Universidad de Murcia | Departamento Matemáticas

Measurability and semicontinuity of multi-functions

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- **First 2 lecures: Upper and lower semi-continuity for multi-functions:**
    - generation of K-analytic structures, applications to functional analysis;
    - Michael's selection theorem; distances to spaces of continuous functions; quantitative perspective of compactness.
- **3rd lecture: Measurability for multi-functions**
    - Kuratowski-Ryll-Narzesdky selection theorem; extension to non separable Banach spaces.
    - integration of multifunction

Notation

- $L, M, \dots X, Y, \dots$ topological spaces; E, F Banach or sometimes lcs;
- K compact Hausdorff space;
- 2^X subsets; $\mathscr{K}(X)$ family of compact sets; if E Banach then $wk(E)$ weakly compact sets and $cwk(E)$ convex weakly compact sets;
- $C(X)$ continuous functions; $C_p(X)$ continuous functions endowed with the pointwise convergence topology τ_p;

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- $C(X)$ continuous functions; $C_p(X)$ continuous functions endowed with the pointwise convergence topology τ_p;
- $\Omega \subset \mathbb{C}$ open set; $\mathscr{H}(\Omega)$ space of holomorphic functions with the topology of uniform convergence on compact sets;
- $\Omega \subset \mathbb{R}^n$ open set; $\mathscr{D}'(\Omega)$ space of distributions;
- $\varinjlim E_n$ inductive limit of a sequence of Fréchet spaces.

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- $\Omega \subset \mathbb{R}^n$ open set; $\mathscr{D}'(\Omega)$ space of distributions;
- $\varinjlim E_n$ inductive limit of a sequence of Fréchet spaces.
- (Ω, Σ, μ) complete probability space;
- Σ^+ measurable sets of positive measure; for $A \in \Sigma$, Σ_A^+ measurable subsets of A of positive measure;
- measurability for scalar function $f : \Omega \to \mathbb{R}$ standard; measurability for $F : \Omega \to 2^E$ will be defined;

First sample... our goal is to understand

Theorem

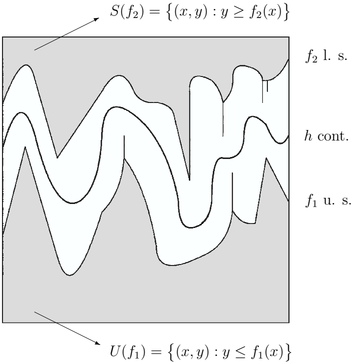
Let K be a compact space and Δ its diagonal. TFAE:

1. K is metrizable;
2. $(C(K), \|\cdot\|_\infty)$ is separable;
3. Δ is a G_δ;
4. $\Delta = \cap_n G_n$ with G_n open and $\{G_n\}_n$ a basis of neighb. of Δ;
5. $(K \times K) \setminus \Delta = \cup_n F_n$, with $\{F_n\}$ an increasing fundamental family of compact sets in $(K \times K) \setminus \Delta$;
6. $(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ with each $\{A_\alpha\}$ a fundamental family of compact sets such that $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;
7. $(K \times K) \setminus \Delta$ is Lindelöf.

and apply it to Functional Analysis.

Second sample... our goal is to understand

how to use Michael's selection theorem to prove that if



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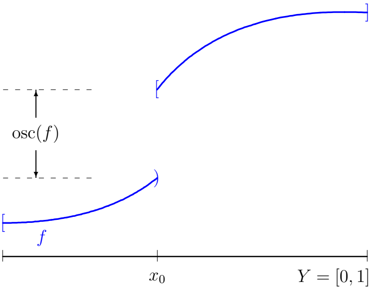

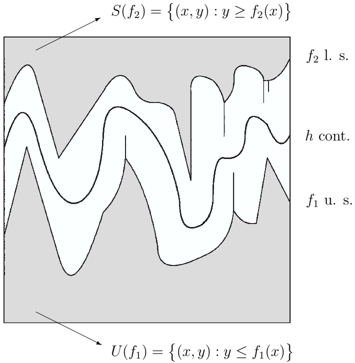

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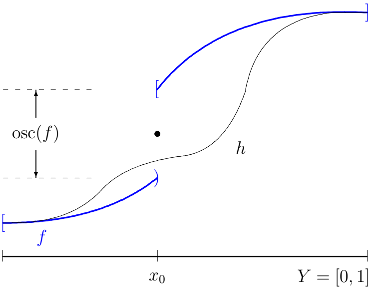

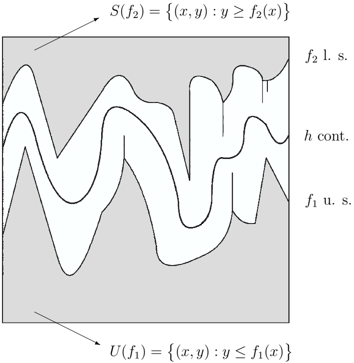

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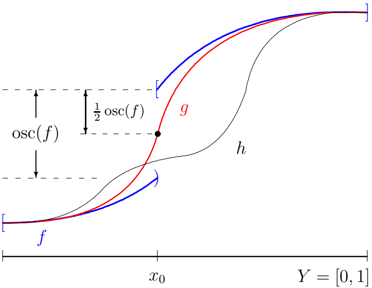

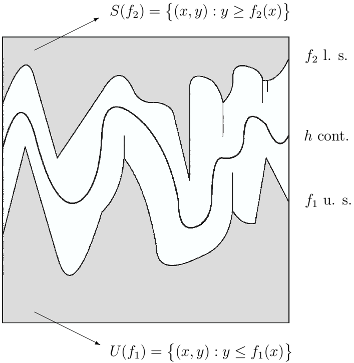

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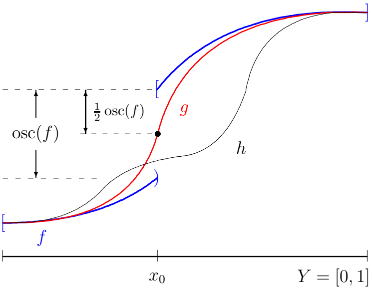
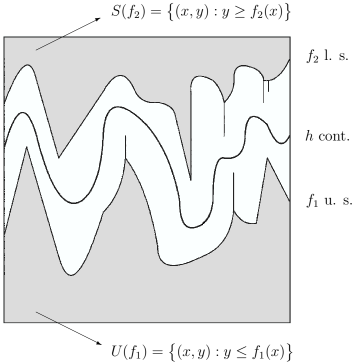
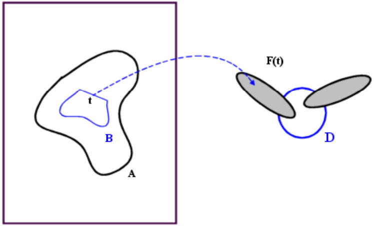




and apply it to define measures of non compactness in spaces of continuous functions and Banach spaces.

Third sample... our goal is to understand how the notion



Definition

$F : \Omega \to 2^E$ satisfies property (P) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma_A^+$ and $D \subset E$ with $\mathrm{diam}(D) < \varepsilon$ such that

$$F(t) \cap D \neq \emptyset \text{ for every } t \in B.$$

helps to produce measurable selectors beyond the separable case and to extend integration of multi-function to this general setting.

