Universidad de Murcia | Departamento Matemáticas

A biased view of topology as a tool for analysis

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THE BISHOP-PHELPS-BOLLOBAS THEOREM AND ASPLUND OPERATORS

B. CASCALES

In this three lectures series we plan to present a strengthening of the Bishop-Phelps property for operators that in the literature is called the Bishop-Phelps-Bollobás property. Let X be a Banach space and L a locally compact Hausdorff space. We will prove that if $T : X \to C_0(L)$ is an Asplund operator and $\|T(x_0)\| \approxeq \|T\|$ for some $\|x_0\| = 1$, then there is an norm attaining Asplund operator $S : X \to C_0(L)$ and $\|u_0\| = 1$ with $\|S(u_0)\| = \|S\| = \|T\|$ such that $u_0 \approxeq x_0$ and $S \approxeq T$. As particular cases we obtain: (A) if T is weakly compact, then S can also be taken being weakly compact; (B) if X is Asplund (for instance, $X = c_0$), the pair $(X, C_0(L))$ has the Bishop-Phelps-Bollobás property for all L; (C) if L is scattered, the pair $(X, C_0(L))$ has the Bishop-Phelps-Bollobás property for all Banach spaces X.

Our idea is to present our results mostly in a self-contained way and consequently the plan will be:

Lecture 1: To recall the classical Bishop-Phelps theorem, Bollobás observation and their relationship with Ekeland's variational principle;

Lecture 2: To recall the notion of Asplund space, Asplund operator and establish the main ideas behind the characterization of Asplund spaces and operators via the Radon-Nikodym property and fragmentability;

Lecture 3: To use the tools presented in the two previous lectures and then give a self-contained proof of the results announced in the abstract.

Key words: Bishop-Phelps, Bollobás, fragmentability, Asplund operator, weakly compact operator, norm attaining.
AMS classification: 46B22, 47B07

When presenting these lectures we are strongly motivated by the fact that general topology and functional analysis continuously benefit from cross-fertilization between them. Our starting point for these two lectures, intended for students, are the two exercises below.

Exercise 1 *A compact Hausdorff topological space K is metrizable if, and only, if $(C(K), \|\cdot\|_\infty)$ is separable.*

Exercise 2 *From Engelking's book [3]:*

> **4.2.B** (Šneĭder [1945]). **Show that a compact space X is metrizable if and only if the diagonal Δ is a G_δ-set in the Cartesian product $X \times X$ (see Problem 3.12.22(e); cf. Problem 4.5.15 and Exercise 5.1.I).**

Both exercises are connected. From Exercise 1 we will motivate some classical results about weak compactness in Banach spaces. Exercise 2 can be easily rephrased as follows: a compact Hausdorff topological space K is metrizable if, and only if, $(K \times K) \setminus \Delta = \bigcup_{n \in \mathbb{N}} F_n$ with each F_n a closed subset of $K \times K$. From here we will move to some other more intriguing cases. To name one, if $(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ where each A_α is compact and $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$, we shall prove that the latter assumption also implies metrizability when either $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental family of compact subsets for $(K \times K) \setminus \Delta$ or when $\mathrm{MA}(\omega_1)$ is assumed. The success when proving these results relies upon the generation of *u*sco maps. We provide applications (old and new) of the results and techniques presented here to functional analysis: metrizability of compact subsets in inductive limits, Lindelöf property of WCG Banach spaces and classification of compact topological spaces, separability of Fréchet-Montel spaces, Lindelöf-Σ character of spaces $C_p(X)$, etc. For the students is a good objective to learn all the details of how to solve both exercises. Furthermore, the lectures will stress on how these simple but tricky ideas have motivated recent Ph. D. dissertations as a well as some new results and applications published elsewhere.

1. The proto-ideas
2. Starting Point
3. Domination by Polish Spaces: applications to FA
4. Domination by Second Countable Spaces
5. Further extensions and open questions

Notation

- $L, M, \ldots X, Y, \ldots$ topological spaces; E, F Banach or sometimes lcs;
- K compact Hausdorff space;
- 2^X subsets; $\mathscr{K}(X)$ family of compact sets;
- $C(X)$ continuous functions; $C_p(X)$ continuous functions endowed with the pointwise convergence topology τ_p;

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- $C(X)$ continuous functions; $C_p(X)$ continuous functions endowed with the pointwise convergence topology τ_p;
- $\Omega \subset \mathbb{C}$ open set; $\mathscr{H}(\Omega)$ space of holomorphic functions with the topology of uniform convergence on compact sets;
- $\Omega \subset \mathbb{R}^n$ open set; $\mathscr{D}'(\Omega)$ space of distributions;
- $\varinjlim E_n$ inductive limit of a sequence of Fréchet spaces.

First of Two inspiring papers

Valdivia, J. London Math. Soc. 1987

QUASI-LB-SPACES

MANUEL VALDIVIA

We shall see later that properties (1) and (2) are important in order to obtain some results on the closed graph theorem. This is the reason for introducing the following definitions. *A quasi-*LB*-representation in a topological vector space F is a family* $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *of Banach discs satisfying the following conditions:*

1. $\bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = F$;
2. *if* $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ *and* $\alpha \leqslant \beta$ *then* $A_\alpha \subset A_\beta$.

M. Valdivia, *Quasi-LB-spaces*, J. London Math. Soc. (2) **35** (1987), no. 1, 149–168. MR 88b:46012

Second of Two inspiring papers

Talagrand, Ann. of Math. 1979

Annals of Mathematics, 110 (1979), 407-438

Espaces de Banach faiblement $\mathcal{K}$-analytiques

Par Michel Talagrand

Proposition 6.13. *Soit K un espace compact. Les assertions suivantes sont équivalentes:*

a). *K est de type $\mathcal{E}_1$.*

b). *Il existe une application croissante $\sigma \to A_\sigma$ de Σ (muni de l'ordre produit) dans l'ensemble des compacts de $C_p(K)$ telle que $\bigcup_{\sigma \in \Sigma} A_\sigma$ sépare les points de K.*

Démonstration. Nous savons déjà que a) implique b) l'application

M. Talagrand, *Espaces de Banach faiblement $\mathscr{K}$-analytiques*, Ann. of Math. (2) **110** (1979), no. 3, 407–438, MR 81a:46021

Simple facts to keep in mind

1. $\mathbb{N}^{\mathbb{N}}$ endowed with the product of discrete topology on $\mathbb{N}$ is separable and metrizable with a complete metric (*i.e.* $\mathbb{N}^{\mathbb{N}}$ is a Polish space).

