:
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- Given $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ and $m \in \mathbb{N}$, define

$$\alpha|_m := (n_1, n_2, \ldots, n_m).$$

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Proposition, B. C., 1987

Given X and $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ as above, if we define $\psi : \mathbb{N}^\mathbb{N} \to 2^X$ given by

$$\psi(\alpha) := \bigcap_{m=1}^{\infty} \bigcup \{A_\beta : \beta|_m = \alpha|_m\}$$

then:

- each $\psi(\alpha)$ is countably compact (even more, all cluster points of any sequence in $\psi(\alpha)$ remain in $\psi(\alpha)$).
- if $\psi(\alpha)$ is compact then $\alpha \to \psi(\alpha)$ gives K-analytic structure to X.

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X has K-analytic structure if *countably compact subsets=compact subsets*.

Talagrand's solution to a conjecture Corson

Theorem, Talagrand 1975, Bull. Sci. Math.

Every WCG Banach space E is weakly Lindelöf.

Proof.-

- Fix $W \subset E$ absolutely convex w-compact with $E = \overline{\operatorname{span} W}$.
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$,

$$A_\alpha := (n_1 W + B_{E^{**}}) \cap (n_2 W + \frac{1}{2} B_{E^{**}}) \cap \cdots \cap (n_1 W + \frac{1}{k} B_{E^{**}}) \cap \ldots$$

- Proposition $\Rightarrow (E, w)$ K-analytic $\Rightarrow (E, w)$ Lindelöf. ♣

Fréchet-Montel spaces

Theorem, Dieudonné 1954

Every Fréchet-Montel space E is separable (in particular $\mathscr{H}(\Omega)$ is separable).

Proof.-

- Fix $V_1 \supset V_2 \supset \cdots \supset V_n \ldots$ a basis of closed neighborhoods of 0.
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$,

$$A_\alpha := \bigcap_{k=1}^{\infty} n_k V_k.$$

- $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ fundamental family of bdd closed sets=compact;
- Proposition $\Rightarrow$ E K-analytic +metrizable $\Rightarrow$ E Lindelöf + metrizable $\Rightarrow$ E separable. ♣

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The strong dual of every inductive limit of Fréchet-Montel spaces is analytic.

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\begin{array}{ccccccccccc}
 & E_1 & \hookrightarrow & E_2 & \hookrightarrow \cdots \hookrightarrow & E_m & \hookrightarrow & \cdots & \hookrightarrow & E \\
 & U_1^1 & & U_1^2 & & U_1^m & & & & \\
 & \cup & & \cup & & \cup & & & & \\
 & \vdots & & \vdots & & \vdots & & & & \\
 & \cup & & & & \cup & & & & \\
 & & & & & \boxed{U_{n_m}^m} & & & & \\
\alpha = (n_k)_k & \boxed{U_{n_1}^1} & & \vdots & & \cup & & & & \\
 & \cup & & \cup & & \vdots & & & & \\
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 & & & \vdots & & & & & & & &
\end{array}
$$

$$U_\alpha := \overline{\mathrm{aco}\left(\bigcup_{k=1}^{\infty} U_{n_k}^{k}\right)}$$

- $U_\beta \subset U_\alpha$ si $\alpha \leq \beta$; $\mathscr{U} := \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ neigh. basis of 0 en E.
- $A_\alpha := U_\alpha^\circ$ compact & $A_\alpha \subset A_\beta$, $\alpha \leq \beta$;
- $E' = \cup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and E' sub-metrizable $\Rightarrow$ E' K-analytic sub-metrizable $\Rightarrow$ E' analytic.

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\end{array}
$$

Schwartz, 1964

Any Borel linear map from a separable Banach space into $\mathscr{D}'(\Omega)$ is continuous. In particular, the Closed Graph Theorem holds for linear maps

$$T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega).$$

Metrizability of compact sets (I)

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   (B) $B_\alpha \subset B_\beta$ whenever $\alpha \le \beta$;
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5. $(C(K), \|\ \|_\infty)$ is K-analytic +metrizable $\Rightarrow$ E Lindelöf + metrizable $\Rightarrow$ E separable $\Rightarrow$ K is metrizable. ♣

Metrizability of compact sets (I): A different formulation

We didn't stated our result below as presented.