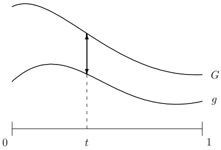
Multi-functions. Examples

Definition

A multi-function (set-valued map, multi-map, etc.) is a map ψ from a set X into the family of subsets 2^Y of another set Y, *i.e.*, for each $x \in X$ the image $\psi(x)$ is a subset of Y.

Examples:

- the map $log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that sends every $z \in \mathbb{C} \setminus \{0\}$ to the set of its logarithms;
- if $g, G : [0, 1] \to \mathbb{R}$ are functions with $g(x) \leq G(x)$ then $\psi(x) := [g(x), G(x)]$ is a multifunction;



Multi-functions. Examples

Examples:

- if $f : Y \to X$ is onto, then $\psi(x) := f^{-1}(x)$, $x \in X$ is a multi-function;
- $f \to \{x \in K : |f(x)| = \|f\|_\infty\}$ is multi-function defined in $C(K)$;
- if E is a Banach space $J : B_E \to 2^{B_E^*}$ given by

$$J(x) := \{x^* \in B_E^* : \|x\| = x^*(x)\} \text{ (duality map)}$$

is a multi-function;

Multi-functions. Examples

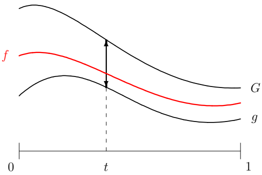
. . . more examples:

- if E is a Banach space and $F \subset E$ is closed proximinal, then $x \to \{y \in F : \|x - y\| = d(x, F)\}$ is a multi-function (metric projection);
- If E is a Fréchet space, and $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$ is a basis of neighb. of 0 then $\psi : \mathbb{N}^{\mathbb{N}} \to 2^E$ given by

$$\psi(\alpha) := \bigcap_{k=1}^{\infty} n_k V_k, \text{ with } \alpha = (n_k)_k$$

is a multi-function with $\psi(\mathbb{N}^{\mathbb{N}}) = E$, $\psi(\alpha) \subset \psi(\beta)$ if $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$ and $\{\psi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ fundamental family of bounded sets.

Multi-functions. Selectors



Definition

Given a multi-function $\psi : X \to 2^Y$ a selector is a single-valued map $f : X \to Y$ such that $f(x) \in \psi(x)$ for each $x \in X$.

(Jayne-Rogers) E is Asplund if, and only if, the duality map has a Baire-1 selector;

(Michael) if $\psi : M \to 2^E$ (E Banach, M metric) is lower semi-continuous, takes convex closed values, then ψ has a continuos selector;

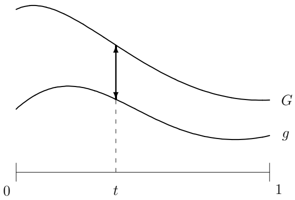
(Kuratowski-Ryll Nardzewski, 1965) Let $F : \Omega \to 2^E$ be a multi-function with closed non empty values of E. If E is separable and F satisfies that

$$\{t \in \Omega : F(t) \cap O \neq \emptyset\} \in \Sigma \text{ for each open set } O \subset E. \tag{E}$$

Then F admits a μ-measurable selector f.

The need of good selectors...integration of multi-functions

$F : \Omega \longrightarrow cwk(E)$ –convex w-compact



There are several possibilities to define the integral of F:

1. to take a reasonable embedding j from $cwk(E)$ into the Banach space $Y (= \ell_\infty(B_{E^*}))$ and then study the integrability of $j \circ F$;

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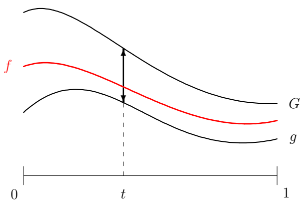

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$$\int F\, d\mu = \left\{ \int f\, d\mu : f \text{ integra. sel.} F \right\}.$$

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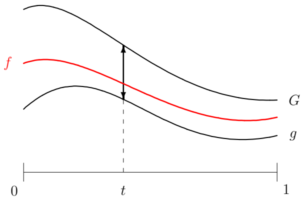

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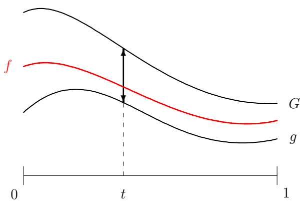

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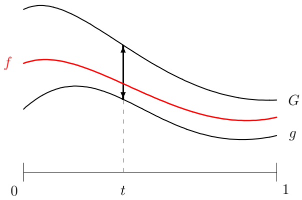

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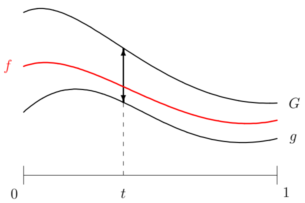

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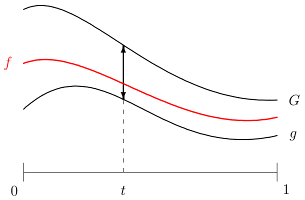

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The non-separable case

1. Pettis integration theory was stuck in the separable case for the lack of a selection result in the general case.

4. Pettis integration for multi-functions was developed in the separable case.

The need of knowing the implications of semi-continuity properties of multi-functions

Theorem, Talagrand 1975

Every WCG Banach space E is weakly Lindelöf.

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Theorem, Talagrand 1975

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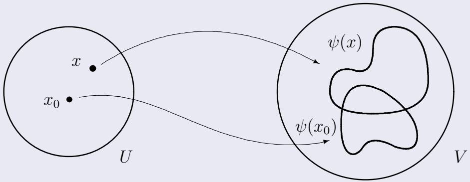
. . . we will see several more applications.

Upper-semicontinuity

Definitions

Upper semi-continuous set-valued map (multi-function)

$$X \xrightarrow{\ \psi\ } 2^Y$$



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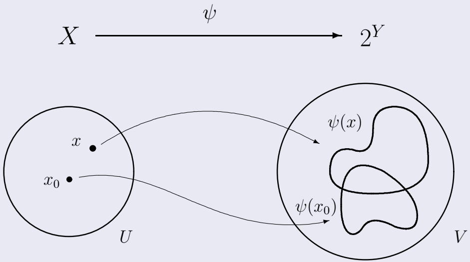

1. Y is K-analytic if there is $\psi : \mathbb{N}^{\mathbb{N}} \to 2^Y$ that is Upper semi-continuous compact-valued and such that $Y = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \psi(\alpha)$;

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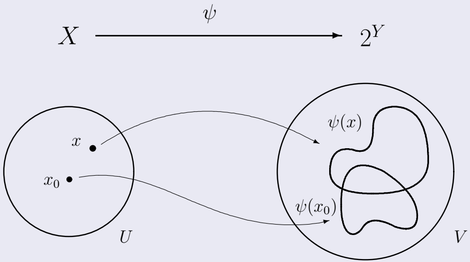

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2. Y is countably K-determined if there is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and $\psi : \Sigma \to 2^Y$ that is upper semi-continuous compact-valued and such that $Y = \bigcup_{\alpha \in \Sigma} \psi(\alpha)$.

