THE GELFAND INTEGRAL FOR MULTI-VALUED FUNCTIONS

B. CASCALES, V. KADETS, AND J. RODRÍGUEZ

ABSTRACT. We study the existence of w^*-scalarly measurable selectors and almost selectors for w^*-scalarly measurable multi-functions with values in dual Banach spaces. These selection results are used to study Gelfand and Dunford integrals for multi-functions: our non separable setting extends previous studies that have been done for separable Banach spaces. Pettis integral for multi-functions, already studied by different authors, naturally appears as a particular case of Dunford integral. We also study when the Gelfand integral of a multi-function is not only w^*-compact but w-compact.

1. INTRODUCTION

Gelfand integral was first studied by Gelfand in 1936, [15]. Gelfand integral for single and multi-valued functions has been extensively studied and applied over the years, see amongst others [2, 16, 17, 18, 21, 23, 24, 25]; a common motivation for some of these papers comes from game theory and mathematical economy, where the need of studying infinite dimensional Banach spaces is motivated, for instance, by the need of dealing with infinite dimensional commodity spaces. Another common thing in the papers above is that their studies are restricted to duals of separable Banach spaces in the range. As far as we see it, this limitation has only been made because one has to ensure the existence of *scalarly measurable selectors* for given *measurable multi-valued functions*: the (measurable) selection results at hand always relied on arguments that require separability. In this paper we overcome this technical difficulty and get rid of the separability hypothesis. To do so we prove a general w^**-almost selection* result and then we study Gelfand integral for multi-valued functions in full generality and some of its consequences.

Throughout this paper (Ω, Σ, μ) is a complete probability space, X is a Banach space and X^* its dual space; the weak (resp. weak*) topology on X (resp. X^*) is denoted by w (resp. w^*). By $cw^*k(X^*)$ we denote the family of all non-empty convex w^*-compact subsets of X^*.

Recall that a function $f : \Omega \to X^*$ is said to be w^**-scalarly measurable* (resp. *Gelfand integrable*) if, for each $x \in X$, the function $\langle f, x\rangle : \Omega \to \mathbb{R}$ given by $t \mapsto \langle f(t), x\rangle$ is measurable (resp. integrable). If f is Gelfand integrable, then for each $A \in \Sigma$ there exists a vector $\int_A f\, d\mu \in X^*$ (called the *Gelfand integral* of f over A) satisfying $\langle \int_A f\, d\mu, x\rangle = \int_A \langle f, x\rangle\, d\mu$ for all $x \in X$. For basic information on the Gelfand integral, see [2, 11.9] and [13, p. 53].

A multi-function from Ω to X is a multi-valued map sending each $t \in \Omega$ to a subset $F(t) \in 2^X$. Here one should note that in the literature multi-functions are also referred to as correspondences, set valued functions, set valued maps, and random sets.

2000 *Mathematics Subject Classification.* 28B05, 28B20, 46G10.

Key words and phrases. Multi-function; measurable selector; Gelfand integral for multi-functions; Dunford integral for multi-functions; Pettis integral for multi-functions.

B. Cascales and J. Rodríguez were supported by MEC and FEDER (project MTM2008-05396) and Fundación Séneca (project 08848/PI/08).

This paper is organized as follows. In Section 2 we study what we call w^*-almost selector for w^*-scalarly measurable multi-functions $F:\Omega\to 2^{X^*}$. F is said to be w^*-scalarly measurable, see Definition 2.1, if for every $x\in X$ the function

$$\delta^*(x,F):\Omega\to\mathbb{R}\cup\{+\infty\},\quad \delta^*(x,F)(t):=\sup\{\langle x^*,x\rangle:\ x^*\in F(t)\}$$

is measurable. Our main result in this section, Theorem 2.7, establishes that for every w^*-scalarly measurable multi-function with bounded values $F:\Omega\to 2^{X^*}$ there is a w^*-scalarly measurable single-valued function $f:\Omega\to X^*$ such that, for each $x\in X$,

$$\langle f,x\rangle\le\delta^*(x,F)\ \ \mu\text{-a.e.}\quad \text{(the exceptional } \mu\text{-null set depending on } x).$$

f is what we called a w^*-scalarly measurable w^*-almost selector.

Section 3 is devoted to study the existence of w^*-scalarly measurable selectors for certain w^*-scalarly measurable multi-functions. We start by noting that when X is separable and $F(\Omega)\subset cw^*k(X^*)$, the w^*-scalarly measurable w^*-almost selector f, found above, can be readily modified into a w^*-scalarly measurable selector for F, see Corollary 3.1. In Theorem 3.8 we establish that if K is a compact and metrizable space and $G:K\to cw^*k(X^*)$ is a multi-function such that $t\mapsto\delta^*(x,G)(t)$ is continuous for all $x\in X$, then G admits a w^*-scalarly measurable selector. In Theorem 3.10 we prove that if $F:\Omega\to 2^{X^*}$ is a w^*-scalarly measurable multi-function with w^*-compact values such that $\overline{\mathrm{co}}^{w^*}(F(t))$ has the Radon-Nikodým Property (RNP for short) for all $t\in\Omega$, then F admits a w^*-scalarly measurable selector. In particular we obtain that if X^* has the RNP then every w^*-scalarly measurable multi-function with w^*-compact values has a w^*-scalarly measurable selector, Corollary 3.11.

In Section 4 we define Gelfand integral for multi-functions $F:\Omega\to cw^*k(X^*)$ (no restriction on X). Our approach nicely extends the single-valued case and also the previously studied multi-valued case when X is separable. Our main result here is Theorem 4.5 where the Gelfand integral of a multi-function is characterized as the set of the integrals of its Gelfand integrable w^*-almost selectors. Section 5 contains the natural definitions and links with Dunford and Pettis integrals for multi-functions; here one should note that Dunford integral appears as a particular case of Gelfand integral and that when defining *naturally* Pettis integral as it is usually done with single-valued functions, we recover the usual notions studied in [4, 5, 7, 9, 10, 11, 29] and [30].

Section 6 deals with the following question: given a Gelfand integrable multi-function $F:\Omega\to cw^*k(X^*)$ with norm compact values, can we expect the integrals $\int_A F\,d\mu$ to be norm or w-compact? We give examples proving that $\int_A F\,d\mu$ need not be norm compact, Examples 6.1 and 6.2, and we prove that $\int_A F\,d\mu$ is always w-compact whenever F is bounded, see Theorem 6.8.

Some open problems are included in the last Section of the paper.

Terminology. Our unexplained terminology can be found in our standard references for multi-functions [8, 19], Banach spaces [14] and vector integration [13, 26].

All vector spaces here are assumed to be real. Given a subset S of a vector space, we write $\mathrm{co}(S)$ and $\mathrm{span}(S)$ to denote, respectively, the convex and linear hull of S. By letters X and Y we always denote Banach spaces. B_X and S_X are the closed unit ball and the unit sphere of X, respectively. X^* stands for the topological dual of X. Given $x^*\in X^*$ and $x\in X$, we write either $\langle x^*,x\rangle$ or $x^*(x)$ to denote the evaluation of x^* at x. Given a non-empty set Γ (resp. a compact topological space K), we write $\ell^\infty(\Gamma)$ (resp. $C(K)$) to denote the Banach space of all bounded (resp. continuous) real-valued functions on Γ

(resp. K), equipped with the supremum norm. Given a Banach space X, we denote by 2^X the set of all non-empty subsets of X. We will consider the following families of sets:

- $w^*k(X^*)$ = all w^*-*compact* non-empty subsets of X^*,
- $wk(X)$ = all w-*compact* non-empty subsets of X,
- $k(X)$ = all *norm compact* non-empty subsets of X.