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2. If $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ then there is $\beta \in \mathbb{N}^{\mathbb{N}}$ such that

$$\alpha_n, \alpha \leq \beta$$

(here $\leq$ stands for the natural order for the coordinates)

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A few words about descriptive set theory

Descriptive set theory
From Wikipedia, the free encyclopedia

In mathematical logic, descriptive set theory is the study of certain classes of "well-behaved" subsets of the real line and other Polish spaces. As well as being one of the primary areas of research in set theory, it has applications to other areas of mathematics such as functional analysis, ergodic theory, the study of operator algebras and group actions, and mathematical logic.

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Contents

The origin of descriptive set theory

Leçons sur les Ensembles Analytiques et leurs Applications. By Nicolas Lusin. With a preface by Henri Lebesgue and a note by Waclaw Sierpinski. Paris, Gauthier-Villars, 1930. xvi+328 pages.

This volume in the Borel series contains a systematic survey of the present knowledge of analytic sets, a knowledge which is chiefly due to the researches of the Russian mathematician who is the author of this book. In fact the only results which are not due to Lusin or his pupils come from members of the Polish school of Sierpinski and Mazurkiewicz. The analytic sets of Lusin, which are a generalization of Borel sets, have been briefly mentioned previously in several books (Hausdorff's *Mengenlehre*, for instance), but this is the first book devoted entirely to their study.

Lebesgue in his preface humorously points out that the origin of the problems considered by Lusin lies in an error made by Lebesgue himself in his 1905 memoir on functions representable analytically. Lebesgue stated there that the projection of a Borel set is always a Borel set. Lusin and his colleague Souslin constructed an example showing that this statement was false, thus discovering a new domain of point sets, a domain which includes as a proper part the domain of Borel sets. Lebesgue expresses his joy that he was inspired to commit such a fruitful error.

A few definitions from descriptive set theory

a.- Souslin operation: $\bigcup_{\alpha=(n_1,n_2,\ldots)} \bigcap C_{n_1 n_2 \ldots n_k} =: Z$

$C_{n_1,n_2,\ldots,n_k}$ sets in a topological space X,
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EXAMPLE:= if X is metrizable and $C_{n_1,n_2,\dots,n_k}$ are closed then Z's are called ANALYTIC.

- $X=\mathbb{R}$ analytic + co-analytic $\Leftrightarrow$ Borel
- Continuous image Borel $\Rightarrow$ analytic.
- Borel $\subsetneq$ analytic (Lebesgue Mistake).

A few definitions from descriptive set theory

b.- Polish space ≡ separable, metrizable and complete for some compatible metric.

- G_δ -sets of Polish are Polish; • every Polish is image of $\mathbb{N}^{\mathbb{N}}$.
- $\mathbb{N}^{\mathbb{N}}$ Polish ∧ irrationals are Polish.

c.- (Analytic in general).- Continuous images of Polish spaces

- analytic spaces are separable;
- class stable under usual operations.

D. L. Cohn, *Measure theory*, Birkhäuser, Boston, Mass., 1980. MR 81k:28001

Starting point

A characterization of metrizability

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2. Let $(E, \| \, \|)$ be a Banach space. Then, $(B_E, w))$ is metrizable if, and only if, $(E^* \| \, \|)$ is separable.
3. (Šmulian, 1940) Let E be a Banach space. The w-compact subsets of E are w-sequentially compact, i.e., if $H \subset E$ w-compact, then each sequence $(x_n)_n$ en H has a subsequence that w-converges to a point in H.

Another characterization of metrizability

Exercise. . . from Engelking's book

4.2.B (Šneĭder [1945]). Show that a compact space X is metrizable if and only if the diagonal Δ is a G_δ-set in the Cartesian product $X \times X$ (see Problem 3.12.22(e); cf. Problem 4.5.15 and Exercise 5.1.I).

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Hint. Define a countable family $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of finite open covers of the space X such that for every pair x, y of distinct points of X there exists a natural number i with the property that the closure of no member of $\mathscr{V}_i$ contains both x and y. Check that the family of all finite intersections $V_1 \cap V_2 \cap \ldots \cap V_k$, where $V_i \in \mathscr{V}_i$ for $i = 1, 2, \ldots, k$, is a base for X.

We are indeed a bit more greedy

The goal

For a compact space K TFAE:

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7. $(K \times K) \setminus \Delta$ is Lindelöf.

The proof. . .

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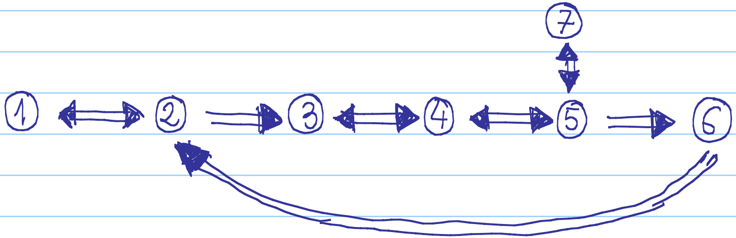


$(K \times K) \setminus \Delta$ is Lindelöf$\Rightarrow$ Δ is a G_δ

It works even for X Hausdorff regular space.

