Domination by second countable spaces

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- Starting point
- 2 Domination by Polish Spaces
- 3 Domination by Second Countable Spaces
- Open questions

Starting point

A characterization of metrizability

Starting point

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Exercise...from Engelking's book

4.2.B (Šnešder [1945]). Show that a compact space X is metrizable if and only if the diagonal Δ is a G_0 -set in the Cartesian product $X \times X$ (see Problem 3.12.22(e); cf. Problem 4.5.15 and Exercise 5.1.I).

Open questions

K compact space & $\{A_\alpha:\alpha\in\mathbb{N}^\mathbb{N}\}$ subsets of $(K\times K)\setminus\Delta$. We write:

- (A) each A_{α} is compact;
- **(B)** $A_{\alpha} \subset A_{\beta}$ whenever $\alpha \leq \beta$;
- (C) $(K \times K) \setminus \Delta = \bigcup \{A_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}} \}.$

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Theorem (Orihuela, B.C. 1987)

(A) + **(B)** + **(C)** + **(D)**
$$\Rightarrow$$
 K is metrizable.

(D) For each compact set $F \subset (K \times K) \setminus \Delta$, there is A_{α} such that $F \subset A_{\alpha}$.

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$$MA(\omega_1) \Rightarrow K$$
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(A) + (B) + (C)
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The references

Domination by Polish Spaces



M. Talagrand, Espaces de Banach faiblement \mathcal{K} -analytiques, Ann. of Math. (2) 110 (1979), no. 3, 407-438. MR 81a:46021



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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 126, 1-10 (1987)

On a Theorem of Choquet and Dolecki

IWO LABUDA*

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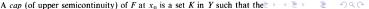
Submitted by Ky Fan

Received June 6, 1985

The results of this paper clarify and extend slightly the previous work of Dolecki and Lechicki (C. R. Acad. Sci. Paris 293 (1981), 219-221; J. Math. Anal. Appl. 88 (1982), 547-584) and Hansell, Jayne, Rogers and the author (Math. Z. 189 (1985), 297–318). Let X, Y be Hausdorff spaces and F: $X \rightarrow Y$ an upper semicontinuous setvalued map. A subset K of F(x) is said to be a peak of F at x, if, for every open set V containing K, there exists a neighbourhood U of x such that $F(U) \setminus F(x) \subseteq V$. Criteria ("Choquet-Dolecki Theorems") are given in order that F has the smallest possible peak. It turns out that in unexpectedly general situations an upper semicontinuous map F has, for every x in X, a peak which is the smallest possible at x and moreover compact. © 1987 Academic Press, Inc.

1 THE RESULTS

Let X, Y be Hausdorff spaces, and let $F: X \to Y$ be a set-valued map. In what follows the term "map" means always such a set-valued map.



The references

ROCKY MOUNTAIN JOURNAL OF MATHEMATICS Volume 39, Number 2, 2009

INHERENT COMPACTNESS OF HPPER CONTINUOUS SET VALUED MAPS

BRIAN L. DAVIS AND IWO LABUDA

The following theorem is due to Cascales and Orihuela [8], see also [7].

- **3.4.** Theorem. The following are equivalent for \mathcal{B} with a countable base.
 - (i) For each $(y_n) \geq \mathcal{B}$, the closure $\{y_n : n \in \mathbb{N}\}$ is countably compact.
- (ii) B is countably compact at adh B which itself is countably compact.

The proof given here follows [26] rather than [8]. What is really interesting though, is the link between this result and the Vainštein-Choquet-Dolecki theorem. Cascales and Orihuela knew [22], but their applications do not go beyond points of countable character treated in [25]. On the other hand, a part of the arguments in [26] could have been skipped using the above theorem.

Starting point

- L, M, ..., X, Y, ... topological spaces; E, F Banach or sometimes lcs:
- K compact Hausdorff space;
- 2^X subsets; $\mathcal{K}(X)$ family of compact sets;
- C(X) continuous functions; $C_{D}(X)$ continuous functions endowed with the pointwise convergence topology τ_p ;