By $cw^*k(X^*)$, $cwk(X)$ and $ck(X)$ we denote, respectively, the subfamilies of $w^*k(X^*)$, $wk(X)$ and $k(X)$ made up of *convex* sets. Given a set $C \subset X$ and $x^* \in X^*$, we write

$$\delta^*(x^*, C) := \sup\{\langle x^*, x\rangle : \ x \in C\} \in \mathbb{R} \cup \{+\infty\}.$$

A multi-function $F : \Omega \to 2^X$ is called *bounded* if $\bigcup_{t\in\Omega} F(t)$ is a bounded subset of X.

We write Σ^+ to denote the subfamily of Σ made up of sets of positive measure. Given $A \in \Sigma$, the subfamily of Σ^+ made up of subsets of A is denoted by Σ^+_A. As usual, $\mathcal{L}^p(\mu)$ and $L^p(\mu)$ will denote, respectively, the Lebesgue spaces of functions and equivalence classes of functions. For a function $h : \Omega \to \mathbb{R}$ we denote by h^+ the function defined by $h^+(t) := \max\{h(t), 0\}$ and $h^- := (-h)^+$. The symbol $\mathbb{1}_A$ stands for the characteristic function of A.

2. w^*-SCALARLY MEASURABLE MULTI-FUNCTIONS AND w^*-ALMOST SELECTORS

A multi-function $F : \Omega \to cwk(X)$ is said to be scalarly measurable [8] if the real-valued map

$$t \mapsto \delta^*(x^*, F(t)) := \sup\{\langle x^*, x\rangle : \ x \in F(t)\}$$

is measurable for all $x^* \in X^*$. Hence, the natural definition for w^*-*scalarly measurable* multi-function is:

Definition 2.1. *A multi-function $F : \Omega \to 2^{X^*}$ is said to be w^*-scalarly measurable if for every $x \in X$ the function*

$$\delta^*(x, F) : \Omega \to \mathbb{R} \cup \{+\infty\}, \quad \delta^*(x, F)(t) := \sup\{\langle x^*, x\rangle : \ x^* \in F(t)\}$$

is measurable.

Note that if F is w^*-scalarly measurable then the function $\delta_*(x, F) : \Omega \to \mathbb{R} \cup \{-\infty\}$ defined by $\delta_*(x, F)(t) := \inf\{\langle x^*, x\rangle : \ x^* \in F(t)\}$, $t \in \Omega$, is also measurable for every $x \in X$. The functions $\delta^*(x, F)$ and $\delta_*(x, F)$ are real-valued for any $x \in X$ whenever $F : \Omega \to 2^{X^*}$ takes bounded values.

A multi-function $F : \Omega \to cwk(X)$ is scalarly measurable if, and only if, F is w^*-scalarly measurable when *naturally* considered with values $F : \Omega \to cw^*k(X^{**})$.

Definition 2.2. *A single valued function $f : \Omega \to X^*$ is a w^*-almost selector of a multi-function $F : \Omega \to 2^{X^*}$ if for every $x \in X$ we have $\langle f, x\rangle \leq \delta^*(x, F)$ μ-a.e. (the exceptional μ-null set depending on x).*

If $F : \Omega \to cwk(X)$ is a multi-function and $f : \Omega \to X$ is a w^*-almost selector of F when *naturally* considered with values $F : \Omega \to cw^*k(X^{**})$, we will say that f *is a w-almost selector of F*.

Next proposition collects a first nice quality of w^*-almost selectors. In the proof we use the Mackey topology $\mu(X^*, X)$ of the dual pair $\langle X^*, X\rangle$, that is, the topology in X^* of uniform convergence on absolutely convex weakly compact subsets of X, [20, §21.4.(1)]. We note that according to [20, §21.4.(2)], $\mu(X^*, X)$ is the finest locally convex topology for which the dual $(X^*, \mu(X^*, X))' = X = (X^*, w^*)'$.

Proposition 2.3. *Suppose X is separable. The following properties hold:*

(i) *If $F : \Omega \to cw^*k(X^*)$ is a multi-function and $f : \Omega \to X^*$ is a w^*-almost selector of F, then $f(t) \in F(t)$ for μ-a.e. $t \in \Omega$.*

(ii) *If $F : \Omega \to cwk(X)$ is a multi-function and $f : \Omega \to X$ is a w-almost selector of F, then $f(t) \in F(t)$ for μ-a.e. $t \in \Omega$.*

Proof. We prove (i) first. Let (x_n) be a dense sequence in X. For each $n \in \mathbb{N}$, let $E_n \in \Sigma$ with $\mu(E_n) = 1$ such that $\langle f(t), x_n\rangle \leq \delta^*(x_n, F)(t)$ for every $t \in E_n$. Then the set $E := \bigcap_{n\in\mathbb{N}} E_n \in \Sigma$ satisfies $\mu(E) = 1$ and

$$\langle f(t), x_n\rangle \leq \delta^*(x_n, F)(t) \quad \text{for every } t \in E \text{ and every } n \in \mathbb{N}. \tag{1}$$

Since F takes bounded values and (x_n) is dense in X, inequality (1) implies that

$$\langle f(t), x\rangle \leq \delta^*(x, F)(t) \quad \text{for every } t \in E \text{ and every } x \in X.$$

Since F takes convex w^*-compact values, from the separation Hahn-Banach theorem it follows now that $f(t) \in F(t)$ for every $t \in E$ and (i) is proved.

To prove (ii) we use ideas similar to those in the proof of (i) but slightly modified. Since X is separable, (X^*, w^*) is also separable. So we can take $D \subset X^*$ countable and w^*-dense in X^*. We have

$$X^* = \overline{D}^{w^*} = \overline{\operatorname{span}_{\mathbb{R}} D}^{w^*} \stackrel{(a)}{=} \overline{\operatorname{span}_{\mathbb{R}} D}^{\mu(X^*,X)} = \overline{\operatorname{span}_{\mathbb{Q}} D}^{\mu(X^*,X)},$$

where equality (a) follows from [20, §20.8.(6)]. Let $\{x_n^* : n \in \mathbb{N}\}$ be an enumeration of $\operatorname{span}_{\mathbb{Q}} D$. Then, proceeding as we did in the proof of (i) above, we find a set $E \in \Sigma$ with $\mu(E) = 1$ such that

$$\langle f(t), x_n^*\rangle \leq \delta^*(x_n^*, F)(t) \quad \text{for every } t \in E \text{ and every } n \in \mathbb{N}. \tag{2}$$

Given $t \in E$ and any $x^* \in X^*$, since $F(t)$ is convex and weakly compact the equality $X^* = \overline{\operatorname{span}_{\mathbb{Q}} D}^{\mu(X^*,X)}$ implies that x^* can be approximated as much as we want by some x_n^* uniformly on $F(t) \cup \{f(t)\}$. Therefore, from inequality (2) we deduce that $\langle x^*, f(t)\rangle \leq \delta^*(x^*, F)(t)$ for every $t \in E$ and every $x^* \in X^*$. The separation Hahn-Banach theorem implies that $f(t) \in F(t)$ for every $t \in E$ and (ii) is proved. □

Remark 2.4. The statements in the previous Proposition fail in general for non-separable spaces. For instance, the function $f : [0,1] \to \ell^2([0,1])$ given by $f(t) := e_t$ is a w-almost selector of the multi-function $F : [0,1] \to 2^{\ell^2([0,1])}$ given by $F(t) := \{0\}$. Here $\{e_t\}_{t\in[0,1]}$ denotes the usual orthonormal basis of $\ell^2([0,1])$.