If $x \neq y$, there exist two closed neighbourhoods F_x and F_y of x and y, respective, such that

$$F_x \times F_y \subset (X \times X)\setminus\Delta.$$

The Lindelöf property enables to determine a sequence $(x_n, y_n)_n$ such that

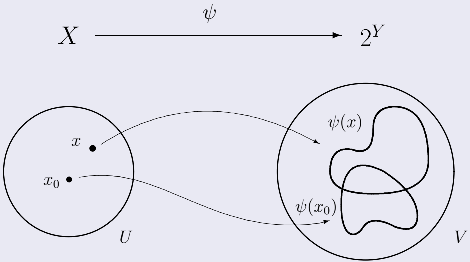
$$X \times X\setminus\Delta = \bigcup_n F_{x_n} \times F_{y_n}.$$

Therefore Δ is a G_δ-subset of $X \times X$ since $\Delta = \bigcap_n G_n$, where

$$G_n = (X \times X)\setminus(F_{x_n} \times F_{y_n}).$$

Definitions

Upper semi-continuous set-valued map (multi-function)



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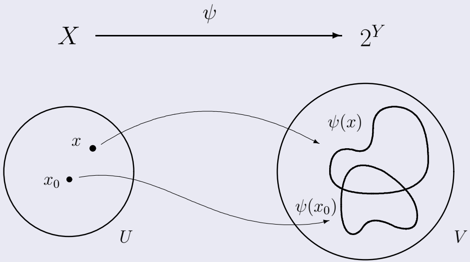

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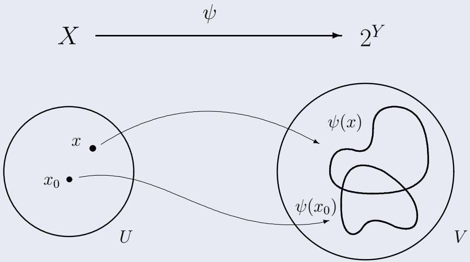

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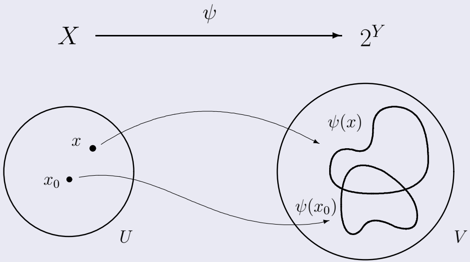

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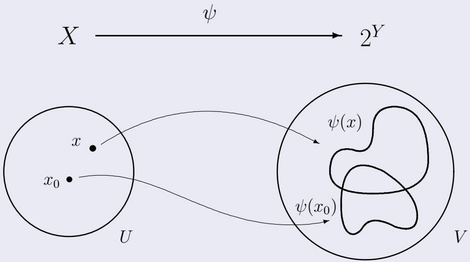

$\Sigma \Leftrightarrow$ any second countable space M (Lindelöf Σ)

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Easy known facts

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4. countably K-determined $+$ metrizable $\Rightarrow$ separable;

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3. K-analytic $\Rightarrow$ countably K-determined $\Rightarrow$ Lindelöf;
4. countably K-determined + metrizable $\Rightarrow$ separable;
5. if X is K-analytic ($\psi : \mathbb{N}^{\mathbb{N}} \to 2^X$) and $A_\alpha := \psi(\{\beta : \beta \leq \alpha\})$ then:

   (A) each A_α is compact;
   (B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;
   (C) $X = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

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6. ditto, if X is countably K-determined, there is a second countable space M and a family $\{A_K : K \in \mathscr{K}(M)\}$ such that:
   - **(A)** each A_K is compact;
   - **(B)** $A_K \subset A_F$ whenever $K \subset F$;
   - **(C)** $X = \bigcup\{A_K : K \in \mathscr{K}(M)\}$.

To keep in mind

Proposition

Let X be a metric space and $\psi : X \to 2^Y$ multi-valued . TFAE:

1. ψ is usco;
2. ψ is compact valued + For every sequence $x_n \to x$ in X if $y_n \in \psi(x_n)$, $n \in \mathbb{N}$ then $(y_n)_n$ has a cluster point $y \in \psi(x)$.

Domination by Polish Spaces

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Definition

A topological space X is dominated by a Polish space, if there is a Polish space P and a family $\{A_K : K \in \mathcal{K}(P)\} \subset X$ such that:

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Proposition, Orihuela-Tkachuk-C, 2011

For a topological space X the TFAE:

1. X is dominated by a Polish space;
2. There is a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of X with:
   (A) each A_α is compact;
   (B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;
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Two nice previous cases

Talagrand, Ann. of Math. 1979

Annals of Mathematics, 110 (1979), 407-438

Espaces de Banach faiblement $\mathcal{K}$-analytiques

Par MICHEL TALAGRAND

PROPOSITION 6.13. *Soit K un espace compact. Les assertions suivantes sont équivalentes:*

a). *K est de type $\mathcal{E}_1$.*

b). *Il existe une application croissante $\sigma \to A_\sigma$ de Σ (muni de l'ordre produit) dans l'ensemble des compacts de $\mathcal{C}_p(K)$ telle que $\bigcup_{\sigma \in \Sigma} A_\sigma$ sépare les points de K.*

Démonstration. Nous savons déjà que a) implique b) l'application

Two nice previous cases

Valdivia, J. London Math. Soc. 1987

QUASI-LB-SPACES

MANUEL VALDIVIA

We shall see later that properties (1) and (2) are important in order to obtain some results on the closed graph theorem. This is the reason for introducing the following definitions. *A quasi-*LB*-representation in a topological vector space F is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of Banach discs satisfying the following conditions:*

1. $\bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = F$;
2. *if* $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ *and* $\alpha \leqslant \beta$ *then* $A_\alpha \subset A_\beta$.