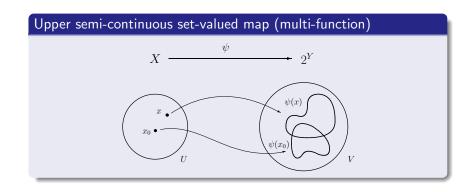
Notation

Starting point

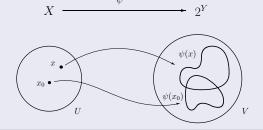
- L, M,...X, Y,... topological spaces; E, F Banach or sometimes lcs;
- K compact Hausdorff space;
- 2^X subsets; $\mathcal{K}(X)$ family of compact sets;
- C(X) continuous functions; $C_p(X)$ continuous functions endowed with the pointwise convergence topology τ_p ;
- $\Omega \subset \mathbb{C}$ open set; $\mathscr{H}(\Omega)$ space of holomorphic functions with the topology of uniform convergence on compact sets;
- $\Omega \subset \mathbb{R}^n$ open set; $\mathscr{D}'(\Omega)$ space of distributions;
- $\underline{\lim} E_n$ inductive limit of a sequence of Fréchet spaces.



Open questions

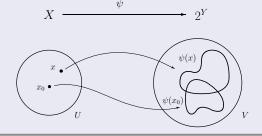


Upper semi-continuous set-valued map (multi-function)



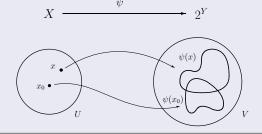
① Y is K-analytic if there is $\psi: \mathbb{N}^{\mathbb{N}} \to 2^Y$ that is upper semi-continuous compact-valued and such that $Y = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \psi(\alpha)$;

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- ② Y is countably K-determined if there is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and $\psi : \Sigma \to 2^Y$ that is upper semi-continuous compact-valued and such that $Y = \bigcup_{\alpha \in \Sigma} \psi(\alpha)$.

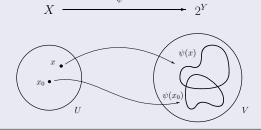
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① Y is K-analytic if there is $\psi: \mathbb{N}^{\mathbb{N}} \to 2^Y$ that is upper semi-continuous compact-valued and such that $Y = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \psi(\alpha)$;

 $\mathbb{N}^{\mathbb{N}} \Leftrightarrow \text{any Polish space } P$

Upper semi-continuous set-valued map (multi-function)



$\Sigma \Leftrightarrow$ any second countable space M (Lindelöf Σ)

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- **3** K-analytic \Rightarrow countably K-determined \Rightarrow Lindelöf;

Starting point

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- **4** countably K-determined + metrizable \Rightarrow separable;

Open questions

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- **3** K-analytic \Rightarrow countably K-determined \Rightarrow Lindelöf;
- **4** countably K-determined + metrizable \Rightarrow separable;
- **5** if X is K-analytic $(\psi : \mathbb{N}^{\mathbb{N}} \to 2^X)$ and $A_{\alpha} := \psi(\{\beta : \beta < \alpha\})$ then:
 - (A) each A_{α} is compact;
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 - (C) $X = \bigcup \{A_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}}\}.$
- \bullet ditto, if X is countably K-determined, there is a second countable space M and a family $\{A_K : K \in \mathcal{K}(M)\}$ such that:
 - (A) each A_K is compact;
 - **(B)** $A_K \subset A_F$ whenever $K \subset F$:
 - (C) $X = \bigcup \{A_K : K \in \mathcal{K}(M)\}.$

Domination by Polish Spaces

Domination by Second Countable Spaces

Domination by Polish Spaces

Definition

A topological space X is dominated by a Polish space, if there is a Polish space P and a family $\{A_K : K \in \mathcal{K}(P)\} \subset X$ such that:

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- (C) $X = \bigcup \{A_K : K \in \mathcal{K}(P)\}.$

Proposition

For a topological space X the TFAE:

- ① X is dominated by a Polish space;
- ② There is a family $\{A_{\alpha}: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of X with:
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K-analyticity implies domination by Polish spaces

Observation

If X is K-analytic $(\psi : \mathbb{N}^{\mathbb{N}} \to 2^X)$ and $A_{\alpha} := \psi(\{\beta : \beta \leq \alpha\})$ then:

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The question

• When domination by Polish spaces implies K-analyticity?

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The question

- When domination by Polish spaces implies K-analyticity?
- How useful is a positive answer to the above?

A nice old case

Talagrand, Ann. of Math. 1979

Annals of Mathematics, 110 (1979), 407-438

Espaces de Banach faiblement X-analytiques

Par Michel, Talagrand

PROPOSITION 6.13. Soit K un espace compact. Les assertions suivantes sont équivalentes:

- B). K est de type &..
- b). Il existe une application croissante $\sigma \sim A$, de Σ (muni de l'ordre produit) dans l'ensemble des compacts de $C_{\mathfrak{p}}(K)$ telle que $\bigcup_{s\in S} A_s$ sépare les points de K.