Remark 2.5. We also note that the conclusion in statement (ii) in the previous Proposition does not hold if we only assume that $f : \Omega \to X^{**}$ is a w^*-almost selector when we look at $F : \Omega \to cw^*k(X^{**})$. Indeed, consider $\Omega = [0,1]$ with the standard Lebesgue measure and $X = C[0,1]$. Then X^{**} contains in the natural way the space of all bounded Borel measurable functions on $[0,1]$. Take the trivial multi-function $F : \Omega \to cwk(X)$ given by $F(t) := \{0\}$, and take as $f(t)$ the characteristic function of the singleton $\{t\}$. Then $f : \Omega \to X^{**}$ is a w^*-almost selector of F, but there is no point in which $f(t) \in F(t)$.

Theorem 2.7 below ensures the existence of w^*-scalarly measurable w^*-almost selectors for w^*-scalarly measurable multi-functions. We first need a lemma which will be used several times throughout the paper.

Lemma 2.6. *Let $F : \Omega \to 2^{X^*}$ be a w^*-scalarly measurable multi-function with bounded values. Then there exist a countable partition (E_n) of Ω in Σ and a sequence (C_n) of positive real numbers such that, for each $x \in X$ and each $n \in \mathbb{N}$, we have*

$$|\delta^*(x, F)| \leq C_n \|x\| \quad \mu\text{-a.e. on } E_n.$$

Proof. Observe that $\{\delta^*(x, F) : x \in B_X\} \subset \mathbb{R}^\Omega$ is a pointwise bounded family of measurable functions. Then there exists a measurable function $h : \Omega \to [0, \infty)$ such that, for each $x \in B_X$, one has $|\delta^*(x, F)| \leq h$ μ-a.e. (see *e.g.* [22, Proposition 3.1]). Now it is enough to take $C_n := n$ and $E_n := \{t \in \Omega : n - 1 \leq h(t) < n\}$ for every $n \in \mathbb{N}$. □

Theorem 2.7. *Every w^*-scalarly measurable multi-function $F : \Omega \to 2^{X^*}$ with bounded values admits a w^*-scalarly measurable w^*-almost selector.*

Proof. In view of Lemma 2.6, without loss of generality we may assume that there is $C > 0$ such that for every $x \in X$ we have $|\delta^*(x, F)| \leq C\|x\|$ μ-a.e.

Fix an arbitrary selector $g : \Omega \to X^*$ of F. Denote by E the quotient Banach space of $\ell^\infty(\Omega)$ over the closed subspace of all bounded functions vanishing μ-a.e. We consider the operator (*i.e.* linear continuous mapping) $T : X \to E$ that satisfies $T(x) = \langle g, x\rangle$ μ-a.e. for every $x \in X$. Clearly, for every $x \in X$ we have

$$T(x) \leq \delta^*(x, F) \quad \mu\text{-a.e.} \tag{3}$$

Since $L^\infty(\mu)$ is isometrically isomorphic to a subspace of E, we can find a norm-one projection $P : E \to L^\infty(\mu)$ (see *e.g.* [1, Proposition 4.3.8]). We claim that P preserves inequalities. For if $u \in E$ and $u \geq 0$, then

$$\left\| u - \frac{\|u\|_E}{2} \mathbb{1}_\Omega \right\|_E \leq \frac{\|u\|_E}{2}.$$

Since P is a norm-one projection, we have

$$\left\| P(u) - \frac{\|u\|_E}{2} \mathbb{1}_\Omega \right\|_{L^\infty(\mu)} \leq \frac{\|u\|_E}{2}$$

and consequently $P(u) \geq 0$, as claimed.

¿From inequality (3) it follows that for any $x \in X$ we have $(P \circ T)(x) \leq \delta^*(x, F)$ μ-a.e. Let $\rho : L^\infty(\mu) \to \mathcal{L}^\infty(\Omega)$ be a norm-one linear lifting preserving inequalities (see *e.g.* [28, Theorem G.1]). Then for every $x \in X$ we have

$$(\rho \circ P \circ T)(x) \leq \delta^*(x, F) \quad \mu\text{-a.e.} \tag{4}$$

For each $t \in \Omega$, denote by $\delta_t \in \ell^\infty(\Omega)^*$ the evaluation functional at t. Define $f : \Omega \to X^*$ as

$$f(t) := \delta_t \circ \rho \circ P \circ T.$$

Clearly, $\langle f, x\rangle = (\rho \circ P \circ T)(x)$ is measurable for every $x \in X$ and inequality (4) ensures that f is a w^*-almost selector of F. The proof is finished. □

Remark 2.8. In the previous proof, we can assume further that $\rho(\mathbb{1}_\Omega) = \mathbb{1}_\Omega$. Since for each $x \in B_X$ one has $|(P \circ T)(x)| \leq C$ μ-a.e., the assumption $\rho(\mathbb{1}_\Omega) = \mathbb{1}_\Omega$ yields $|\langle f(t), x\rangle| \leq C$ for every $t \in \Omega$. Therefore, $\|f(t)\| \leq C$ for every $t \in \Omega$. In particular, this argument shows that *every bounded w^*-scalarly measurable multi-function $F : \Omega \to 2^{X^*}$ admits a bounded w^*-scalarly measurable w^*-almost selector*.

3. Existence of w^*-scalarly measurable selectors

This section is devoted to prove the existence, in some cases, of w^*-scalarly measurable selectors for w^*-scalarly measurable multi-functions. The first positive result that we can prove appears as the natural outcome of our work in the previous section, *cf.* [2, Theorem 18.33] and [27, Proposition 7].

Corollary 3.1. *Suppose X is separable. Every w^*-scalarly measurable multi-function $F : \Omega \to cw^*k(X^*)$ admits a w^*-scalarly measurable selector.*

Proof. Combining Theorem 2.7 and (i) in Proposition 2.3 we obtain a w^*-scalarly measurable function $f : \Omega \to X^*$ such that $f(t) \in F(t)$ for μ-a.e. $t \in \Omega$. Modifying f in a set of μ-measure zero if needed we will end up with the stated w^*-scalarly measurable selector for F. □

Throughout this section K is a compact Hausdorff topological space, μ is a Radon probability on K, Σ is the σ-algebra on K of all μ-measurable sets and $X := \ell^\infty(K)$. For each $t \in K$ we write δ_t to denote the functional on $C(K)$ given by $\delta_t(h) = h(t)$ and we denote by $\mathcal{E}_t$ the family of all open neighborhoods of t. We study now the existence of w^*-scalarly measurable selectors for the multi-function $F : K \to 2^{X^*}$ given by

$$F(t) := \{x^* \in B_{X^*} : \ x^*|_{C(K)} = \delta_t\}, \quad t \in K. \tag{5}$$

The lemmata that follow provide us with the technicalities needed to prove the w^*-scalar measurability of F, the existence of w^*-scalarly measurable selectors and their consequences.