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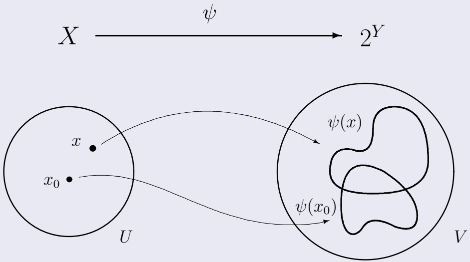

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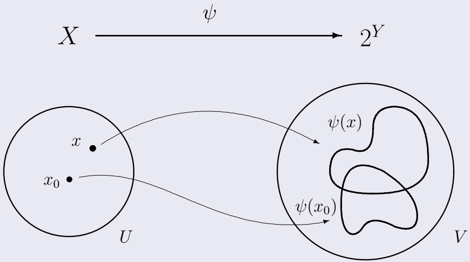

$\Sigma \Leftrightarrow$ any second countable space M (Lindelöf Σ)

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Easy known facts

1. If $\psi : X \to 2^Y$ that is upper semi-continuous compact-valued, then $K \subset X$ is compact $\Rightarrow$ $\psi(K)$ is compact;

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3. K-analytic $\Rightarrow$ countably K-determined $\Rightarrow$ Lindelöf;
4. countably K-determined $+$ metrizable $\Rightarrow$ separable;
5. if X is K-analytic ($\psi : \mathbb{N}^{\mathbb{N}} \to 2^X$) and $A_\alpha := \psi(\{\beta : \beta \leq \alpha\})$ then:

   (A) each A_α is compact;
   (B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;
   (C) $X = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

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6. ditto, if X is countably K-determined, there is a second countable space M and a family $\{A_K : K \in \mathscr{K}(M)\}$ such that:

   (A) each A_K is compact;
   (B) $A_K \subset A_F$ whenever $K \subset F$;
   (C) $X = \bigcup\{A_K : K \in \mathscr{K}(M)\}$.

To keep in mind

Proposition

Let X be a metric space and $\psi : X \to 2^Y$ multi-valued . TFAE:

1. ψ is usco;
2. ψ is compact valued $+$ For every sequence $x_n \to x$ in X if $y_n \in \psi(x_n)$, $n \in \mathbb{N}$ then $(y_n)_n$ has a cluster point $y \in \psi(x)$.

Simple facts to keep in mind

1. $\mathbb{N}^{\mathbb{N}}$ endowed with the product of discrete topology on $\mathbb{N}$ is separable and metrizable with a complete metric (*i.e.* $\mathbb{N}^{\mathbb{N}}$ is a Polish space).

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2. If $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ then there is $\beta \in \mathbb{N}^{\mathbb{N}}$ such that

$$\alpha_n, \alpha \leq \beta$$

(here $\leq$ stands for the natural order for the coordinates)

2nd LECTURE/SEP 15 2011/Things to remember

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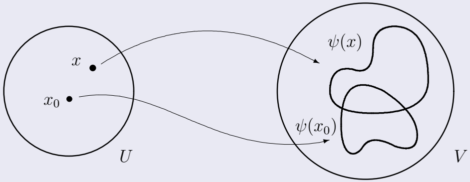

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2nd LECTURE/SEP 15 2011/Things to remember

1. K-analytic $\Rightarrow$ Lindelöf;
2. K-analytic + metrizable (or sub-metrizable) $\Rightarrow$ separable;
3. if X is K-analytic ($\psi : \mathbb{N}^{\mathbb{N}} \to 2^X$) and $A_\alpha := \psi(\{\beta : \beta \leq \alpha\})$ then:

   (A) each A_α is compact;
   (B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;
   (C) $X = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

From increasing compact coverings to K-analyticity

- Let X be a topological space $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of X with:

  (A) each A_α is compact;

  (B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;

  (C) $X = \bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and $m \in \mathbb{N}$, define

$$C_{n_1, n_2, \ldots, n_k} := \bigcup \{A_\beta : \beta|_k = (n_1, n_2, \ldots, n_k)\}$$

From increasing compact coverings to K-analyticity

- Let X be a topological space $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of X with:
  - **(A)** each A_α is compact;
  - **(B)** $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;
  - **(C)** $X = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.
- Given $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ and $m \in \mathbb{N}$, define

$$C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta|_k = (n_1, n_2, \ldots, n_k)\}$$

Proposition, B. C., 1987

Given X and $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ as above, if we define $\varphi : \mathbb{N}^\mathbb{N} \to 2^X$ given by

$$\psi(\alpha) := \bigcap_{k=1}^{\infty} C_{n_1,n_2,\ldots,n_k}$$

then:

- $A_\alpha \subset \psi(\alpha)$, hence $\psi(\mathbb{N}^\mathbb{N}) = X$;
- if $\alpha_n \to \alpha$ in $\mathbb{N}^\mathbb{N}$ and $y_n \in \psi(\alpha_n)$ then (y_n) has a cluster point in $\psi(\alpha)$;

X has K-analytic structure if *countably compact subsets=compact subsets*.

From where the ideas were inspired

Talagrand, Ann. of Math. 1979

Annals of Mathematics, 110 (1979), 407-438

Espaces de Banach faiblement $\mathcal{K}$-analytiques

Par Michel Talagrand

Proposition 6.13. *Soit K un espace compact. Les assertions suivantes sont équivalentes:*

a). *K est de type $\mathcal{E}_1$.*

b). *Il existe une application croissante $\sigma \to A_\sigma$ de Σ (muni de l'ordre produit) dans l'ensemble des compacts de $\mathcal{C}_p(K)$ telle que $\bigcup_{\sigma\in\Sigma} A_\sigma$ sépare les points de K.*

Démonstration. Nous savons déjà que a) implique b) l'application

Two nice previous cases

Valdivia, J. London Math. Soc. 1987

QUASI-LB-SPACES

MANUEL VALDIVIA

We shall see later that properties (1) and (2) are important in order to obtain some results on the closed graph theorem. This is the reason for introducing the following definitions. *A quasi-*LB*-representation in a topological vector space F is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of Banach discs satisfying the following conditions:*

1. $\bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = F$;
2. *if* $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ *and* $\alpha \leqslant \beta$ *then* $A_\alpha \subset A_\beta$.

Talagrand's solution to a conjecture Corson

Theorem, Talagrand 1975

Every WCG Banach space E is weakly Lindelöf.

Proof.-

- Fix $W \subset E$ absolutely convex w-compact with $E = \overline{\operatorname{span} W}$.
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$,

$$A_\alpha := \left(n_1 W + B_{E^{**}}\right) \cap \left(n_2 W + \frac{1}{2} B_{E^{**}}\right) \cap \cdots \cap \left(n_1 W + \frac{1}{k} B_{E^{**}}\right) \cap \ldots$$

- Proposition $\Rightarrow$ (E,w) K-analytic $\Rightarrow$ (E,w) Lindelöf. ♣

Fréchet-Montel spaces

Theorem, Dieudonné 1954

Every Fréchet-Montel space E is separable (in particular $\mathscr{H}(\Omega)$ is separable).

Proof.-

- Fix $V_1 \supset V_2 \supset \cdots \supset V_n \ldots$ a basis of closed neighborhoods of 0.
- Given $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$,

$$A_\alpha := \bigcap_{k=1}^{\infty} n_k V_k .$$

- $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ fundamental family of bdd closed sets=compact;
- Proposition $\Rightarrow$ E K-analytic +metrizable $\Rightarrow$ E Lindelöf + metrizable $\Rightarrow$ E separable. ♣

$\mathscr{D}'(\Omega)$ is analytic

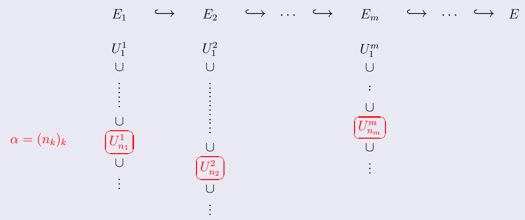
Theorem, $\mathscr{D}'(\Omega)$ is analytic.