Domination by Polish implies (many times) K-analyticity

- Let X be a topological space $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of X with:
  - **(A)** each A_α is compact;
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- Given $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ and $m \in \mathbb{N}$, define

$$\alpha|_m := (n_1, n_2, \ldots, n_m).$$

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Given X and $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ as above, if we define $\varphi : \mathbb{N}^{\mathbb{N}} \to 2^X$ given by

$$\varphi(\alpha) := \bigcap_{k=1}^{\infty} \bigcup\{A_\beta : \beta|_k = \alpha|_k\}$$

then:

- each $\varphi(\alpha)$ is countably compact (even more, all cluster points of any sequence in $\varphi(\alpha)$ remains in $\varphi(\alpha)$).
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X has K-analytic structure if *countably compact subsets=compact subsets*.

The proof

Given $\alpha = (n_1, n_2, \dots, n_m, \dots)$ we write

$$C_{n_1,n_2,\dots,n_k} := \bigcup \{A_\beta : \beta|_k = \alpha|_k\}$$

Define now for $\alpha = (n_1, n_2, \dots, n_m, \dots) \in \mathbb{N}^{\mathbb{N}}$

$$\varphi(\alpha) := \bigcap_{k=1}^{+\infty} C_{n_1 n_2 \dots n_k}$$

CLAIMS := [1] $\displaystyle\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{+\infty} C_{n_1 \dots n_k} = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \varphi(\alpha) = X$

because $A_\alpha \subset \varphi(\alpha)$.

[2] φ has the property that $\alpha_j \longrightarrow \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and we take $x_j \in \varphi(\alpha_j)$ then $x_j \rightsquigarrow x \in \varphi(\alpha)$

WHY??

The proof

$\alpha_1 = (n^1_1; n^1_2, \ldots, n^1_k, \ldots) \wedge x_1 \in A_{\beta_1}$ $\qquad \beta_1 = (n^1_1,$ ___ whatever $)$

$\alpha_2 = (n^2_1, n^2_2, \ldots, n^2_k, \ldots) \wedge x_2 \in A_{\beta_2}$ $\qquad \beta_2 = (n^2_1, n^2_2,$ ___ whatever $)$

$\vdots$

$\alpha_j = (n^j_1, n^j_2, \ldots, n^j_k, \ldots) \wedge x_j \in A_{\beta_j}$ $\qquad \beta_j = (n^j_1, \ldots, n^j_j,$ ___ whatever $)$

$\downarrow \qquad\qquad \downarrow \; \downarrow$

$\alpha = (n_1, n_2, \ldots, n_k, \ldots)$ $\qquad \alpha = (n_1, n_2 \ldots)$

$\exists\, \beta \geq \beta_j \; \forall j \implies (x_j)_j \subset A_\beta$ hence

$(x_j)_j \to x \in X$ but x actually belongs to $\varphi(\alpha)$

because $x_j \in A_{\beta_j}$ but eventually ($j \geq$ something)

$\beta_j = \{n_1, \text{—}\}$ whatever ...

The proof

$x_j \in A_{(n_1, \underline{\text{whatever}} \dots)}$ $\quad j \geq$ something

$\curvearrowright \quad x \in A_{(n_1, \underline{\text{whatever}})} \subset C_{n_1}$

We repeat the argument and

$$x \in \bigcap_{k=1}^{\infty} C_{n_1 n_2 \dots n_k} = \varphi(\alpha)$$

and therefore the claim is proved.

The proof

The claim proved is the second part of (2) below:

Keep in mind Proposition

Let X be a metric space and $\psi : X \to 2^Y$ multi-valued . TFAE:

1. ψ is usco;
2. ψ is compact valued + For every sequence $x_n \to x$ in X if $y_n \in \psi(x_n)$, $n \in \mathbb{N}$ then $(y_n)_n$ has a cluster point $y \in \psi(x)$.

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Given X and $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ as above, if we define $\varphi_\infty : \mathbb{N}^\mathbb{N} \to 2^X$ given by

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Talagrand's solution to a conjecture Corson

Theorem, Talagrand 1975

Every WCG Banach space E is weakly Lindelöf.

Proof.-

- Fix $W \subset E$ absolutely convex w-compact with $E = \overline{\operatorname{span} W}$.
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$,

$$A_\alpha := (n_1 W + B_{E^{**}}) \cap (n_2 W + \frac{1}{2} B_{E^{**}}) \cap \cdots \cap (n_1 W + \frac{1}{k} B_{E^{**}}) \cap \ldots$$

- Proposition $\Rightarrow (E, w)$ K-analytic $\Rightarrow (E, w)$ Lindelöf. ♣

Fréchet-Montel spaces

Theorem, Dieudonné 1954

Every Fréchet-Montel space E is separable (in particular $\mathscr{H}(\Omega)$ is separable).

Proof.-

- Fix $V_1 \supset V_2 \supset \cdots \supset V_n \ldots$ a basis of closed neighborhoods of 0.
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$,

$$A_\alpha := \bigcap_{k=1}^{\infty} n_k V_k.$$

- $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ fundamental family of bdd closed sets=compact;
- Proposition $\Rightarrow$ E K-analytic +metrizable $\Rightarrow$ E Lindelöf + metrizable $\Rightarrow$ E separable. ♣

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$$
\begin{array}{ccccccccccc}
 & E_1 & \hookrightarrow & E_2 & \hookrightarrow \cdots \hookrightarrow & E_m & \hookrightarrow & \cdots & \hookrightarrow & E \\
 & U_1^1 & & U_1^2 & & U_1^m & & & & \\
 & \cup & & \cup & & \cup & & & & \\
 & \vdots & & \vdots & & \vdots & & & & \\
 & \cup & & & & \cup & & & & \\
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$$

$$U_\alpha := \overline{\operatorname{aco}\left(\bigcup_{k=1}^{\infty} U_{n_k}^k\right)}$$

- $U_\beta \subset U_\alpha$ si $\alpha \leq \beta$; $\mathscr{U} := \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ neigh. basis of 0 en E.
- $A_\alpha := U_\alpha^\circ$ compact & $A_\alpha \subset A_\beta$, $\alpha \leq \beta$;
- $E' = \cup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and E' sub-metrizable $\Rightarrow$ E' K-analytic sub-metrizable $\Rightarrow$ E' analytic.