Lemma 3.2. *Let Γ be a set. If $\varphi \in B_{\ell^\infty(\Gamma)^*}$ satisfies $\varphi(\mathbb{1}_\Gamma) = 1$, then φ is positive.*

Proof. Take $x \in \ell^\infty(\Gamma)$ with $x \geq 0$ and set $a := \|x\|_\infty$. Then $\|x - \frac{a}{2}\mathbb{1}_\Gamma\|_\infty \leq a/2$. So $|\varphi(x) - a/2| = |\varphi(x - \frac{a}{2}\mathbb{1}_\Gamma)| \leq \|x - \frac{a}{2}\mathbb{1}_\Gamma\|_\infty \leq a/2$, hence $\varphi(x) \geq 0$. □

Lemma 3.3. *Let $t \in K$ and $x \in X$. Then*

$$\inf_{U \in \mathcal{E}_t} \sup_{\tau \in U} x(\tau) = \inf_{g \in C(K)} \big(\|x + g\|_\infty - g(t)\big). \tag{6}$$

Proof. Let α be the left hand side of (6) and β the right hand side. We prove first that $\alpha \leq \beta$. Take $g \in C(K)$ and fix $\varepsilon > 0$. There is $U \in \mathcal{E}_t$ such that $|g(t) - g(\tau)| \leq \varepsilon$ for all $\tau \in U$, hence for each $\tau \in U$ we have

$$x(\tau) \leq \big(x(\tau) + g(\tau)\big) - g(\tau) \leq \|x + g\|_\infty - g(\tau) \leq \|x + g\|_\infty - g(t) + \varepsilon.$$

It follows that $\alpha \leq \sup_{\tau \in U} x(\tau) \leq \|x + g\|_\infty - g(t) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $\alpha \leq \|x + g\|_\infty - g(t)$. Being $g \in C(K)$ arbitrary we obtain $\alpha \leq \beta$.

The inequality $\alpha \geq \beta$ is proved as follows. Fix $\varepsilon > 0$. There is a neighborhood $U \in \mathcal{E}_t$ such that $\sup_{\tau \in U} x(\tau) \leq \alpha + \varepsilon$. Let $V \in \mathcal{E}_t$ such that $\overline{V} \subset U$. By Urysohn's lemma, there is $g \in C(K)$ with $0 \leq g \leq 2\|x\|_\infty$ such that g vanishes on $K \setminus U$ and $g(\tau) = 2\|x\|_\infty$ for all $\tau \in V$. If $\tau' \in U$, then:

(i) $g(\tau') \leq 2\|x\|_\infty = g(t)$ and so $x(\tau') + g(\tau') \leq \sup_{\tau \in U} x(\tau) + g(t)$;

(ii) $-(x(\tau') + \sup_{\tau \in U} x(\tau)) \leq 2\|x\|_\infty \leq g(\tau') + g(t)$, hence

$$-(x(\tau') + g(\tau')) \leq \sup_{\tau \in U} x(\tau) + g(t).$$

It follows that $|x(\tau')+g(\tau')| \leq \sup_{\tau\in U} x(\tau)+g(t) \leq \alpha+\varepsilon+g(t)$ for all $\tau' \in U$. On the other hand, for each $\tau \in K \setminus U$ we have

$$|x(\tau)+g(\tau)| = |x(\tau)| \leq -\|x\|_\infty + g(t) \leq x(t)+g(t) \leq \alpha+\varepsilon+g(t).$$

We conclude that $\|x+g\|_\infty \leq \alpha+\varepsilon+g(t)$. This shows that

$$\alpha+\varepsilon \geq \|x+g\|_\infty - g(t) \geq \beta.$$

Since $\varepsilon > 0$ is arbitrary, we get the desired inequality and the proof is over. □

Lemma 3.4. *If $F : K \to 2^{X^*}$ is the multi-function defined by* (5)*, then:*

(i) *$F(t) \in cw^*k(X^*)$ for every $t \in K$;*

(ii) *$F(t)$ is made up of positive functionals for every $t \in K$;*

(iii) *for each $x \in X$ we have $\delta^*(x, F(t)) = \inf_{U\in\mathcal{E}_t} \sup_{\tau\in U} x(\tau)$ for every $t \in K$.*

Proof. (i) $F(t)$ is non-empty by the Hahn-Banach theorem. Clearly, $F(t)$ is convex. Moreover, $F(t)$ is w^*-closed in B_{X^*} and so it is w^*-compact.

(ii) This follows from Lemma 3.2.

(iii) Fix $x \in X$ and $t \in K$ and set $\alpha := \inf_{U\in\mathcal{E}_t} \sup_{\tau\in U} x(\tau)$. To prove (iii) we distinguish two cases:

CASE 1. $x \in C(K)$.
We clearly have $\alpha = x(t)$ and $x^*(x) = x(t)$ for all $x^* \in F(t)$, hence $\delta^*(x, F(t)) = \alpha$.

CASE 2. $x \in X \setminus C(K)$.
Observe first that

$$\|x+g\|_\infty \geq x^*(x+g) = x^*(x)+g(t) \quad \text{whenever } x^* \in F(t) \text{ and } g \in C(K),$$

which together with Lemma 3.3 yields $\alpha \geq \delta^*(x, F(t))$. We prove now the other inequality.

Define $S := \mathrm{span}(C(K) \cup \{x\})$ and define a linear mapping $\phi : S \to \mathbb{R}$ by declaring $\phi|_{C(K)} = \delta_t$ and $\phi(x) = \alpha$. We claim that $|\phi(y)| \leq \|y\|_\infty$ for all $y \in S$. Indeed, let $y \in S$ and write $y = \lambda x + h$, where $h \in C(K)$ and $\lambda \in \mathbb{R}$. Assume that $\lambda > 0$. Set $g := h/\lambda \in C(K)$. By Lemma 3.3 we have that $\alpha \leq \|x+g\|_\infty - g(t)$, and therefore $\lambda(\alpha+g(t)) \leq \|\lambda(x+g)\|_\infty = \|y\|_\infty$. But $\phi(y) = \lambda\alpha + h(t) = \lambda(\alpha+g(t))$, hence $\phi(y) \leq \|y\|_\infty$. On the other hand, since $\alpha \geq x(t)$, we also have

$$\|y\|_\infty \geq -(\lambda x(t)+h(t)) \geq -(\lambda\alpha+h(t)) = -\phi(y).$$

Therefore $|\phi(y)| \leq \|y\|_\infty$, in the case $\lambda > 0$. If $\lambda < 0$, the previous argument applied to $-y = -\lambda x - h \in S$ yields $|\phi(y)| = |\phi(-y)| \leq \|-y\|_\infty = \|y\|_\infty$.

Hence, $|\phi(y)| \leq \|y\|_\infty$ for every $y \in S$ and the Hahn-Banach theorem implies that there is $x^* \in B_{X^*}$ such that $x^*|_S = \phi$. In other words $x^* \in F(t)$ and $x^*(x) = \alpha$. Hence $\alpha \leq \delta^*(x, F(t))$ and the proof is finished. □

Corollary 3.5. *The multi-function $F : K \to cw^*k(X^*)$ defined by* (5) *is w^*-scalarly measurable.*

Proof. Fix $x \in X$. Given any $a \in \mathbb{R}$, by Lemma 3.4(iii) we can write

$$\{t \in K :\ \delta^*(x, F(t)) < a\} = \Big\{t \in K :\ \text{there is } U \in \mathcal{E}_t \text{ such that } \sup_{\tau\in U} x(\tau) < a\Big\},$$

which is open. So the function $\delta^*(x, F)$ is $\mathrm{Borel}(K)$-measurable. □

Proposition 3.6. *If K is metrizable, then the multi-function $F : K \to cw^*k(X^*)$ defined by* (5) *admits a w^*-scalarly measurable selector.*

Proof. Let $J : C(K) \to L^\infty(\mu)$ be the canonical operator (that sends each function to its equivalence class). J can be extended to a norm-one operator $W : X \to L^\infty(\mu)$ because $L^\infty(\mu)$ is isometrically injective (*cf.* [1, Proposition 4.3.8]).