The strong dual of every inductive limit of Fréchet-Montel spaces is analytic.

$\mathscr{D}'(\Omega)$ is analytic

Theorem, $\mathscr{D}'(\Omega)$ is analytic.

The strong dual of every inductive limit of Fréchet-Montel spaces is analytic.



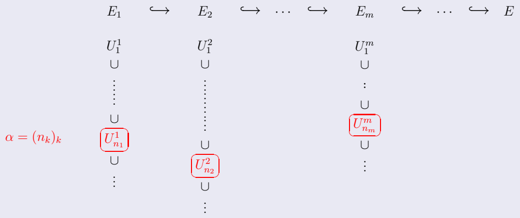
$$U_\alpha := \overline{\operatorname{aco}\left(\bigcup_{k=1}^{\infty} U_{n_k}^k\right)}$$

- $U_\beta \subset U_\alpha$ si $\alpha \leq \beta$; $\mathscr{U} := \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ neigh. basis of 0 en E.
- $A_\alpha := U_\alpha^\circ$ compact & $A_\alpha \subset A_\beta$, $\alpha \leq \beta$;
- $E' = \cup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and E' sub-metrizable $\Rightarrow$ E' K-analytic sub-metrizable $\Rightarrow$ E' analytic.

$\mathscr{D}'(\Omega)$ is analytic

Theorem, $\mathscr{D}'(\Omega)$ is analytic.

The strong dual of every inductive limit of Fréchet-Montel spaces is analytic.



Schwartz, 1964

Any Borel linear map from a separable Banach space into $\mathscr{D}'(\Omega)$ is continuous. In particular, the Closed Graph Theorem holds for linear maps

$$T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega).$$

Metrizability of compact sets (I)

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ subsets of $(K \times K) \setminus \Delta$. We write:

(A) each A_α is compact;

(B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;

(C) $(K \times K) \setminus \Delta = \bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

Theorem (Orihuela, B.C. 1987)

(A) + **(B)** + **(C)** + **(D)** $\Rightarrow$ *K is metrizable.*

(D) *For each compact set $F \subset (K \times K) \setminus \Delta$, there is A_α such that $F \subset A_\alpha$.*

Me

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ subsets of $(K\times K)\setminus\Delta$. We write:

- **(A)** each A_α is compact;
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Theorem (Orihuela, B.C. 1987)

(A) + **(B)** + **(C)** + **(D)** $\Rightarrow$ *K is metrizable.*

(D) *For each compact set $F \subset (K\times K)\setminus\Delta$, there is A_α such that $F \subset A_\alpha$.*

Proof.-

1. Given $\alpha \in \mathbb{N}^\mathbb{N}$, define $N_\alpha := (K\times K)\setminus A_\alpha$.
2. N_α is a basis of open neighborhoods of Δ;
3. $B_\alpha := \{f \in C(K) : \|f\|_\infty \le n_1, |f(x)-f(y)| \le \frac{1}{m},$ whenever $(x,y) \in N_{\alpha|^m}\}$; for $\alpha|^m := (n_m, n_{m+1}, \dots)$, $m \in \mathbb{N}$.
4.
   - **(A)** each B_α is $\|\ \|_\infty$-bdd & closed & equicontinuous $\overset{Ascoli}{\Rightarrow}$ B_α is $\|\ \|_\infty$-compact;
   - **(B)** $B_\alpha \subset B_\beta$ whenever $\alpha \le \beta$;
   - **(C)** $C(K) = \bigcup\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.
5. $(C(K), \|\ \|_\infty)$ is K-analytic +metrizable $\Rightarrow$ E Lindelöf + metrizable $\Rightarrow$ E separable $\Rightarrow$ K is metrizable. ♣

The original paper

Math. Z. 195, 365–381 (1987)

Mathematische Zeitschrift

On Compactness in Locally Convex Spaces

B. Cascales and J. Orihuela

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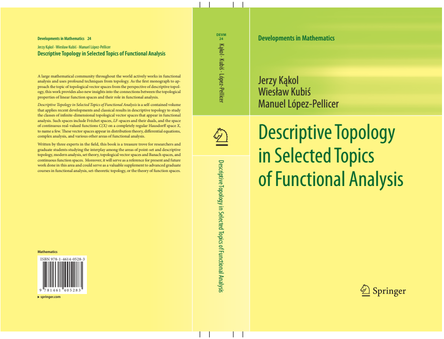
1. Introduction and Terminology

The purpose of this paper is to show that the behaviour of compact subsets in many of the locally convex spaces that usually appear in Functional Analysis is as good as the corresponding behaviour of compact subsets in Banach spaces. Our results can be intuitively formulated in the following terms: *Dealing with metrizable spaces or their strong duals, and carrying out any of the usual operations of countable type with them, we ever obtain spaces with their precompact subsets metrizable, and they even give good performance for the weak topology, indeed they are weakly angelic,* [14], *and their weakly compact subsets are metrizable if and only if they are separable.*

The techniques seems to be useful yet



INDEX

5 Strongly web-compact spaces and Closed Graph Theorem 167
5.1 Strongly web-compact spaces . . . 167
5.2 Products of strongly web-compact spaces . . . 168
5.3 A Closed Graph Theorem for strongly web-compact spaces . . . 170

6 Weakly analytic spaces 175
6.1 Something about analytic spaces . . . 175
6.2 Christensen theorem . . . 182
6.3 Subspaces of analytic spaces . . . 189
6.4 Trans-separable topological spaces . . . 192
6.5 Weakly analytic spaces need not be analytic . . . 200
6.6 When a weakly analytic locally convex space is analytic? . . . 204
6.7 Weakly compact density condition . . . 206
6.8 More examples of non-separable weakly analytic tvs . . . 215

7 K-analytic Baire spaces 223
7.1 Baire tvs with a bounded resolution . . . 223
7.2 Continuous maps on spaces with resolutions . . . 229

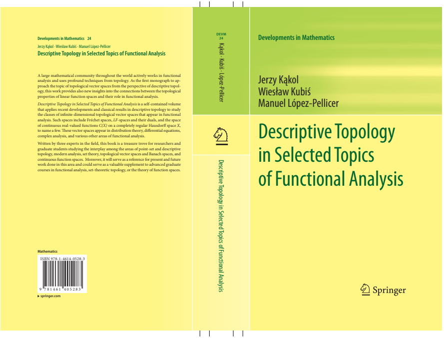
8 A three-space property for analytic spaces 235
8.1 Corson's example . . . 235
8.2 A positive result and a counterexample . . . 239

9 K-analytic and analytic spaces $C_p(X)$ 245
9.1 Talagrand's theorem for spaces $C_p(X)$ . . . 245
9.2 Christensen and Calbrix theorem for $C_p(X)$ . . . 249
9.3 Bounded resolutions for spaces $C_p(X)$ . . . 257
9.4 More examples of K-analytic spaces $C_p(X)$ and $C_p(X,E)$ . . . 278
9.5 K-analytic spaces $C_p(X)$ over locally compact groups X . . . 280
9.6 K-analytic group $X_p^\wedge$ of homomorphisms . . . 284