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\end{array}
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Schwartz, 1964

Any Borel linear map from a separable Banach space into $\mathscr{D}'(\Omega)$ is continuous. In particular, the Closed Graph Theorem holds for linear maps

$$T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega).$$

Metrizability of compact sets (I)

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ subsets of $(K \times K) \setminus \Delta$. We write:

(A) each A_α is compact;

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Theorem (Orihuela, B.C. 1987)

(A) + **(B)** + **(C)** + **(D)** $\Rightarrow$ *K is metrizable.*

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   (C) $C(K) = \bigcup\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.
5. $(C(K), \| \ \|_\infty)$ is K-analytic +metrizable $\Rightarrow$ E Lindelöf + metrizable $\Rightarrow$ E separable $\Rightarrow$ K is metrizable. ♣

Metrizability of compact sets (I): A different formulation

We didn't stated our result below as presented.

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(D) For each compact set $F \subset (K \times K) \setminus \Delta$, there is A_α such that $F \subset A_\alpha$.

Theorem (Orihuela, B.C. 1987)

$(K, \mathfrak{U})$ a compact uniform space with a basis for the uniformity $\mathscr{B}_\mathfrak{U} = \{N_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ satisfying:

$$N_\beta \subset N_\alpha \text{ si } \alpha \leq \beta \text{ whenever } \alpha, \beta \in \mathbb{N}^\mathbb{N}.$$

Then K is metrizable.

The original paper

Math. Z. 195, 365–381 (1987)

Mathematische Zeitschrift

On Compactness in Locally Convex Spaces

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1. Introduction and Terminology

The purpose of this paper is to show that the behaviour of compact subsets in many of the locally convex spaces that usually appear in Functional Analysis is as good as the corresponding behaviour of compact subsets in Banach spaces. Our results can be intuitively formulated in the following terms: *Dealing with metrizable spaces or their strong duals, and carrying out any of the usual operations of countable type with them, we ever obtain spaces with their precompact subsets metrizable, and they even give good performance for the weak topology, indeed they are weakly angelic,* [14], *and their weakly compact subsets are metrizable if and only if they are separable.*

A nice applications

Theorem 16. *If* $E[\mathfrak{T}] = \varinjlim E_n[\mathfrak{T}_n]$ *is an inductive limit of an increasing sequence of subspaces* $E_n[\mathfrak{T}_n]$ *belonging to the class* $\mathfrak{G}$, *then the following statements are equivalent:*

(i) $E[\mathfrak{T}]$ *is sequentially retractive.*
(ii) $E[\mathfrak{T}]$ *is sequentially compact-regular.*
(iii) $E[\mathfrak{T}]$ *is compact-regular.*
(iv) $E[\mathfrak{T}]$ *is precompactly retractive.*

If every $E_n[\mathfrak{T}_n]$ *is complete, the former conditions are also equivalent to the following:*

(v) *For every precompact subset* A *of* $E[\mathfrak{T}]$ *there is a positive integer* n *such that* A *is contained in* $E_n[\mathfrak{T}_n]$ *and it is precompact in this space.*

The techniques seems to be useful yet

The results have been used by many authors over the years: Bonet, Dierolf, Maestre, Bistrom, Robertson, Valdivia, Wengeroth, Lindstrom, Bierstedt, etc.

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Applications

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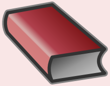

Descriptive Topology in Selected Topics of Functional Analysis

J. Kąkol, W. Kubiś, M. Lopez Pellicer

December 29, 2009

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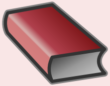

Descriptive Topology in Selected Topics of Functional Analysis

J. Kąkol, W. Kubiś, M. Lopez Pellicer

December 29, 2009

Tkachuk's views about the metrizability result

A selection of recent results and problems in C_p-theory ☆

V.V. Tkachuk

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Received 19 December 2006; accepted 18 March 2007

He wrote 20 years later

Cascales and Orihuela introduced a stronger notion in [29]; say that a space X is *strongly dominated by the irrationals* if it has an ω^ω-ordered compact cover $\mathcal{K} = \{K_f : f \in \omega^\omega\}$ and, for any compact subspace $K \subset X$ there is $f \in \omega^\omega$ such that $K \subset K_f$, i.e., the family $\mathcal{K}$ "swallows" all compact subsets of X. To show that strong domination by the irrationals is important for topologists it suffices to look at the following two results.

Theorem 2.8. *(See Christensen [31].) A second countable space is strongly dominated by the irrationals if and only if it is completely metrizable.*

Theorem 2.9. *(See Cascales and Orihuela [30].) A compact space K is metrizable if and only if the space $(K \times K)\setminus\Delta$ is strongly dominated by the irrationals. Here $\Delta = \{(x, x) : x \in K\}$ is the diagonal of the space K.*

An interesting thing about Theorem 2.9 is that Cascales and Orihuela proved this purely topological metrization result dealing with function spaces and nowadays no direct topological proof is known.

We came back 25 years later

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Domination by second countable spaces and Lindelöf Σ-property

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ABSTRACT

Given a space M, a family of sets $\mathcal{A}$ of a space X is *ordered by* M if $\mathcal{A} = \{A_K\colon K$ is a compact subset of $M\}$ and $K \subset L$ implies $A_K \subset A_L$. We study the class $\mathcal{M}$ of spaces which have compact covers ordered by a second countable space. We prove that a space $C_p(X)$ belongs to $\mathcal{M}$ if and only if it is a Lindelöf Σ-space. Under MA(ω_1), if X is compact and $(X \times X)\backslash\Delta$ has a compact cover ordered by a Polish space then X is metrizable; here $\Delta = \{(x,x)\colon x \in X\}$ is the diagonal of the space X. Besides, if X is a compact space of countable tightness and $X^2\backslash\Delta$ belongs to $\mathcal{M}$ then X is metrizable in ZFC.
We also consider the class $\mathcal{M}^*$ of spaces X which have a compact cover $\mathcal{F}$ ordered by a second countable space with the additional property that, for every compact set $P \subset X$ there exists $F \in \mathcal{F}$ with $P \subset F$. It is a ZFC result that if X is a compact space and $(X \times X)\backslash\Delta$ belongs to $\mathcal{M}^*$ then X is metrizable. We also establish that, under CH, if X is compact and $C_p(X)$ belongs to $\mathcal{M}^*$ then X is countable.