Since K is a compact metric space, we can find a sequence (Π_n) of partitions of K into finitely many Borel sets such that each Π_{n+1} is finer than Π_n and

$$\lim_{n\to\infty} \max_{A\in\Pi_n} \operatorname{diam}(A) = 0. \tag{7}$$

For each $n \in \mathbb{N}$, define the operator $E_n : L^\infty(\mu) \to X$ by

$$E_n(h) := \sum_{A\in\Pi_n} \Big(\frac{1}{\mu(A)} \int_A h\, d\mu\Big) \mathbb{1}_A$$

(with the convention $0/0 = 0$); that is, $E_n(h)$ is just the conditional expectation of h with respect to the sub-σ-algebra Σ_n generated by Π_n.

Claim: for each $h \in C(K)$ and each $t \in K$ we have $\lim_{n\to\infty} E_n(J(h))(t) = h(t)$. Indeed, for each $n \in \mathbb{N}$ there is $A_n \in \Pi_n$ such that $t \in A_n$. Fix $\varepsilon > 0$. By the continuity of h at t and (7), there is $n_\varepsilon \in \mathbb{N}$ such that $|h(\tau)-h(t)| \le \varepsilon$ whenever $\tau \in A_n$ and $n \ge n_\varepsilon$. Therefore

$$|E_n(J(h))(t) - h(t)| = \Big|\frac{1}{\mu(A_n)} \int_{A_n} h\, d\mu - h(t)\Big| = \Big|\frac{1}{\mu(A_n)} \int_{A_n} (h - h(t))\, d\mu\Big| \le \varepsilon$$

for all $n \ge n_\varepsilon$. This proves the claim.

Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Given $x \in X$ and $t \in K$, we have

$$|E_n(W(x))(t)| \le \|W(x)\|_{L^\infty(\mu)} \le \|x\|_\infty \quad \text{for all } n \in \mathbb{N}$$

and so we can define a function $f : K \to B_{X^*}$ by the formula

$$\langle f(t), x\rangle := \lim_{\mathcal{U}} E_n(W(x))(t).$$

The previous *Claim* ensures that, for each $t \in K$ and $h \in C(K)$, we have

$$\langle f(t), h\rangle = \lim_{\mathcal{U}} E_n(W(h))(t) = \lim_{\mathcal{U}} E_n(J(h))(t) = h(t).$$

Hence f is a selector of F. In order to prove that f is w^*-scalarly measurable, fix $x \in X$. Clearly $(E_n(W(x)), \Sigma_n)$ is a martingale in $L^1(\mu)$ and we can conclude that

$$\lim_{n\to\infty} E_n(W(x)) = W(x) \ \ \mu\text{-a.e.},$$

see [13, V.2, Corollary 2 and Theorem 8]. It follows that $\langle f, x\rangle = W(x)$ μ-a.e. and so $\langle f, x\rangle$ is measurable. □

Next lemma shows that the existence of w^*-scalarly measurable selectors for the multi-function F defined by (5) would ensure that certain multi-functions $G : K \to cw^*k(Y^*)$ also have such selectors.

Lemma 3.7. *Let Y be a Banach space and $G : K \to cw^*k(Y^*)$ a bounded multi-function such that $\delta^*(y, G)$ is continuous for all $y \in Y$. If the multi-function $F : K \to cw^*k(X^*)$ defined by* (5) *admits a w^*-scalarly measurable selector, then G admits a w^*-scalarly measurable selector.*

Proof. Since G is bounded, we can choose $C > 0$ such that $\delta^*(y, G(t)) \le C$ for all $y \in B_Y$ and $t \in K$. Let $h : K \to Y^*$ be an arbitrary selector of G and let $f : K \to X^*$ be a w^*-scalarly measurable selector of F. For every $t \in K$ and every $y \in Y$ we have

$$|\langle h(t), y\rangle| \le \max\{\delta^*(y, G(t)), \delta^*(-y, G(t))\} \le C\|y\|.$$

Therefore we can define an operator $T : Y \to X$ by the formula

$$T(y)(t) := \langle h(t), y\rangle, \quad y \in Y,\ t \in K.$$

Let $U : X \to X$ be the operator defined by

$$U(x)(t) := \langle f(t), x\rangle, \quad x \in X,\ t \in K.$$

Clearly, $U(x) = x$ whenever $x \in C(K)$. Since $T(y) \leq \delta^*(y, G)$ pointwise for all $y \in Y$ and U is positive (by Lemma 3.4(ii)), we infer that

$$U(T(y))(t) \leq \delta^*(y, G(t)) \quad \text{for all } y \in Y \text{ and } t \in K. \tag{8}$$

Define $g : K \to Y^*$ by $g(t) := \eta_t \circ U \circ T$, where $\eta_t \in X^*$ is given by $\eta_t(x) = x(t)$. For each $y \in Y$ we have

$$\langle g(t), y\rangle = U(T(y))(t) = \langle f(t), T(y)\rangle \quad \text{for all } t \in K$$

and so the function $\langle g, y\rangle$ is measurable (because f is w^*-scalarly measurable). Therefore g is w^*-scalarly measurable. Moreover, g is a selector of G. Indeed, fix $t \in K$ and observe that inequality (8) says that $\langle g(t), y\rangle \leq \delta^*(y, G(t))$ for all $y \in Y$. The separation Hahn-Banach theorem and the fact that $G(t)$ is convex and w^*-closed in Y^* ensure that $g(t) \in G(t)$. □

Theorem 3.8. *If K is metrizable, Y is a Banach space and $G : K \to cw^*k(Y^*)$ is a multi-function such that $\delta^*(y, G)$ is continuous for all $y \in Y$, then G admits a w^*-scalarly measurable selector.*

Proof. By a standard exhaustion argument, it suffices to prove that for each $A \in \Sigma^+$ there is $B \in \Sigma_A^+$ such that the restriction $G|_B$ admits a w^*-scalarly measurable selector.

By Lemma 2.6 there exist $B \in \Sigma_A^+$ and $C > 0$ such that, for each $y \in B_Y$, we have $|\delta^*(y, G)| \leq C$ μ-a.e. on B. Since μ is a Radon measure, we can assume further that B is closed and that $\mu(O \cap B) > 0$ whenever $O \subset K$ is an open set such that $O \cap B \neq \emptyset$.

Take any $y \in B_Y$ and consider the open set $O_y := \{t \in K : |\delta^*(y, G(t))| > C\}$. Then $\mu(O_y \cap B) = 0$ and so $O_y \cap B = \emptyset$. Since $y \in B_Y$ is arbitrary, $G|_B$ is bounded. Lemma 3.7 and Proposition 3.6 applied to the compact metric space B ensure that $G|_B$ admits a w^*-scalarly measurable selector. The proof is over. □

Remark 3.9. A similar argument allows us to obtain the following result: *Let K_1 be a compact Hausdorff topological space equipped with a Radon probability. Suppose that for each closed set $K \subset K_1$ the multi-function defined by* (5) *admits a w^*-scalarly measurable selector. Let Y be a Banach space and $G : K_1 \to cw^*k(Y^*)$ a multi-function such that $\delta^*(y, G)$ is continuous for all $y \in Y$. Then G admits a w^*-scalarly measurable selector.*

In [6, Theorem 3.8] we proved that every $cwk(X)$-valued scalarly measurable multi-function admits a scalarly measurable selector. Our proof relied strongly on the Radon-Nikodým property (RNP for short) of weakly compact convex sets in Banach spaces. The arguments used there can be straightforwardly adapted to obtain the following result. For complete information on sets with the RNP, we refer the reader to [3].