Démonstration. Nous savons déjà que a) implique b) l'application

Another case with domination

Valdivia, J. London Math. Soc. 1987

QUASI-LB-SPACES

MANUEL VALDIVIA

We shall see later that properties (1) and (2) are important in order to obtain some results on the closed graph theorem. This is the reason for introducing the following definitions. A quasi-LB-representation in a topological vector space F is a family $\{A_{\alpha}: \alpha \in \mathbb{N}^{\mathbb{N}}\}\$ of Banach discs satisfying the following conditions:

- 1. $\{A_{\alpha}: \alpha \in \mathbb{N}^{\mathbb{N}}\} = F$
- 2. if $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ and $\alpha \leq \beta$ then $A_{\alpha} \subset A_{\beta}$.

- Let X be a topological space $\{A_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of X with:
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 - (C) $X = \bigcup \{A_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}}\}.$
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and $m \in \mathbb{N}$, define

$$\alpha|_m:=(n_1,n_2,\ldots,n_m).$$

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Proposition, B. C., 1987

Given X and $\{A_\alpha:\alpha\in\mathbb{N}^\mathbb{N}\}$ as above, if we define $\psi:\mathbb{N}^\mathbb{N}\to 2^X$ given by

$$\psi(\alpha) := \bigcap_{m=1} \bigcup \{A_{\beta} : \beta|_{m} = \alpha|_{m}\}$$

then:

- each $\psi(\alpha)$ is countably compact (even more, all cluster points of any sequence in $\psi(\alpha)$ remain in $\psi(\alpha)$).
- if $\psi(\alpha)$ is compact then $\alpha \to \psi(\alpha)$ gives K-analytic structure to X.

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- if $\psi(\alpha)$ is compact then $\alpha \to \psi(\alpha)$ gives K-analytic structure to X.

X has K-analytic structure if countably compact subsets=compact subsets.

Talagrand's solution to a conjecture Corson

Theorem, Talagrand 1975, Bull. Sci. Math.

Every WCG Banach space E is weakly Lindelöf.

Proof.-

- Fix $W \subset E$ absolutely convex w-compact with $E = \operatorname{span} W$.
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$,

$$A_{\alpha}:=\left(\textit{n}_{1}\textit{W}+\textit{B}_{\textit{E}^{**}}\right)\cap\left(\textit{n}_{2}\textit{W}+\frac{1}{2}\textit{B}_{\textit{E}^{**}}\right)\cap\cdots\cap\left(\textit{n}_{1}\textit{W}+\frac{1}{k}\textit{B}_{\textit{E}^{**}}\right)\cap\ldots$$

• Proposition \Rightarrow (E, w) K-analytic \Rightarrow (E, w) Lindelöf.



Open questions

Fréchet-Montel spaces

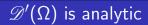
Theorem. Dieudonné 1954

Every Fréchet-Montel space E is separable (in particular $\mathcal{H}(\Omega)$ is separable).

- Fix $V_1 \supset V_2 \supset \cdots \supset V_n \ldots$ a basis of closed neighborhoods of 0.
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$,

$$A_{\alpha}:=\bigcap_{k=1}^{\infty}n_{k}V_{k}.$$

- $\{A_{\alpha}: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ fundamental family of bdd closed sets=compact;
- Proposition \Rightarrow E K-analytic +metrizable \Rightarrow E Lindelöf + metrizable \Rightarrow E separable.



Theorem, $\mathscr{D}'(\Omega)$ is analytic.

The strong dual of every inductive limit of Fréchet-Montel spaces is analytic.

$\mathcal{D}'(\Omega)$ is analytic

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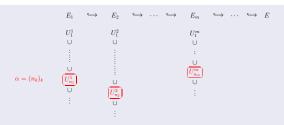
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Theorem, $\mathcal{D}'(\Omega)$ is analytic.

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$$U_{\alpha} := \overline{\operatorname{aco}(\bigcup_{k=1}^{\infty} U_{n_k}^k)}$$

- $U_{\beta} \subset U_{\alpha}$ si $\alpha \leq \beta$; $\mathscr{U} := \{U_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ neigh. basis of 0 en E.
- $A_{\alpha} := U_{\alpha}^{\circ}$ compact & $A_{\alpha} \subset A_{\beta}$, $\alpha \leq \beta$;
- $E' = \bigcup \{A_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and E' sub-metrizable $\Rightarrow E'$ K-analytic sub-metrizable $\Rightarrow E'$ analytic.