10 Precompact sets in (LM)-spaces and dual metric spaces 289
10.1 The case of (LM)-spaces, elementary approach . . . 289
10.2 The case of dual metric spaces, elementary approach . . . 292

11 Metrizability of compact sets in class $\mathfrak{G}$ 295
11.1 Spaces in class $\mathfrak{G}$ and examples . . . 295
11.2 Cascales-Orihuela theorem and applications . . . 298

The techniques seems to be useful yet



INDEX

12 Weakly and *-weakly realcompact locally convex spaces 305
- 12.1 Tightness and quasi-Suslin weak duals . . . 305
- 12.2 A Kaplansky type theorem about tightness . . . 309
- 12.3 More about K-analytic spaces in class $\mathfrak{G}$ . . . 314
- 12.4 Every (WCG) Fréchet space is weakly K-analytic . . . 318
- 12.5 About a theorem of Amir-Lindenstrauss and (CSPP) property 325
- 12.6 An example of R. Pol . . . 331
- 12.7 One more fact about Banach spaces $C_c(X)$ over compact scattered X . . . 337

13 Corson property (C) and tightness 341
- 13.1 Property (C) and weakly Lindelöf Banach spaces . . . 341
- 13.2 Property (C) for Banach spaces $C(K)$ . . . 348

14 Fréchet-Urysohn spaces and topological groups 353
- 14.1 Fréchet-Urysohn topological spaces . . . 353
- 14.2 A few facts about Fréchet-Urysohn topological groups . . . 356
- 14.3 Sequentially complete Fréchet-Urysohn lcs are Baire . . . 362
- 14.4 Three-space property for spaces Fréchet-Urysohn . . . 366
- 14.5 Topological vector spaces with bounded tightness . . . 369

15 Sequential conditions in class $\mathfrak{G}$ 373
- 15.1 Fréchet-Urysohn lcs are metrizable in class $\mathfrak{G}$ . . . 373
- 15.2 Sequential (LM)-spaces and dual metric spaces . . . 380
- 15.3 (LF)-spaces with property C_3^- and φ . . . 392

16 Tightness and distinguished Fréchet spaces 401
- 16.1 A characterization of distinguished spaces in term of tightness 401
- 16.2 $\mathfrak{G}$-bases and tightness . . . 409
- 16.3 $\mathfrak{G}$-bases, bounding and dominating cardinals and tightness . . 414
- 16.4 More about the Wulbert-Morris space $C_c(\omega_1)$ . . . 427

17 Banach spaces with many projections 433
- 17.1 Preliminaries, model-theoretic tools . . . 433
- 17.2 Projections from elementary submodels . . . 441
- 17.3 Lindelöf property of weak topologies . . . 444
- 17.4 Separable complementation property . . . 445
- 17.5 Projectional skeletons . . . 450

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Domination by second countable spaces and Lindelöf Σ-property

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Compact cover
Function spaces
Cosmic spaces

ABSTRACT

Given a space M, a family of sets $\mathcal{A}$ of a space X is *ordered by* M if $\mathcal{A} = \{A_K\colon\ K$ is a compact subset of $M\}$ and $K \subset L$ implies $A_K \subset A_L$. We study the class $\mathcal{M}$ of spaces which have compact covers ordered by a second countable space. We prove that a space $C_p(X)$ belongs to $\mathcal{M}$ if and only if it is a Lindelöf Σ-space. Under $\mathrm{MA}(\omega_1)$, if X is compact and $(X \times X)\backslash\Delta$ has a compact cover ordered by a Polish space then X is metrizable; here $\Delta = \{(x,x)\colon\ x \in X\}$ is the diagonal of the space X. Besides, if X is a compact space of countable tightness and $X^2\backslash\Delta$ belongs to $\mathcal{M}$ then X is metrizable in ZFC.
We also consider the class $\mathcal{M}^*$ of spaces X which have a compact cover $\mathcal{F}$ ordered by a second countable space with the additional property that, for every compact set $P \subset X$ there exists $F \in \mathcal{F}$ with $P \subset F$. It is a ZFC result that if X is a compact space and $(X \times X)\backslash\Delta$ belongs to $\mathcal{M}^*$ then X is metrizable. We also establish that, under CH, if X is compact and $C_p(X)$ belongs to $\mathcal{M}^*$ then X is countable.

Metrizability of compact sets (II)

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ subsets of $(K \times K) \setminus \Delta$. We write:

(A) each A_α is compact;

(B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;

(C) $(K \times K) \setminus \Delta = \bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

Theorem (Orihuela, Tkachuk, B.C. 2011)

(A) + **(B)** + **(C)** + *MA(ω_1)* $\Rightarrow$ *K is metrizable.*

2.12. Theorem. *Assume* MA(ω_1) *and suppose that X is a compact space such that $X^2 \setminus \Delta$ is $\mathbb{P}$-dominated. Then X has a small diagonal and hence $t(X) = \omega$.*

Proof. Suppose that $A = \{z_\alpha \colon \alpha < \omega_1\} \subset X^2 \setminus \Delta$ and $\alpha \neq \beta$ implies $z_\alpha \neq z_\beta$. Fix a $\mathbb{P}$-directed cover $\{K_p \colon p \in \mathbb{P}\}$ of compact subsets of $X^2 \setminus \Delta$. Take $p_\alpha \in \mathbb{P}$ such that $z_\alpha \in K_{p_\alpha}$ for any $\alpha < \omega_1$.

It follows from MA(ω_1) that there exists $p \in \mathbb{P}$ such that $p_\alpha \leqslant^* p$ for any $\alpha < \omega_1$. The set $P = \bigcup \{K_q \colon q \in \mathbb{P}$ and $q =^* p\}$ is σ-compact and $A \subset P$. Consequently, there is $q \in \mathbb{P}$ for which $K_q \cap A$ is uncountable; therefore the set $K_q \cap A$ witnesses the small diagonal property of X. Since no space with a small diagonal can have a convergent ω_1-sequence, it follows from [16, Theorem 1.2] that X has no free sequences of length ω_1, i.e., $t(X) \leqslant \omega$. $\square$

Open questions

More problems...here!

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Domination by second countable spaces and Lindelöf Σ-property

B. Cascales [a,1,2], J. Orihuela [a,1,2], V.V. Tkachuk [b,*,3,4]

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$\aleph_0$-spaces
Lindelöf Σ-space
Compact space
Metrizable space

A B S T R A C T

Given a space M, a family of sets $\mathcal{A}$ of a space X is *ordered by* M if $\mathcal{A} = \{A_K\colon K$ is a compact subset of $M\}$ and $K \subset L$ implies $A_K \subset A_L$. We study the class $\mathcal{M}$ of spaces which have compact covers ordered by a second countable space. We prove that a space $C_p(X)$ belongs to $\mathcal{M}$ if and only if it is a Lindelöf Σ-space. Under MA(ω_1), if X is compact and $(X \times X)\setminus\Delta$ has a compact cover ordered by a Polish space then X is metrizable; here $\Delta = \{(x,x)\colon x \in X\}$ is the diagonal of the space X. Besides, if X is a compact space of countable tightness and $X^2\setminus\Delta$ belongs to $\mathcal{M}$ then X is metrizable in ZFC.