Theorem 3.10. *Let $F : \Omega \to w^*k(X^*)$ be a w^*-scalarly measurable multi-function such that $\overline{\text{co}}^{w^*}(F(t))$ has the RNP for all $t \in \Omega$. Then F admits a w^*-scalarly measurable selector.*

As immediate consequences we have:

Corollary 3.11. *Suppose X^* has the RNP. Then every w^*-scalarly measurable multi-function $F : \Omega \to w^*k(X^*)$ admits a w^*-scalarly measurable selector.*

Corollary 3.12. *Let $F : \Omega \to wk(X^*)$ be a w^*-scalary measurable multi-function. Then F admits a w^*-scalary measurable selector.*

We stress that for $k(X^*)$-valued multi-functions, the assertion of Corollary 3.12 can be obtained more easily by adapting the proof of [6, Theorem 3.6].

4. THE SET-VALUED GELFAND INTEGRAL

We start this section with the notion of Gelfand integrable multi-function.

Definition 4.1. *A multi-function $F : \Omega \to cw^*k(X^*)$ is said to be Gelfand integrable if for every $x \in X$ the function $\delta^*(x, F)$ is integrable. In this case, the Gelfand integral of F over $A \in \Sigma$ is defined as*

$$\int_A F\, d\mu := \bigcap_{x \in X} \Big\{ x^* \in X^* : \int_A \delta_*(x, F)\, d\mu \le \langle x^*, x\rangle \le \int_A \delta^*(x, F)\, d\mu \Big\}.$$

For a Gelfand integrable multi-function F the set $\int_A F\, d\mu$ is convex and w^*-closed; moreover, an appeal to the uniform boundedness principle ensures that $\int_A F\, d\mu$ is bounded, hence w^*-compact. It satisfies

$$\delta^*\Big(x, \int_A F\, d\mu\Big) \le \int_A \delta^*(x, F)\, d\mu \tag{9}$$

for every $x \in X$.

Our goal now is to prove that $\int_A F\, d\mu$ is not empty and that it behaves *nicely*, meaning, it can be described via the Gelfand integral of the w^*-almost selectors of F and that indeed $\delta^*(x, \int_A F\, d\mu) = \int_A \delta^*(x, F)\, d\mu$ for every $x \in X$.

Lemma 4.2. *Let $f : \Omega \to X^*$ be a w^*-scalarly measurable w^*-almost selector of a Gelfand integrable multi-function $F : \Omega \to cw^*k(X^*)$. Then f is Gelfand integrable and $\int_A f\, d\mu \in \int_A F\, d\mu$ for all $A \in \Sigma$.*

Proof. For each $x \in X$, we have $\delta_*(x, F) \le \langle f, x\rangle \le \delta^*(x, F)$ μ-a.e., hence $\langle f, x\rangle$ is integrable and $\int_A \delta_*(x, F)\, d\mu \le \int_A \langle f, x\rangle\, d\mu \le \int_A \delta^*(x, F)\, d\mu$ for all $A \in \Sigma$. □

We need to quote the following result (particular case of [27, Lemme 3]).

Lemma 4.3 (Valadier)**.** *Let $F : \Omega \to cw^*k(X^*)$ be a w^*-scalarly measurable multi-function. Then for each $x \in X$ the multi-function*

$$F|^x : \Omega \to cw^*k(X^*), \quad F|^x(t) := \{x^* \in F(t) : \langle x^*, x\rangle = \delta^*(x, F)(t)\}$$

is w^*-scalarly measurable.

Remark 4.4. The previous result also holds true for $w^*k(X^*)$-valued multi-functions, by an argument similar to that of [6, Lemma 3.3] (now dealing with the w^*-topology in X^*).

Theorem 4.5. *Let $F : \Omega \to cw^*k(X^*)$ be a w^*-scalarly measurable multi-function. Then F is Gelfand integrable if and only if every w^*-scalarly measurable w^*-almost selector of F is Gelfand integrable. In this case, for each $A \in \Sigma$, the set $\int_A F\, d\mu$ is non-empty, convex, w^*-compact and:*

(i) $\int_A F\, d\mu = \Big\{ \int_A f\, d\mu : f$ *is a Gelfand integrable w^*-almost selector of F*$\Big\}$.
(ii) $\delta^*(x, \int_A F\, d\mu) = \int_A \delta^*(x, F)\, d\mu$ *for every $x \in X$.*

Proof. The *only if* part has been proved in Lemma 4.2. In order to prove the *if* part, fix $x \in X$. By Lemma 4.3, the multi-function $F|^x$ is w^*-scalarly measurable and so we can apply Theorem 2.7 to find a w^*-scalarly measurable w^*-almost selector f of $F|^x$. Of course, f is a w^*-almost selector of F and so it is Gelfand integrable. In particular, $\langle f, x\rangle$ is integrable. Observe that $\delta_*(x, F|^x) = \delta^*(x, F|^x) = \delta^*(x, F)$ and therefore $\langle f, x\rangle = \delta^*(x, F)$ μ-a.e. It follows that $\delta^*(x, F)$ is integrable. As $x \in X$ is arbitrary, F is Gelfand integrable.

Now, we prove (i) and (ii). Let us define

$$S := \Big\{ \int_A f \, d\mu : \; f \text{ is a Gelfand integrable } w^*\text{-almost selector of } F \Big\}.$$

¿From Lemma 4.3 it follows that $\int_A F\, d\mu \supset S$. It will be clear at the end of the proof that the converse inclusion $\int_A F\, d\mu \subset S$ can be established when proving that S is w^*-compact. Once this is done both the inclusion $\int_A F\, d\mu \subset S$ and the proof for (ii) are provided in a single shot below.

¿From now on we assume without loss of generality that $A = \Omega$. Let Q be the collection of all Gelfand integrable w^*-almost selectors of F and consider

$$\tilde{Q} := \Big\{ \big(\langle f, x\rangle\big)_{x\in X} \in L^1(\mu)^X : \; f \in Q \Big\}$$

equipped with the product topology $\mathfrak{T}$ induced by the weak topology on $L^1(\mu)$, where for notational convenience we denote in the same way the composition $\langle f, x\rangle$ and its equivalence class in $L^1(\mu)$. Let $T : \tilde{Q} \to X^*$ be the mapping defined by

$$T\Big(\big(\langle f, x\rangle\big)_{x\in X}\Big) := \int_\Omega f\, d\mu.$$

We claim that:

(α) $\tilde{Q}$ is $\mathfrak{T}$-compact;
(β) T is $\mathfrak{T}$-to-w^*-continuous;
(γ) $T(\tilde{Q}) = S$, hence S is w^*-compact.