$\mathscr{D}'(\Omega)$ is analytic

Theorem, $\mathscr{D}'(\Omega)$ is analytic.

The strong dual of every inductive limit of Fréchet-Montel spaces is analytic.



Schwartz, 1964

Any Borel linear map from a separable Banach space into $\mathscr{D}'(\Omega)$ is continuous. In particular, the Closed Graph Theorem holds for linear maps

$$T: \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega).$$

K compact space & $\{A_{\alpha}: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ subsets of $(K \times K) \setminus \Delta$. We write:

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Theorem (Orihuela, B.C. 1987)

- $(A) + (B) + (C) + (D) \Rightarrow K$ is metrizable.
- **(D)** For each compact set $F \subset (K \times K) \setminus \Delta$, there is A_{α} such that $F \subset A_{\alpha}$.

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Proof.-

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(D) For each compact set $F \subset (K \times K) \setminus \Delta$, there is A_{α} such that $F \subset A_{\alpha}$.

- **1** Given $\alpha \in \mathbb{N}^{\mathbb{N}}$, define $N_{\alpha} := (K \times K) \setminus A_{\alpha}$.
- **2** N_{α} is a basis of open neighborhoods of Δ ;

- (A) each A_{α} is compact;
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Theorem (Orihuela, B.C. 1987)

$$(A) + (B) + (C) + (D) \Rightarrow K \text{ is metrizable.}$$

(D) For each compact set $F \subset (K \times K) \setminus \Delta$, there is A_{α} such that $F \subset A_{\alpha}$.

- **①** Given $\alpha \in \mathbb{N}^{\mathbb{N}}$, define $N_{\alpha} := (K \times K) \setminus A_{\alpha}$.
- **2** N_{α} is a basis of open neighborhoods of Δ ;
- ③ $B_{\alpha} := \{ f \in C(K) : ||f||_{\infty} \le n_1, |f(x) f(y)| \le \frac{1}{m}, \text{ whenever } (x, y) \in N_{\alpha|^m} \};$ for $\alpha|^m := (n_m, n_{m+1}, ...), m \in \mathbb{N}.$

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- (C) $C(K) = \bigcup \{B_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}}\}.$
- **⑤** $(C(K), \| \|_{\infty})$ is *K*-analytic +metrizable ⇒ *E* Lindelöf + metrizable ⇒ *E* separable ⇒ *K* is metrizable.



Metrizability of compact sets (1): A different formulation

We didn't stated our result below as presented.

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Theorem (Orihuela, B.C. 1987)

 (K,\mathfrak{U}) a compact uniform space with a basis for the uniformity $\mathscr{B}_{\mathfrak{U}} = \{ N_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}} \}$ satisfying:

$$N_{eta}\subset N_{lpha}$$
 si $lpha\leq eta$ whenever $lpha,eta\in\mathbb{N}^{\mathbb{N}}.$

Then K is metrizable.

Applications



Descriptive Topology in Selected Topics of Functional Analysis

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Applications

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Open questions

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Proposition, B. C., 1987

Given X and $\{A_\alpha:\alpha\in\mathbb{N}^\mathbb{N}\}$ as above, if we define $\psi:\mathbb{N}^\mathbb{N}\to 2^{(K\times K)\setminus\Delta}$ given by

$$\psi(\alpha) := \bigcap_{m=1}^{\infty} \bigcup \{A_{\beta} : \beta|_{m} = \alpha|_{m}\}$$

then:

- ullet each $\psi(lpha)$ is countably compact (even more, all cluster points of any sequence in $\psi(lpha)$ remain in $\psi(lpha)$).
- if $\psi(\alpha)$ is compact then $\alpha \to \psi(\alpha)$ gives K-analytic structure to $(K \times K) \setminus \Delta$.

that $B \cap \Delta = \emptyset$.

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- $(K \times K) \setminus \Delta$ is K-analytic

$$\overline{\psi(\alpha)} \subset \psi(\alpha)$$
, (closure in $K \times K$)

- take $x \in \overline{\psi(\alpha)}$;
- there is A ⊂ ψ(α) countable with x ∈ A:
- \bullet is $x \in A \Rightarrow x \in \psi(\alpha)$;
- otherwise, $x \in (\overline{A} \setminus A) \Rightarrow x$ is cluster point of a sequence in $\psi(\alpha) \Rightarrow x \in \psi(\alpha)$.