Metrizability of compact sets (II)

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ subsets of $(K \times K) \setminus \Delta$. We write:

(A) each A_α is compact;

(B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;

(C) $(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.

Theorem (Orihuela, Tkachuk, B.C. 2011)

(A) + **(B)** + **(C)** + $MA(\omega_1) \Rightarrow K$ *is metrizable.*

Metrizability of compact sets (II)

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Theorem (Orihuela, Tkachuk, B.C. 2011)

(A) + **(B)** + **(C)** + *MA*(ω_1) $\Rightarrow$ K *is metrizable.*

Proof.-

1. **(A)** + **(B)** + **(C)** + MA(ω_1) $\Rightarrow$ K has small diagonal, *i.e.*, for any uncountable set $A \subset (K \times K) \setminus \Delta$ there exists an uncountable $B \subset A$ such that $\overline{B} \cap \Delta = \emptyset$.

Metrizability of compact sets (II)

2.12. Theorem. *Assume* MA(ω_1) *and suppose that* X *is a compact space such that* $X^2\setminus\Delta$ *is* $\mathbb{P}$*-dominated. Then* X *has a small diagonal and hence* $t(X)=\omega$.

Proof. Suppose that $A=\{z_\alpha\colon \alpha<\omega_1\}\subset X^2\setminus\Delta$ and $\alpha\neq\beta$ implies $z_\alpha\neq z_\beta$. Fix a $\mathbb{P}$-directed cover $\{K_p\colon p\in\mathbb{P}\}$ of compact subsets of $X^2\setminus\Delta$. Take $p_\alpha\in\mathbb{P}$ such that $z_\alpha\in K_{p_\alpha}$ for any $\alpha<\omega_1$.

It follows from MA(ω_1) that there exists $p\in\mathbb{P}$ such that $p_\alpha\leqslant^* p$ for any $\alpha<\omega_1$. The set $P=\bigcup\{K_q\colon q\in\mathbb{P}$ and $q=^* p\}$ is σ-compact and $A\subset P$. Consequently, there is $q\in\mathbb{P}$ for which $K_q\cap A$ is uncountable; therefore the set $K_q\cap A$ witnesses the small diagonal property of X. Since no space with a small diagonal can have a convergent ω_1-sequence, it follows from [16, Theorem 1.2] that X has no free sequences of length ω_1, i.e., $t(X)\leqslant\omega$. □

Theorem (Orihuela, Tkachuk, B.C. 2011)

(A) + **(B)** + **(C)** + *MA*(ω_1) $\Rightarrow$ K *is metrizable.*

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1. **(A)** + **(B)** + **(C)** + MA(ω_1) $\Rightarrow$ K has small diagonal, *i.e.*, for any uncountable set $A\subset(K\times K)\setminus\Delta$ there exists an uncountable $B\subset A$ such that $\overline{B}\cap\Delta=\emptyset$.
2. K has small diagonal $\Rightarrow$ K has countable tightness $\Rightarrow$ $K\times K$ has countable tightness;

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Proposition, B. C., 1987

Given X and $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ as above, if we define $\psi : \mathbb{N}^\mathbb{N} \to 2^{(K\times K)\setminus\Delta}$ given by

$$\psi(\alpha) := \bigcap_{m=1}^{\infty} \bigcup \{A_\beta : \beta|_m = \alpha|_m\}$$

then:

- each $\psi(\alpha)$ is countably compact (even more, all cluster points of any sequence in $\psi(\alpha)$ remains in $\psi(\alpha)$).
- if $\psi(\alpha)$ is compact then $\alpha \to \psi(\alpha)$ gives K-analytic structure to $(K \times K) \setminus \Delta$.

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Me

$$\overline{\psi(\alpha)} \subset \psi(\alpha), \text{ (closure in } K \times K)$$

- take $x \in \overline{\psi(\alpha)}$;
- there is $A \subset \psi(\alpha)$ countable with $x \in \overline{A}$;
- is $x \in A \Rightarrow x \in \psi(\alpha)$;
- otherwise, $x \in (\overline{A} \setminus A) \Rightarrow x$ is cluster point of a sequence in $\psi(\alpha) \Rightarrow x \in \psi(\alpha)$.

Proposition, B. C., 1987

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(A) + **(B)** + **(C)** + *MA(ω_1)* $\Rightarrow$ *K is metrizable.*

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2. K has small diagonal $\Rightarrow$ K has countable tightness $\Rightarrow$ $K \times K$ has countable tightness;
3. $(K \times K) \setminus \Delta$ is K-analytic $\Rightarrow$ $(K \times K) \setminus \Delta$ is Lindelöf $\Rightarrow$ Δ is G_δ $\Rightarrow$ K is metrizable.

♣

Domination by Second Countable Spaces

Domination by Second Countable Spaces

Definition

A topological space X is dominated by a second countable space, if there is a second countable space M and a family $\{A_K : K \in \mathscr{K}(M)\} \subset X$ such that:

(A) each A_K is compact;

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Theorem (Orihuela, Tkachuk, B.C. 2011)

For a topological space TFAE:

1. *X is countably K-determined;*
2. *X is Dieudonné complete and dominated by a second countable space.*

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For a topological space TFAE:

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2. *X is Dieudonné complete and dominated by a second countable space.*

The class of spaces dominated by a second countable space enjoy the usual stability properties we might expect.