Being (β) and (γ) obvious we simply prove (α). To this end, we will establish first that $\tilde{Q}$ is $\mathfrak{T}$-closed in $L^1(\mu)^X$. Let $\{f_\alpha\}$ be a net in Q such that $(\langle f_\alpha, x\rangle)_{x\in X} \to (f^x)_{x\in X} \in L^1(\mu)^X$ in the topology $\mathfrak{T}$, that is, for each $x \in X$ the net $\{\langle f_\alpha, x\rangle\}$ converges to f^x in the weak topology of $L^1(\mu)$. Fix $A \in \Sigma$. For every $x \in X$ the net of integrals $\{\int_A \langle f_\alpha, x\rangle\, d\mu\}$ is convergent; observe also that $\int_A \langle f_\alpha, x\rangle\, d\mu = \langle \int_A f_\alpha\, d\mu, x\rangle$. Since $\{\int_A f_\alpha\, d\mu\}$ is a net contained in the w^*-compact set $\int_A F\, d\mu \subset X^*$, it follows at once that there exists the w^*-limit of $\{\int_A f_\alpha\, d\mu\}$ in X^*, say $\nu(A) \in \int_A F\, d\mu$. Clearly, the set function $\nu : \Sigma \to X^*$ is finitely additive, vanishes on all μ-null sets and satisfies

$$\langle \nu(A), x\rangle = \lim \Big\langle \int_A f_\alpha\, d\mu, x \Big\rangle = \lim \int_A \langle f_\alpha, x\rangle\, d\mu = \int_A f^x\, d\mu$$

for every $x \in X$.

Bearing in mind Lemma 2.6 and the fact that $\langle \nu(A), x\rangle \le \int_A \delta^*(x, F)\, d\mu$ for every $A \in \Sigma$ and every $x \in X$, we can find a countable partition (E_n) of Ω in Σ and positive constants (C_n) such that, for each $n \in \mathbb{N}$, we have $\|\nu(A)\| \le C_n \mu(A)$ for all $A \subset E_n$, $A \in \Sigma$. An appeal to [28, Proposition 6.2] ensures the existence of a Gelfand integrable function $g_n : E_n \to X^*$ such that $\nu(A) = \int_A g_n\, d\mu$ for all $A \subset E_n$, $A \in \Sigma$. Define $g : \Omega \to X^*$ by $g(t) := g_n(t)$ if $t \in E_n$, $n \in \mathbb{N}$. Observe that for each $x \in X$ and $n \in \mathbb{N}$

we have

$$\int_A \langle g, x\rangle \, d\mu = \langle \nu(A), x\rangle \leq \int_A \delta^*(x, F)\, d\mu \quad \text{for all } A \subset E_n,\ A \in \Sigma.$$

It follows that $\langle g, x\rangle \leq \delta^*(x, F)$ μ-a.e. and so g is a w^*-almost selector of F. Since F is Gelfand integrable, we infer that g is Gelfand integrable after Lemma 4.2. Take any $x \in X$. Then for each $A \subset E_n$, $A \in \Sigma$, we have

$$\int_A \langle g, x\rangle \, d\mu = \langle \nu(A), x\rangle = \lim \int_A \langle f_\alpha, x\rangle \, d\mu = \int_A f^x \, d\mu,$$

and therefore $\langle g, x\rangle = f^x$ in $L^1(\mu)$. Hence $(\langle g, x\rangle)_{x\in X} = (f^x)_{x\in X}$ in $L^1(\mu)^X$. This shows that $\tilde{Q}$ is $\mathfrak{T}$-closed in $L^1(\mu)^X$. Moreover, for each $x \in X$ the set

$$K_x := \{h \in L^1(\mu) :\ \delta_*(x, F) \leq h \leq \delta^*(x, F)\}$$

is weakly compact in $L^1(\mu)$ because it is bounded, uniformly integrable, convex and norm closed. Hence $\prod_{x\in X} K_x$ is compact in $(L^1(\mu)^X, \mathfrak{T})$. Since $\tilde{Q} \subset \prod_{x\in X} K_x$, it follows that $\tilde{Q}$ is $\mathfrak{T}$-compact, as claimed.

We already know that $S \subset \int_\Omega F\, d\mu$. Since both sets are convex and w^*-compact, in order to finish the proof of (i) we only have to check (by the separation Hahn-Banach theorem) that for every $x \in X$ we have $\delta^*(x, \int_\Omega F\, d\mu) \leq \delta^*(x, S)$. To this end, like at the beginning of the proof, let $f : \Omega \to X^*$ be a w^*-scalarly measurable w^*-almost selector of $F|^x$, so that f is Gelfand integrable and $\langle f, x\rangle = \delta^*(x, F)$ μ-a.e. The vector $\int_\Omega f\, d\mu \in S \subset \int_\Omega F\, d\mu$ satisfies

$$\delta^*\Big(x, \int_\Omega F\, d\mu\Big) \geq \Big\langle \int_\Omega f\, d\mu, x\Big\rangle = \int_\Omega \langle f, x\rangle \, d\mu = \int_\Omega \delta^*(x, F)\, d\mu \overset{(9)}{\geq} \delta^*\Big(x, \int_\Omega F\, d\mu\Big).$$

This completes the proof of (i) and (ii). $\square$

For separable spaces, the previous Theorem allows us to deduce (via Proposition 2.3(i)) the following result which improves [2, Corollary 18.37].

Corollary 4.6. *Suppose X is separable. Let $F : \Omega \to cw^*k(X^*)$ be a w^*-scalarly measurable multi-function. Then F is Gelfand integrable if and only if every w^*-scalarly measurable selector of F is Gelfand integrable. In this case, for each $A \in \Sigma$, we have*

$$\int_A F\, d\mu = \Big\{\int_A f\, d\mu :\ f \text{ is a Gelfand integrable selector of } F\Big\}.$$

5. The set-valued Dunford and Pettis integrals

Next definition extends the notion of Dunford integrable vector-valued function to the case of multi-functions.

Definition 5.1. *A multi-function $F : \Omega \to cwk(X)$ is said to be Dunford integrable if $\delta^*(x^*, F)$ is integrable for every $x \in X$.*

We note that the multi-function $F : \Omega \to cwk(X)$ is Dunford integrable if, and only if, it is Gelfand integrable when naturally considered with values $F : \Omega \to cw^*k(X^{**})$. Therefore, for each $A \in \Sigma$ there is a set $\int_A F\, d\mu \in cw^*k(X^{**})$ (called the *Dunford integral* of F over A) such that

$$\delta^*\Big(x^*, \int_A F\, d\mu\Big) = \int_A \delta^*(x^*, F)\, d\mu$$

for every $x^* \in X^*$.

Proceeding with multi-functions as it is usually done when defining Pettis integrability via Dunford integrability for vector-valued functions, we arrive at the following definition.

Definition 5.2. *A multi-function $F : \Omega \to cwk(X)$ is said to be Pettis integrable if it is Dunford integrable and $\int_A F\,d\mu \subset X$ for all $A \in \Sigma$.*

Corollary 5.3. *If the multi-function $F : \Omega \to cwk(X)$ is Pettis integrable, then $\int_A F\,d\mu$ is weakly compact in X for all $A \in \Sigma$.*

Proof. When we look at F as a Gelfand integrable function $F : \Omega \to cw^*k(X^{**})$ the integral $\int_A F\,d\mu \subset X^{**}$ is w^*-compact. If we require now F to be Pettis integrable we have $\int_A F\,d\mu \subset X$, and therefore $\int_A F\,d\mu$ is weakly compact in X. □

As a consequence of the above we conclude that the notion of Pettis integrability introduced here does coincide with the notion of Pettis integrability introduced in the monograph by Castaing and Valadier [8] and that has been studied more recently by different authors, see for instance [4, 5, 7, 9, 10, 11, 29] and [30].