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0. Introduction

Given a space X we denote by $\mathcal{K}(X)$ the family of all compact subsets of X. One of about a dozen equivalent definitions says that X is *a Lindelöf Σ-space* (*or has the Lindelöf Σ-property*) if there exists a second countable space M and a compact-valued upper semicontinuous map $\varphi : M \to X$ such that $\bigcup\{\varphi(x)\colon x \in M\} = X$ (see, e.g., [23, Section 5.1]). It is worth mentioning that in Functional Analysis, the same concept is usually referred to as a *countably K-determined space*.

Suppose that X is a Lindelöf Σ-space and hence we can find a compact-valued upper semicontinuous surjective map $\varphi : M \to X$ for some second countable space M. If we let $F_K = \bigcup\{\varphi(x)\colon x \in K\}$ for any compact set $K \subset M$ then the family $\mathcal{F} = \{F_K\colon K \in \mathcal{K}(M)\}$ consists of compact subsets of X, covers X and $K \subset L$ implies $F_K \subset F_L$. We will say that $\mathcal{F}$ is an M-ordered compact cover of X.

The class $\mathcal{M}$ of spaces with an M-ordered compact cover for some second countable space M, was introduced by Cascales and Orihuela in [9]. They proved, among other things, that a Dieudonné complete space is Lindelöf Σ if and only

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(A) each A_α is compact;

(B) $A_\alpha \subset A_\beta$ whenever $\alpha \le \beta$;

(C) $(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

Open question

(A) + **(B)** + **(C)** $\overset{?}{\Rightarrow}$ K is metrizable.

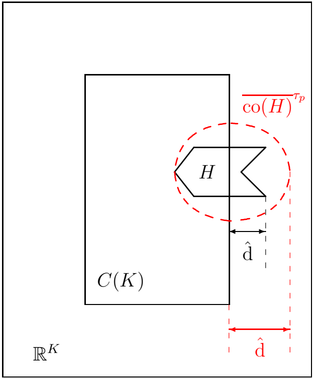
Lower-semicontinuity

...our goal

...goals

- To offer quantitative versions of the results about compactness for spaces $C(K)$, $C(X)$, $B_1(X)$, Banach spaces, etc.

...our goal

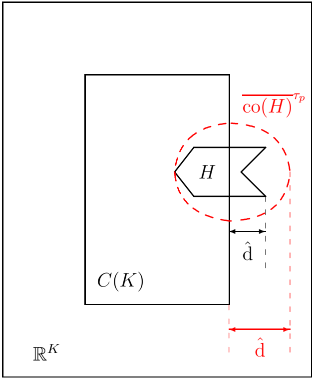


$$\hat{\mathrm{d}} \leq \hat{\mathrm{d}} \leq 5\hat{\mathrm{d}}$$

...goals

- To offer quantitative versions of the results about compactness for spaces $C(K)$, $C(X)$, $B_1(X)$, Banach spaces, etc.

...our goal

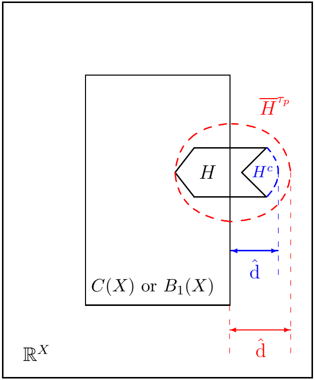


$$\hat{\mathrm{d}} \leq \hat{\mathrm{d}} \leq 5\hat{\mathrm{d}}$$

...goals

- To offer quantitative versions of the results about compactness for spaces $C(K)$, $C(X)$, $B_1(X)$, Banach spaces, etc.
- To offer new applications of these quantitative versions;

...our goal



$$\hat{d} \leq \hat{d} \leq M\hat{d}$$

...goals

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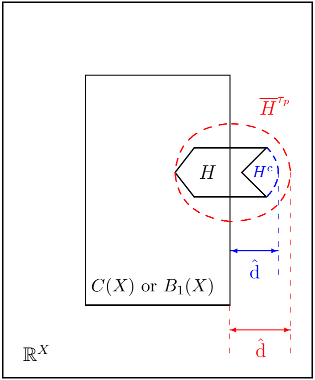
...our goal



$$\hat{d} \leq \hat{d} \leq M\hat{d}$$

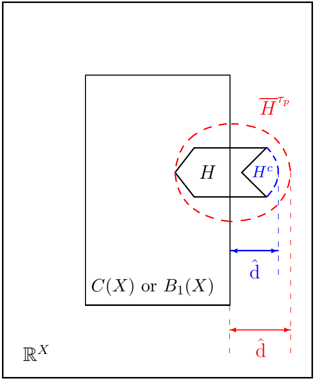
...goals

- To offer quantitative versions of the results about compactness for spaces $C(K)$, $C(X)$, $B_1(X)$, Banach spaces, etc.
- To offer new applications of these quantitative versions;

tools

- new reading of the *classical*;

...our goal



$$\hat{d} \leq \hat{d} \leq M\hat{d}$$

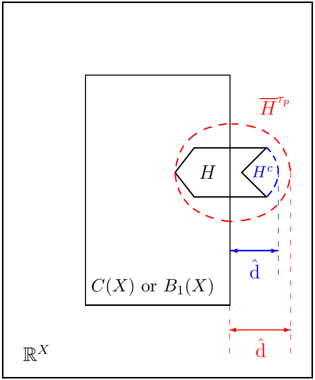
...goals

- To offer quantitative versions of the results about compactness for spaces $C(K)$, $C(X)$, $B_1(X)$, Banach spaces, etc.
- To offer new applications of these quantitative versions;

tools

- new reading of the *classical*;
- *double limits techniques* used by Grothendieck; techniques learnt when dealing with Asplund spaces (fragmentability);

...our goal



$$\hat{d} \leq \hat{d} \leq M\hat{d}$$

...goals

- To offer quantitative versions of the results about compactness for spaces $C(K)$, $C(X)$, $B_1(X)$, Banach spaces, etc.
- To offer new applications of these quantitative versions;

tools

- new reading of the *classical*;
- *double limits techniques* used by Grothendieck; techniques learnt when dealing with Asplund spaces (fragmentability);
- selectors for lower semincontinuos functions;

Lower semicontinuous multi-functions

Definition

If X and Y are topological spaces, a multi-functions $\psi : X \to 2^Y$ is said to be lower semicontinuous if for every open set $G \subset Y$ the set

$$\{x \in X : \psi(x) \cap G \neq \emptyset\}$$

is open in X.

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Example

$f : Y \to X$ onto, then $\psi : X \to 2^Y$ given by

$$\psi(x) := f^{-1}(x), \text{ for every } x \in X$$

is lower semicontinuous iff f is open.

Proof.-

$$\left\{x \in x : f^{-1}(x) \cap G \neq \emptyset\right\} = f(G).$$

Michael's selection theorem

Theorem (Michael, 1956)

If X is paracompact (for instance compact or metric) E a Banach space and $\psi : X \to 2^E$ is lower semicontinuous with closed convex values, then ψ has a continuous selector.

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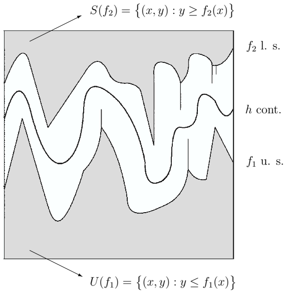
Corollary (Teorema de Bartle-Graves, 1952)

If E and F are Banach spaces, and $T : E \to F$ lineal continuous and onto then there is a continuous map $S : F \to E$ such that $T \circ S = id_F$.