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$(K, \mathfrak{U})$ a compact uniform space with a basis for the uniformity $\mathscr{B}_\mathfrak{U} = \{N_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ satisfying:

$$N_\beta \subset N_\alpha \text{ si } \alpha \leq \beta \text{ whenever } \alpha, \beta \in \mathbb{N}^\mathbb{N}.$$

Then K is metrizable.

Applications

Descriptive Topology in Selected Topics of Functional Analysis

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Applications

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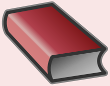

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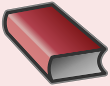

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Proposition, B. C., 1987

Given X and $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ as above, if we define $\psi : \mathbb{N}^\mathbb{N} \to 2^{(K \times K) \setminus \Delta}$ given by

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then:

- each $\psi(\alpha)$ is countably compact (even more, all cluster points of any sequence in $\psi(\alpha)$ remain in $\psi(\alpha)$).
- if $\psi(\alpha)$ is compact then $\alpha \to \psi(\alpha)$ gives K-analytic structure to $(K \times K) \setminus \Delta$.

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$$\overline{\psi(\alpha)} \subset \psi(\alpha), \text{ (closure in } K \times K)$$

- take $x \in \overline{\psi(\alpha)}$;
- there is $A \subset \psi(\alpha)$ countable with $x \in \overline{A}$;
- is $x \in A \Rightarrow x \in \psi(\alpha)$;
- otherwise, $x \in (\overline{A} \setminus A) \Rightarrow x$ is cluster point of a sequence in $\psi(\alpha) \Rightarrow x \in \psi(\alpha)$.

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2. K has small diagonal $\Rightarrow$ K has countable tightness $\Rightarrow$ $K \times K$ has countable tightness;
3. $(K \times K) \setminus \Delta$ is K-analytic $\Rightarrow$ $(K \times K) \setminus \Delta$ is Lindelöf $\Rightarrow$ Δ is G_δ $\Rightarrow$ K is metrizable.

♣

Domination by Second Countable Spaces

Domination by Second Countable Spaces

Definition

A topological space X is dominated by a second countable space, if there is a second countable space M and a family $\{A_K : K \in \mathscr{K}(M)\} \subset X$ such that:

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For a topological space TFAE:

1. *X is countably K-determined;*
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The class of spaces dominated by a second countable space enjoy the usual stability properties we might expect.

Techniques

Generation of usco maps

Let T be a first-countable, X a topological space and let $\varphi : T \to 2^X$ be a set-valued map satisfying the property

$$\bigcup_{n\in\mathbb{N}} \varphi(t_n) \text{ is relatively compact for each convergent sequence } (t_n)_n \text{ in } T. \quad (1)$$

If for each x in X we define

$$C(t) := \{x \in X : \text{there is } t_n \to t \text{ in } T, \text{ for every } n \in \mathbb{N} \text{ there is } x_n \in \varphi(t_n) \text{ and } x \text{ is cluster point of } (x_n)_n\}.$$

Then:

- each $C(t)$ is countably compact.
- if $\psi(t) := \overline{C(t)}$ is compact then $t \to \psi(t)$ is usco $\psi : T \to \mathscr{K}(X)$.

Te...

The following theorem is due to Cascales and Orihuela [8], see also [7].

3.4. Theorem. *The following are equivalent for $\mathcal{B}$ with a countable base.*

(i) *For each* $(y_n) \geq \mathcal{B}$, *the closure* $\overline{\{y_n : n \in \mathbf{N}\}}$ *is countably compact.*

(ii) $\mathcal{B}$ *is countably compact at* $\operatorname{adh}_\omega \mathcal{B}$ *which itself is countably compact.*

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If X is dominated by a second countable space, if there is a second countable space M and a family $\{A_K : K \in \mathscr{K}(M)\}$ such that:

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(C) $X = \bigcup\{A_K : K \in \mathscr{K}(M)\}$.

We take: $T := (\mathscr{K}(M), h)$, $\varphi(K) := A_K$ and we can generate the USCO ψ in many cases.

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Two noticeable results

Theorem (Orihuela, Tkachuk, B.C. 2010)

$C_p(X)$ is countably K-determined iff is dominated by a second countable space.

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Let K be a compact space. If there is a second countable space M and a family $\{A_F : F \in \mathcal{K}(M)\} \subset (K \times K) \setminus \Delta$ such that:

- **(A)** *each A_F is compact;*
- **(B)** *$A_F \subset A_L$ whenever $F \subset L$;*
- **(C)** *$(K \times K) \setminus \Delta = \bigcup\{A_F : F \in \mathcal{K}(M)\}$.*

and

- **(D)** *every compact subset of $(K \times K) \setminus \Delta$ is contained in some A_F.*

Then K is metrizable.