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Domination by Second Countable Spaces

Definition

A topological space X is dominated by a second countable space, if there is a second countable space M and a family $\{A_K : K \in \mathcal{K}(M)\} \subset X$ such that:

- (A) each A_K is compact;
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Theorem (Orihuela, B.C.)

For a topological space TFAE:

- X is countably K-determined;
- 2 X is Dieudonné complete and dominated by a second countable space.

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For a topological space TFAE:

- X is countably K-determined;
- 2 X is Dieudonné complete and dominated by a second countable space.

The class of spaces dominated by a second countable space enjoy the usual stability properties we might expect.

Techniques

Generation of usco maps

Let T be a first-countable, X a topological space and let $\varphi: T \to 2^X$ be a set-valued map satisfying the property

 $\bigcup \varphi(t_n)$ is relatively compact for each convergent sequence $(t_n)_n$ in T. (1) $n \in \mathbb{N}$

If for each x in X we define

 $C(t) := \{x \in X : \text{there is } t_n \to t \text{ in } T, \text{ for every } n \in \mathbb{N} \text{ there is } t \in \mathbb{N} \}$ $x_n \in \varphi(t_n)$ and x is cluster point of $(x_n)_n$.

Then:

- each C(t) is countably compact.
- if $\psi(t) := \overline{C(t)}$ is compact then $t \to \psi(t)$ is usco $\psi : T \to \mathcal{K}(X)$.

- **3.4.** Theorem. The following are equivalent for $\mathcal B$ with a countable base.
 - (i) For each $(y_n) \ge \mathcal{B}$, the closure $\overline{\{y_n : n \in \mathbf{N}\}}$ is countably compact.
- (ii) ${\cal B}$ is countably compact at $adh_{\omega}{\cal B}$ which itself is countably compact.

Let T be a first-countable, X a topological space and let $\varphi: T \to 2^X$ be a set-valued map satisfying the property

 $\bigcup_{n\in\mathbb{N}} \varphi(t_n) \text{ is relatively compact for each convergent sequence } (t_n)_n \text{ in } T. \quad (1)$

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Then:

- each C(t) is countably compact.
- if $\psi(t) := \overline{C(t)}$ is compact then $t \to \psi(t)$ is usco $\psi : T \to \mathcal{K}(X)$.

- If X is dominated by a second countable space, if there is a second countable space M and a family $\{A_K: K \in \mathcal{K}(M)\}$ such that:
 - (A) each A_K is compact;
 - **(B)** $A_K \subset A_F$ whenever $K \subset F$;
 - (C) $X = \bigcup \{A_K : K \in \mathcal{K}(M)\}.$

We take: $T := (\mathcal{K}(M), h), \varphi(K) := A_K$ and we can generate the USCO ψ in many cases.

Let T be a first-countable, X a topological space and let $\varphi: T \to 2^X$ be a set-valued map satisfying the property

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Theorem (Orihuela, Tkachuk, B.C. 2010)

 $C_p(X)$ is countably K-determined iff is dominated by a second countable space.

Two noticeable results

Theorem (Orihuela, Tkachuk, B.C. 2010)

 $C_p(X)$ is countably K-determined iff is dominated by a second countable space.

Theorem (Orihuela, Tkachuk, B.C. 2010)

Let K be a compact space. If there is a second countable space M and a family $\{A_F : F \in \mathcal{K}(M)\} \subset (K \times K) \setminus \Delta$ such that:

- (A) each A_F is compact;
- **(B)** $A_F \subset A_I$ whenever $F \subset L$;
- (C) $(K \times K) \setminus \Delta = \bigcup \{A_F : F \in \mathcal{K}(M)\}.$

and

(D) every compact subset of $(K \times K) \setminus \Delta$ is contained in some A_F .

Then K is metrizable.

Starting point 00000000

Open questions

K compact space &

$${A_{\alpha}: \alpha \in \mathbb{N}^{\mathbb{N}}} \subset (K \times K) \setminus \Delta.$$

We write:

Domination by Second Countable Spaces

- (A) each A_{α} is compact;
- **(B)** $A_{\alpha} \subset A_{\beta}$ whenever $\alpha \leq \beta$;
- (C) $(K \times K) \setminus \Delta = \bigcup \{A_{\alpha} : A_{\alpha} :$ $\alpha \in \mathbb{N}^{\mathbb{N}}$ }.

Open question

(A) + (B) + (C)
$$\stackrel{?}{\Rightarrow}$$
 K is metrizable.