Apply the above to $E \to E/H$ for $H \subset E$ closed subspace.

Sandwich's theorem



Theorem

Let $f_2, f_1 : K \to \mathbb{R}$ be a lower and a upper semicontinuous function with $f_2 \geq f_1$. Then, there exists a function $h \in C(K)$ such that $f_2 \geq h \geq f_1$.

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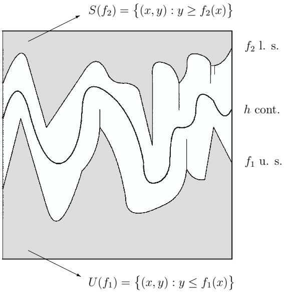

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1. define $\psi(x) = [f_(x), f_2(x)]$ for $x \in K$;
2. $\psi : K \to 2^{\mathbb{R}}$ satisfy Michael's theorem hypothesis. Indeed, if $G = \cup_{i\in I}(a_i, b_i) \subset \mathbb{R}$ is open, then

$$\{x \in X : \psi(x) \cap G \neq \emptyset\} = \\ = \bigcup_{i\in I}\{x \in X : \psi(x) \cap (a_i, b_i) \neq \emptyset\}$$

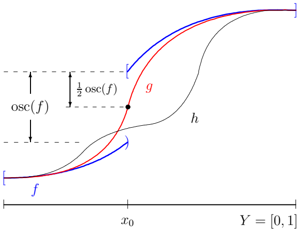
is open.

Distances vs.oscillations

Theorem

Let K be a compact space.
If $f \in \mathbb{R}^K$ is bounded, then

$$d(f, C(K)) = \frac{1}{2}\operatorname{osc}(f).$$



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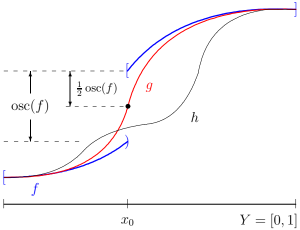

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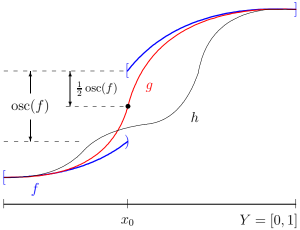

1. It is easy to check that $d(f, C(K)) \geq \operatorname{osc}(f)/2$.
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$$\operatorname{osc}(f) \geq \inf_{U \in \mathscr{U}_x} \sup_{y,z \in U} \big(f(y) - f(z)\big)$$

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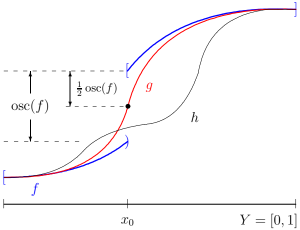

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$$f_2(x) := \sup_{U \in \mathscr{U}_x} \inf_{z \in U} f(z) + \frac{\operatorname{osc}(f)}{2}$$
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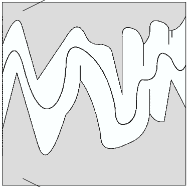

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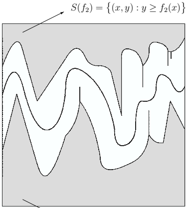

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4. Squeeze h between f_2 and f_1 and $d(f, C(K)) = \|f - h\|_\infty = \operatorname{osc}(f)/2$.

Measures of weak noncompactness

We use the formula $d(f, C(K)) = \frac{1}{2}\operatorname{osc}(f)$ to measure distances to $C(K)$ and the result below to apply what we do to Banach spaces.

Proposition

Let E be a Banach space and let B_{E^*} be the closed unit ball in the dual E^* endowed with the w^*-topology. Let $i : E \to E^{**}$ and $j : E^{**} \to \ell_\infty(B_{E^*})$ be the canonical embedding. Then, for every $x^{**} \in E^{**}$ we have:

$$d(x^{**}, i(E)) = d(j(x^{**}), C(B_{E^*})).$$

Relationship between measures of weak noncompactness

Theorem (... $H \subset E$ bdd...)

$$\operatorname{ck}(H) \leq \operatorname{k}(H) \leq \gamma(H) \leq 2\operatorname{ck}(H) \leq 2\operatorname{k}(H),$$
$$\gamma(H) = \gamma(\operatorname{co}(H)).$$

For any $x^{**} \in \overline{H}^{w^*}$, there is a sequence $(x_n)_n$ in H such that

$$\|x^{**} - y^{**}\| \leq \gamma(H)$$

for any cluster point y^{**} of $(x_n)_n$ in E^{**}. Furthermore, H is weakly relatively compact in E iff it is zero one (all) of the numbers $\operatorname{ck}(H), \operatorname{k}(H), \gamma(H)$.

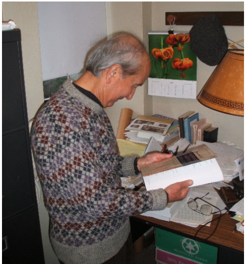
$$\gamma(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset B_{E^*}, (x_n) \subset H\},$$

$$\operatorname{ck}(H) := \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m \in \mathbb{N}} \overline{\{h_n : n > m\}}^{w^*}, E\Big),$$

$$\operatorname{k}(H) := \hat{d}(\overline{H}^{w^*}, E) = \sup_{x^{**} \in \overline{H}^{w^*}} d(x^{**}, E),$$

The result above is the quantitative version of Eberlein-Smulyan and Krein-Smulyan theorems. From $\operatorname{k}(\operatorname{co}(H)) \leq 2\operatorname{k}(H)$ straightforwardly follows Krein-Smulyan theorem.

Other extensions, applications and references



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The quantitative difference between countable compactness and compactness.
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C. Angosto, B. Cascales, and I. Namioka.
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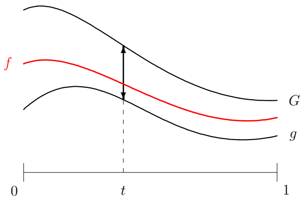
Topology Appl., 153(13):2303–2319, 2006.

Measurability

Our interest in selections: the integral of a multifunction

$F : \Omega \longrightarrow cwk(E)$ –convex w-compact



There are several possibilities to define the integral of F:

1. to take a reasonable embedding j from $cwk(E)$ into the Banach space $Y(=\ell_\infty(B_{E^*}))$ and then study the integrability of $j \circ F$;
2. to take all *integrable* selectors f of F and consider

$$\int F\,d\mu = \left\{ \int f\,d\mu : f \text{ integra. sel.} F \right\}.$$

1. Debreu, [Deb67], used the embedding technique together with Bochner integration for multi-function with values in $ck(E)$ – convex compact subsets of E;

The non-separable case

- Pettis integration theory was stuck in the separable case for the lack of a selection result in the general case.

4. Pettis integration for multi-functions was developed in the separable case.

Naive approach to find measurable selectors

1. start with a nice characterization of measurability for $f : \Omega \to E$;

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How good is this approach going to be?

As good as the real applications you can get!!!