Techniques

Generation of usco maps, Orihuela and B. C. 1991

Let T be a first-countable, X a topological space and let $\varphi : T \to 2^X$ be a set-valued map satisfying the property

$$\bigcup_{n\in\mathbb{N}} \varphi(t_n) \text{ is relatively compact for each convergent sequence } (t_n)_n \text{ in } T. \quad (1)$$

If for each x in X we define

$$C(t) := \{x \in X : \text{there is } t_n \to t \text{ in } T, \text{ for every } n \in \mathbb{N} \text{ there is } x_n \in \varphi(t_n) \text{ and } x \text{ is cluster point of } (x_n)_n\}.$$

Then:

- each $C(t)$ is countably compact.
- if $\psi(t) := \overline{C(t)}$ is compact then $t \to \psi(t)$ is usco $\psi : T \to \mathscr{K}(X)$.

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$$\bigcup_{n\in\mathbb{N}} \varphi(t_n) \text{ is relatively compact for each convergent sequence } (t_n)_n \text{ in } T. \quad (1)$$

If X is dominated by a second countable space, if there is a second countable space M and a family $\{A_K : K \in \mathscr{K}(M)\}$ such that:

- **(A)** each A_K is compact;
- **(B)** $A_K \subset A_F$ whenever $K \subset F$;
- **(C)** $X = \bigcup\{A_K : K \in \mathscr{K}(M)\}$.

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If X is dominated by a second countable space, if there is a second countable space M and a family $\{A_K : K \in \mathscr{K}(M)\}$ such that:

(A) each A_K is compact;

(B) $A_K \subset A_F$ whenever $K \subset F$;

(C) $X = \bigcup\{A_K : K \in \mathscr{K}(M)\}$.

We take: $T := (\mathscr{K}(M), h)$, $\varphi(K) := A_K$ and we can generate the USCO ψ in many cases.

- each $C(t)$ is countably compact.
- if $\psi(t) := \overline{C(t)}$ is compact then $t \to \psi(t)$ is usco $\psi : T \to \mathscr{K}(X)$.

Two noticeable results

Theorem (Orihuela, Tkachuk, B.C. 2011)

$C_p(X)$ is countably K-determined iff is dominated by a second countable space.

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Theorem (Orihuela, Tkachuk, B.C. 2011)

$C_p(X)$ is countably K-determined iff is dominated by a second countable space.

Theorem (Orihuela, Tkachuk, B.C. 2011)

Let K be a compact space. If there is a second countable space M and a family $\{A_F : F \in \mathscr{K}(M)\} \subset (K \times K) \setminus \Delta$ such that:

- **(A)** *each A_F is compact;*
- **(B)** *$A_F \subset A_L$ whenever $F \subset L$;*
- **(C)** $(K \times K) \setminus \Delta = \bigcup \{A_F : F \in \mathscr{K}(M)\}$.

and

- **(D)** *every compact subset of $(K \times K) \setminus \Delta$ is contained in some A_F.*

Then K is metrizable.

Ready to finish... I kept the promise

We know have to solve the exercises and a bit more that become a

Theorem

For a compact space K TFAE: a...

1. K is metrizable;
2. $(C(K), \|\cdot\|_\infty)$ is separable;
3. Δ is a G_δ;
4. $\Delta = \cap_n G_n$ with G_n open and $\{G_n\}_n$ a basis of neighb. of Δ;
5. $(K\times K)\setminus\Delta = \cup_n F_n$, with $\{F_n\}$ an increasing fundamental family of compact sets in $(K\times K)\setminus\Delta$;
6. $(K\times K)\setminus\Delta = \bigcup\{A_\alpha : \alpha\in\mathbb{N}^{\mathbb{N}}\}$ with each $\{A_\alpha\}$ a fundamental family of compact sets such that $A_\alpha\subset A_\beta$ whenever $\alpha\leq\beta$;
7. $(K\times K)\setminus\Delta$ is Lindelöf.

Open questions

Open questions

K compact space &
$\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \subset (K \times K) \setminus \Delta$.
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Open question

(A) + **(B)** + **(C)** $\overset{?}{\Rightarrow}$ K is metrizable.

Open questions

More problems...here!

Domination by second countable spaces and Lindelöf Σ-property

B. Cascales[1,2], J. Orihuela[1,2] and V.V. Tkachuk[3,4]

Abstract. Given a space M, a family of sets $\mathcal{A}$ of a space X is *ordered by* M if $\mathcal{A}=\{A_K : K$ is a compact subset of $M\}$ and $K \subset L$ implies $A_K \subset A_L$. We study the class $\mathcal{M}$ of spaces which have compact covers ordered by a second countable space. We prove that a space $C_p(X)$ belongs to $\mathcal{M}$ if and only if it is a Lindelöf Σ-space. Under MA(ω_1), if X is compact and $(X\times X)\setminus\Delta$ has a compact cover ordered by a Polish space then X is metrizable; here $\Delta=\{(x,x): x\in X\}$ is the diagonal of the space X. Besides, if X is a compact space of countable tightness and $X^2\setminus\Delta$ belongs to $\mathcal{M}$ then X is metrizable in ZFC.

We also consider the class $\mathcal{M}^*$ of spaces X which have a compact cover $\mathcal{F}$ ordered by a second countable space with the additional property that, for every compact set $P\subset X$ there exists $F\in\mathcal{F}$ with $P\subset F$. It is a ZFC result that if X is a compact space and $(X\times X)\setminus\Delta$ belongs to $\mathcal{M}^*$ then X is metrizable. We also establish that, under CH, if X is compact and $C_p(X)$ belongs to $\mathcal{M}^*$ then X is countable.