Open questions

More problems...here!

Domination by second countable spaces and Lindelöf Σ -property

B. Cascales^{1,2}, J. Orihuela^{1,2} and V.V. Tkachuk^{3,4}

We also consider the class \mathcal{M}^* of spaces X which have a compact cover \mathcal{F} ordered by a second countable space with the additional property that, for every compact set $P \in X$ there exists $F \in \mathcal{F}$ with $P \in \mathcal{F}$. It is a $Z \in X$ count that if X is a compact space and $(X \times X) \setminus \Delta$ belongs to \mathcal{M}^* then X is metrizable. We also establish that, under \mathbb{CH}_1 if X is compact and $\mathbb{C}_{\mathcal{F}}(X)$ belongs to \mathcal{M}^* then X is contable.

Kequeords: (strong) domination by irrationals, (strong) domination by a second countable space, discount, metrization, orderings by irrationals, orderings by a second countable space, compact cover, function snaces, cosmic snaces. No-snaces. Lindeld S'S-snace, compact snace, metrizable snace

2000 Mathematics Subject Classification: 54B10, 54C05, 54D30

0. Introduction

Given a space X we denote by K(X) the family of all compact subsets of X. One of about a dozen equivalent definitions asyst that X is a l.midell <math>Y Y-power (or has the l.inidell Y-property) if there exists a second countable space M and a compact-valued upper semicontinuous map Y: $M \to X$ and that $\|V_{Y}(x)^* : x \in M\| = X$ (see, e.g., $\|R\|$.) Section 5.1). It is worth mentioning that in Functional Analysis, the same conceed is usually referred to as a contable K-determined semi-

Suppose that X is a Lindelöf Σ -space and hence we can find a compact-valued upper semicontinuous surjective map $\varphi : M \to X$ for some second countable space M. If we let $F_K = \bigcup \{\varphi(x) : x \in K\}$ for any compact set $K \subset M$ then the family $F = \{F_K : K \in K(M)\}$ consists of compact subsets of X, covers X and $K \subset L$ implies $F_K \subset F_K$. We will say that F is an M - ordered compact cover of X.

The class \mathcal{M} of spaces with an M-ordered compact cover for some second countable space M, was introduced by Cascales and Orinuela in [CO2]. They proved, among other things, that a Dieudonné complete suace is Lindelőf Σ if and K compact space & $\{A_{\alpha}: \alpha \in \mathbb{N}^{\mathbb{N}}\} \subset (K \times K) \setminus \Delta$. We write:

(A) each A_{α} is compact;

(B) $A_{\alpha} \subset A_{\beta}$ whenever $\alpha \leq \beta$;

(C)
$$(K \times K) \setminus \Delta = \bigcup \{A_{\alpha} : \alpha \in \mathbb{N}^{\mathbb{N}} \}.$$

Open question

(A) + (B) + (C)
$$\stackrel{?}{\Rightarrow}$$
 K is metrizable.

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Open questions

More problems...here!

Domination by second countable spaces and Lindelöf Σ-property

B. Cascales^{1,2}, J. Orihuela^{1,2} and V.V. Tkachuk^{3,4}

Address. Given a space M, a family of sets. A of a space X is ordered by M if $A = I_{A/K} X$ is a compact subset of M) and KC L implies, $A_{A/K}$. We storp the class A of spaces with whose compact convex ordered by a mecanic countable space. We prove that a space $C_{A/K}$ belongs to M if and only if it is a Landold Σ space. Under $M(\alpha_{A/K})$, M is a compact and (XXYA), M has a compact ordered by a Palida space then X is metriable; here $\Delta = \{I_{A/K} = I_{A/K} = I_{$

We also consider the class \mathcal{M}^* of spaces X which have a compact cover \mathcal{F} ordered by a second countable space with the additional property that, for every compact set $P \in X$ there exists $F \in \mathcal{F}$ with $P \in \mathcal{F}$. It is a $Z \in X$ count that if X is a compact space and $(X \times X) \setminus \Delta$ belongs to \mathcal{M}^* then X is metrizable. We also establish that, under \mathbb{CH}_1 if X is compact and $\mathbb{C}_{\mathcal{F}}(X)$ belongs to \mathcal{M}^* then X is contable.

Keywords: (strong) domination by irrationals, (strong) domination by a second countable space, discond, metrization, orderings by prirationals, orderings by a second countable space, compact cover, function spaces, cosmic spaces, N₀-spaces, Lindelöf S-space, compact space, metrizable space

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