Starting point. . . an elementary result

Exercise

$f : \Omega \to \mathbb{R}$. TFAE:

1. f is (μ-)measurable;
2. For every $\varepsilon > 0$ $A \in \Sigma^+$ there is $B \in \Sigma_A^+$ such that

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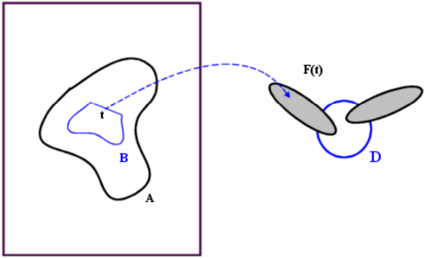

Definition

$F : \Omega \to 2^E$ satisfies property (P) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma_A^+$ and $D \subset E$ with $\operatorname{diam}(D) < \varepsilon$ such that

$$F(t) \cap D \neq \emptyset \text{ for every } t \in B.$$

(P) is the measure theory counterpart of σ-fragmentable multi-functions introduced by Jayne-Pallarés-Orihuela and Vera

Multi-functions

Property (P)

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2. take $\varepsilon > 0$;

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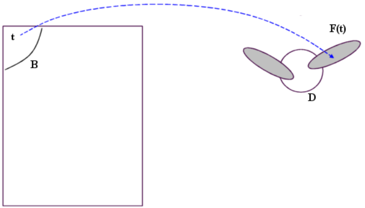

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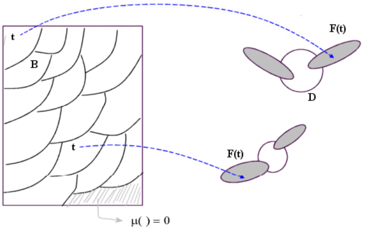

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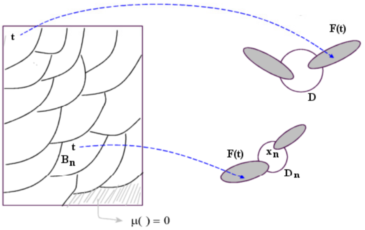

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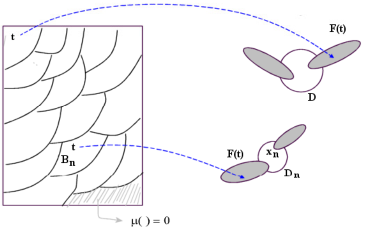

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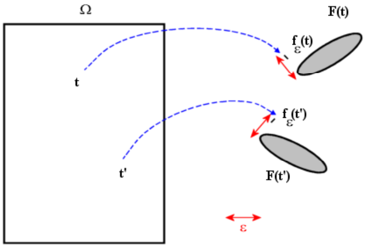

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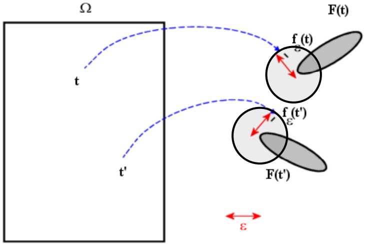

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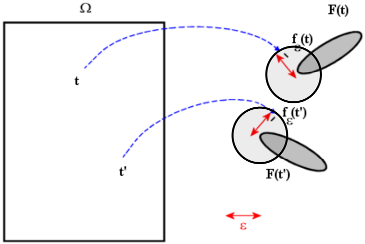

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9. IF F_ε satisfies (P) GOTO 11;
10. STOP;
11. $n := n + 1$;

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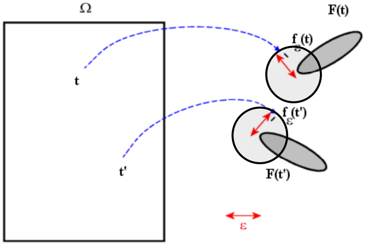
Property (P)

$F:\Omega\to 2^E$ satisfies property (P) if for each $\varepsilon>0$ and each $A\in\Sigma^+$ there exist $B\in\Sigma_A^+$ and $D\subset E$ with $\mathrm{diam}(D)<\varepsilon$ such that $F(t)\cap D\neq\emptyset$ for every $t\in B$.

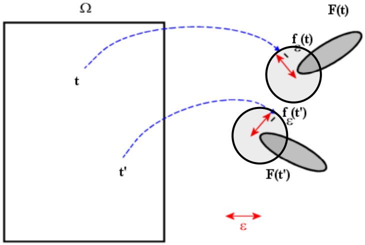


1. Fix $n=0$;
2. take $\varepsilon:=(1/2)^n$;
3. apply (P) for $A=\Omega$, ε and F;
4. a maximality argument produces a partition of $B's$;
5. enumerate $B's$ as $\{B_n\}$ and choose any $x_n\in D_n$;
6. define $f_\varepsilon:=\sum_n \chi_{B_n}x_n$;
7. f_ε is μ-measurable and $d(f_\varepsilon(t),F(t))<\varepsilon$ μ-a.e.;
8. define $F_\varepsilon(t):=F(t)\cap B(f_\varepsilon(t),\varepsilon)$;
9. IF F_ε satisfies (P) GOTO 11;
10. STOP;
11. $n:=n+1$;
12. GOTO 2.

Multi-functions

Property (P)

$F:\Omega\to 2^E$ satisfies property (P) if for each $\varepsilon>0$ and each $A\in\Sigma^+$ there exist $B\in\Sigma^+_A$ and $D\subset E$ with $\mathrm{diam}(D)<\varepsilon$ such that $F(t)\cap D\neq\emptyset$ for every $t\in B$.



1. Fix $n=0$;
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Conclusion

We produce a sequence $(f_n):\Omega\to E$ of μ-measurable functions such that $(f_n(t))$ is Cauchy μ-a.e., hence it is convergent.

Multi-functions: measurable selections

Corollary, Kuratowski-Ryll Nardzewski, 1965

Let $F : \Omega \to 2^E$ be a multi-function with closed non empty values of E. If E is separable and F satisfies that

$$\{t \in \Omega : F(t) \cap O \neq \emptyset\} \in \Sigma \text{ for each open set } O \subset X. \tag{E}$$

Then F admits a μ-measurable selector f.

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Very little is known in the non separable case

Theorem

For a multi-function $F:\Omega\to wk(E)$ TFAE:

(i) *F admits a strongly measurable selector.*

(ii) *There exist a set of measure zero $\Omega_0\in\Sigma$, a separable subspace $Y\subset X$ and a multi-function $G:\Omega\setminus\Omega_0\to wk(Y)$ that is Effros measurable and such that $G(t)\subset F(t)$ for every $t\in\Omega\setminus\Omega_0$;*

(iii) *F satisfies property (P).*

We also developed techniques to prove

Theorem

$F : \Omega \to cwk(E)$ scalarly measurable. Then there is a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(E^*,w^*)}$ of scalarly meas. selectors of F such that

$$F(t) = \overline{\{f_\alpha(t) : \ \alpha < \mathrm{dens}(E^*, w^*)\}} \quad \textit{for every } t \in \Omega.$$

Theorem

If $F : \Omega \to cwk(E)$ a Pettis integrable multi-function, then:

- every scalarly measurable selector is Pettis integrable;
- F admits a scalarly measurable selector.

Furthermore, F admits a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(E^*,w^*)}$ of Pettis integrable selectors such that

$$F(t) = \overline{\{f_\alpha(t) : \ \alpha < \mathrm{dens}(E^*, w^*)\}} \quad \text{for every } t \in \Omega.$$

Moreover, $\int_A F \, d\mu = \overline{IS_F(A)}$ for every $A \in \Sigma$.

Other extensions, applications and references

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