Open questions

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K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \subset (K \times K) \setminus \Delta$.

We write:

(A) each A_α is compact;

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Open question

(A) + **(B)** + **(C)** $\overset{?}{\Rightarrow}$ K is metrizable.

Open questions

More problems...here!

Domination by second countable spaces and Lindelöf Σ-property

B. Cascales[1,2], J. Orihuela[1,2] and V.V. Tkachuk[3,4]

Abstract. Given a space M, a family of sets $\mathcal{A}$ of a space X is *ordered by* M if $\mathcal{A}=\{A_K:K$ is a compact subset of $M\}$ and $K\subset L$ implies $A_K\subset A_L$. We study the class $\mathcal{M}$ of spaces which have compact covers ordered by a second countable space. We prove that a space $C_p(X)$ belongs to $\mathcal{M}$ if and only if it is a Lindelöf Σ-space. Under $\mathrm{MA}(\omega_1)$, if X is compact and $(X\times X)\backslash\Delta$ has a compact cover ordered by a Polish space then X is metrizable; here $\Delta=\{(x,x):x\in X\}$ is the diagonal of the space X. Besides, if X is a compact space of countable tightness and $X^2\backslash\Delta$ belongs to $\mathcal{M}$ then X is metrizable in ZFC.

We also consider the class $\mathcal{M}^*$ of spaces X which have a compact cover $\mathcal{F}$ ordered by a second countable space with the additional property that, for every compact set $P\subset X$ there exists $F\in\mathcal{F}$ with $P\subset F$. It is a ZFC result that if X is a compact space and $(X\times X)\backslash\Delta$ belongs to $\mathcal{M}^*$ then X is metrizable. We also establish that, under CH, if X is compact and $C_p(X)$ belongs to $\mathcal{M}^*$ then X is countable.

Keywords: (strong) domination by irrationals, (strong) domination by a second countable space, diagonal, metrization, orderings by irrationals, orderings by a second countable space, compact cover, function spaces, cosmic spaces, $\aleph_0$-spaces, Lindelöf Σ-space, compact space, metrizable space

2000 Mathematics Subject Classification: 54B10, 54C05, 54D30

0. Introduction.

Given a space X we denote by $\mathcal{K}(X)$ the family of all compact subsets of X. One of about a dozen equivalent definitions says that X is *a Lindelöf Σ-space (or has the Lindelöf Σ-property)* if there exists a second countable space M and a compact-valued upper semicontinuous map $\varphi: M\to X$ such that $\bigcup\{\varphi(x): x\in M\}=X$ (see, e.g., [RJ, Section 5.1]). It is worth mentioning that in Functional Analysis, the same concept is usually referred to as a *countably K-determined space.*

Suppose that X is a Lindelöf Σ-space and hence we can find a compact-valued upper semicontinuous surjective map $\varphi: M\to X$ for some second countable space M. If we let $F_K=\bigcup\{\varphi(x): x\in K\}$ for any compact set $K\subset M$ then the family $\mathcal{F}=\{F_K: K\in\mathcal{K}(M)\}$ consists of compact subsets of X, covers X and $K\subset L$ implies $F_K\subset F_L$. We will say that $\mathcal{F}$ is an M-ordered compact cover of X.

The class $\mathcal{M}$ of spaces with an M-ordered compact cover for some second countable space M, was introduced by Cascales and Orihuela in [CO2]. They proved, among other things, that a Dieudonné complete space is Lindelöf Σ if and

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K compact space & $\{A_\alpha : \alpha\in\mathbb{N}^{\mathbb{N}}\}\subset (K\times K)\setminus\Delta$.

We write:

(A) each A_α is compact;

(B) $A_\alpha\subset A_\beta$ whenever $\alpha\le\beta$;

(C) $(K\times K)\setminus\Delta=\bigcup\{A_\alpha : \alpha\in\mathbb{N}^{\mathbb{N}}\}$.

Open question

(A) + **(B)** + **(C)** $\overset{?}{\Rightarrow}$ K is metrizable.

Open questions

More problems...here!

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We write:

- **(A)** each A_α is compact;
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- **(C)** $(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.

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THANK YOU!