Keywords: (strong) domination by irrationals, (strong) domination by a second countable space, diagonal, metrization, orderings by irrationals, orderings by a second countable space, compact cover, function spaces, cosmic spaces, $\aleph_0$-spaces, Lindelöf Σ-space, compact space, metrizable space

2000 Mathematics Subject Classification: 54B10, 54C05, 54D30

0. Introduction.

Given a space X we denote by $\mathcal{K}(X)$ the family of all compact subsets of X. One of about a dozen equivalent definitions says that X is *a Lindelöf Σ-space (or has the Lindelöf Σ-property)* if there exists a second countable space M and a compact-valued upper semicontinuous map $\varphi : M \to X$ such that $\bigcup\{\varphi(x) : x \in M\} = X$ (see, e.g., [RJ, Section 5.1]). It is worth mentioning that in Functional Analysis, the same concept is usually referred to as a *countably K-determined space.*

Suppose that X is a Lindelöf Σ-space and hence we can find a compact-valued upper semicontinuous surjective map $\varphi : M \to X$ for some second countable space M. If we let $F_K = \bigcup\{\varphi(x) : x \in K\}$ for any compact set $K \subset M$ then the family $\mathcal{F} = \{F_K : K \in \mathcal{K}(M)\}$ consists of compact subsets of X, covers X and $K \subset L$ implies $F_K \subset F_L$. We will say that $\mathcal{F}$ is an M-ordered compact cover of X.

The class $\mathcal{M}$ of spaces with an M-ordered compact cover for some second countable space M, was introduced by Cascales and Orihuela in [CO2]. They proved, among other things, that a Dieudonné complete space is Lindelöf Σ if and

[1] Research supported by FEDER and MEC, Project MTM2008-05396
[2] Research supported by Fundación Séneca de la CARM, Project 08848/PI/08
[3] Research supported by Consejo Nacional de Ciencia y Tecnología de México, Grant U48602-F
[4] Research supported by Programa Integral de Fortalecimiento Institucional (PIFI), Grant 34536-55

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} \subset (K\times K)\setminus\Delta$. We write:

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Open question

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Open questions

More problems...here!

Domination by second countable spaces and Lindelöf Σ-property

B. Cascales[1,2], J. Orihuela[1,2] and V.V. Tkachuk[3,4]

Abstract. Given a space M, a family of sets $\mathcal{A}$ of a space X is *ordered by* M if $\mathcal{A}=\{A_K : K$ is a compact subset of $M\}$ and $K \subset L$ implies $A_K \subset A_L$. We study the class $\mathcal{M}$ of spaces which have compact covers ordered by a second countable space. We prove that a space $C_p(X)$ belongs to $\mathcal{M}$ if and only if it is a Lindelöf Σ-space. Under MA(ω_1), if X is compact and $(X \times X)\setminus\Delta$ has a compact cover ordered by a Polish space then X is metrizable; here $\Delta=\{(x,x):x\in X\}$ is the diagonal of the space X. Besides, if X is a compact space of countable tightness and $X^2\setminus\Delta$ belongs to $\mathcal{M}$ then X is metrizable in ZFC.

We also consider the class $\mathcal{M}^*$ of spaces X which have a compact cover $\mathcal{F}$ ordered by a second countable space with the additional property that, for every compact set $P \subset X$ there exists $F \in \mathcal{F}$ with

Beware!!!

1. This lecture and the paper DO ONLY SHARE the results.
2. None of the proofs presented here are in the paper.

the same concept is usually referred to as a *countably K-determined space.*

Suppose that X is a Lindelöf Σ-space and hence we can find a compact-valued upper semicontinuous surjective map $\varphi : M \to X$ for some second countable space M. If we let $F_K = \bigcup\{\varphi(x) : x \in K\}$ for any compact set $K \subset M$ then the family $\mathcal{F} = \{F_K : K \in \mathcal{K}(M)\}$ consists of compact subsets of X, covers X and $K \subset L$ implies $F_K \subset F_L$. We will say that $\mathcal{F}$ is an M-ordered compact cover of X.

The class $\mathcal{M}$ of spaces with an M-ordered compact cover for some second countable space M, was introduced by Cascales and Orihuela in [CO2]. They proved, among other things, that a Dieudonné complete space is Lindelöf Σ if and

[1] Research supported by FEDER and MEC, Project MTM2008-05396
[2] Research supported by Fundación Séneca de la CARM, Project 08848/PI/08
[3] Research supported by Consejo Nacional de Ciencia y Tecnología de México, Grant U48602-F
[4] Research supported by Programa Integral de Fortalecimiento Institucional (PIFI), Grant 34536-55

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \subset (K \times K)\setminus\Delta$. We write:

(A) each A_α is compact;

(B) $A_\alpha \subset A_\beta$ whenever $\alpha \le \beta$;

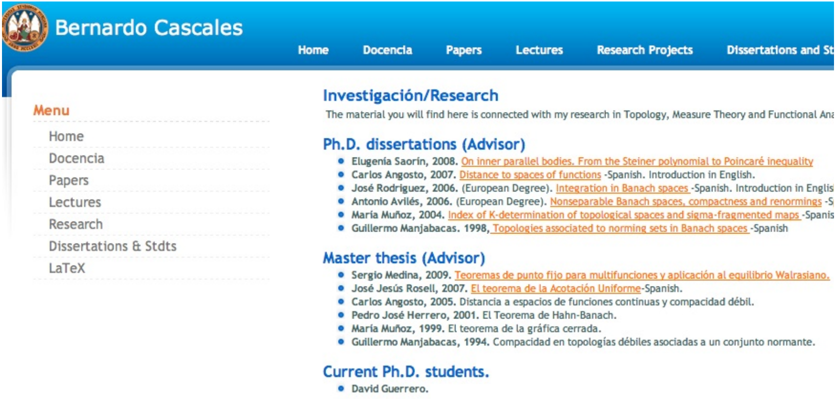
(C) $(K \times K)\setminus\Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.

Open question

(A) + **(B)** + **(C)** $\overset{?}{\Rightarrow}$ K is metrizable.

Further developments and related material at:

http://webs.um.es/beca

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