Proposition 5.4. *Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function. Then F is Dunford integrable if and only if every scalarly measurable selector of F is Dunford integrable. In this case, for each $A \in \Sigma$, we have*

$$\int_A F\,d\mu = \overline{\left\{\int_A f\,d\mu : \ f \text{ is a Dunford integrable selector of } F\right\}}^{w^*}. \tag{10}$$

Proof. We proceed as we did in the proof of Theorem 4.5 but bearing in mind that *every $cwk(X)$-valued scalarly measurable multi-function has always scalarly measurable selectors*, [6, Theorem 3.8]. We note that to prove (10) we can avoid the fuss of dealing with $\tilde{Q}$ as we have to do in Theorem 4.5. Indeed, if we call

$$S := \overline{\left\{\int_A f\,d\mu : \ f \text{ is a Dunford integrable selector of } F\right\}}^{w^*},$$

we easily obtain the inclusion $S \subset \int_A F\,d\mu$. Since S is obviously w^*-compact and convex the converse inclusion $\int_A F\,d\mu \subset S$ can be proved by showing

$$\delta^*\Big(x^*, \int_A F\,d\mu\Big) \leq \delta^*(x^*, S) \quad \text{for every } x^* \in X^*.$$

This inequality is easily established as we did at the end of Theorem 4.5, bearing in mind again that every $cwk(X)$-valued scalarly measurable multi-function has always scalarly measurable selectors. □

Corollary 5.5. *A scalarly measurable multi-function $F : \Omega \to cwk(X)$ is Pettis integrable if and only if every scalarly measurable selector of F is Pettis integrable. In this case, for each $A \in \Sigma$, we have*

$$\int_A F\,d\mu = \overline{\left\{\int_A f\,d\mu : \ f \text{ is a Pettis integrable selector of } F\right\}}. \tag{11}$$

Proof. For the first part, see [5, Theorem 4.2]. The second part is a consequence of Proposition 5.4; see also [5, Theorem 2.6]. □

Corollary 5.6. *Suppose X is separable and contains no isomorphic copy of c_0. Then every Dunford integrable multi-function $F : \Omega \to cwk(X)$ is Pettis integrable.*

Proof. Every scalarly measurable selector of F is Dunford integrable (Proposition 5.4) and so Pettis integrable by the Dimitrov-Diestel theorem, see *e.g.* [13, Theorem 7, p. 54]. □

Replacing selectors by w-almost selectors we can get rid of the closure in equality (11) above. The proof of the following result imitates that of Theorem 4.5 and so we omit it.

Proposition 5.7. *If X has the weak Radon-Nikodým property and $F : \Omega \to cwk(X)$ is a Pettis integrable multi-function, then for each $A \in \Sigma$ we have*

$$\int_A F\,d\mu = \Big\{ \int_A f\,d\mu : \ f \text{ is a Pettis integrable } w\text{-almost selector of } F \Big\}.$$

6. COMPACTNESS OF THE GELFAND INTEGRAL

Next example provides a Gelfand integrable multi-function $F : \Omega \to cw^*k(\ell^\infty)$ with norm compact values whose integral $\int_\Omega F\,d\mu$ is not weakly compact.

Example 6.1. *Let $\Omega := \mathbb{N}$, $\Sigma = \mathcal{P}(\mathbb{N})$ and μ the probability measure on (Ω, Σ) defined by $\mu(A) := \sum_{n\in A} 2^{-n}$. Let $\{e_n\}_{n\in\mathbb{N}}$ denote the canonical basis of c_0 and define*

$$F : \Omega \to ck(c_0), \quad F(n) := \mathrm{co}\{-2^n e_n, 2^n e_n\}.$$

Then F is Dunford integrable and $\int_\Omega F\,d\mu = B_{\ell^\infty}$.

Proof. Take $x^* = (a_n) \in c_0^* = \ell^1$. Then $\delta^*(x^*, F)(n) = |\langle x^*, 2^n e_n\rangle| = 2^n|a_n|$ for all $n \in \mathbb{N}$. So, for any given $A \subset \mathbb{N}$ we have $\int_A |\delta^*(x^*, F)|\,d\mu = \sum_{n\in A} |a_n| < \infty$. It follows that F is Dunford integrable. Observe that

$$\begin{aligned}
\int_A F\,d\mu &= \bigcap_{x^*\in\ell^1} \Big\{ x^{**} \in \ell^\infty : \int_A \delta_*(x^*, F)\,d\mu \le \langle x^{**}, x^*\rangle \le \int_A \delta^*(x^*, F)\,d\mu \Big\} = \\
&= \bigcap_{(a_n)\in\ell^1} \Big\{ (b_n) \in \ell^\infty : -\sum_{n\in A} |a_n| \le \sum_{n\in\mathbb{N}} a_n b_n \le \sum_{n\in A} |a_n| \Big\} = \\
&= \Big\{ (b_n) \in \ell^\infty : b_n = 0 \text{ for all } n \notin A \text{ and } |b_n| \le 1 \text{ for all } n \in A \Big\}.
\end{aligned}$$

Therefore for any infinite $A \subset \mathbb{N}$ we have: (i) the integral $\int_A F\,d\mu$ does not remain in c_0, hence F is not Pettis integrable; (ii) $\int_A F\,d\mu$ is not weakly compact in ℓ^∞. $\square$

The goal of this section is to prove that if $F : \Omega \to cw^*k(X^*)$ is a bounded Gelfand integrable multi-function having norm compact values, then $\int_A F\,d\mu$ is weakly compact for all $A \in \Sigma$, see Theorem 6.8. Our previous Example 6.1 shows that the hypothesis of boundedness of F is really needed in Theorem 6.8.

We start by proving that even for a bounded F the norm compactness of its values does not ensure that $\int_\Omega F\,d\mu$ is norm compact as well.

Example 6.2. *Let $X := C[0, 1]$ and let μ be the Lebesgue measure on $[0, 1]$. The multi-function $F : [0, 1] \to cw^*k(X^*)$ defined by $F(t) := \{\lambda\delta_t : 0 \le \lambda \le 1\}$ satisfies:*

(i) *F is bounded and takes norm compact values;*
(ii) *F is Gelfand integrable;*
(iii) *$\int_A F\,d\mu$ is not norm compact for every Borel set $A \subset [0, 1]$ with $\mu(A) > 0$.*

Proof. (i) is immediate.

(ii) It suffices to check that F is w^*-scalarly measurable. Fix $x \in X = C[0, 1]$. For each $t \in [0, 1]$ we have

$$\delta^*(x, F(t)) = \sup\{\langle \lambda\delta_t, x\rangle : 0 \le \lambda \le 1\} = \sup\{\lambda x(t) : 0 \le \lambda \le 1\} = x^+(t)$$

and similarly $\delta_*(x, F(t)) = -x^-(t)$. Since x is measurable, so is $x^+ = \delta^*(x, F)$.

(iii) Fix a Borel set $A \subset [0,1]$. We shall prove that

$$\int_A F\,d\mu = \big\{f \in L^1[0,1] :\ 0 \le f \le \mathbb{1}_A\big\}, \tag{12}$$

where $L^1[0,1]$ is identified with the closed subspace of X^* made up of all Borel measures which are absolutely μ-continuous.

To prove "$\subset$" in (12), take any $\nu \in \int_A F\,d\mu$. For each $x \in X$, $x \ge 0$, the value $\langle \nu, x\rangle = \int_{[0,1]} x\,d\nu$ lies between:

$$0 = \int_A -x^-\,d\mu = \int_A \delta_*(x,F)\,d\mu \le \langle \nu, x\rangle \le \int_A \delta^*(x,F)\,d\mu = \int_A x^+\,d\mu = \int_A x\,d\mu.$$

It follows from the above that $\nu \ge 0$, $\nu([0,1]\backslash A) = 0$ and $\nu(B) \le \mu(B)$ for every Borel set $B \subset A$. Therefore ν is absolutely μ-continuous and its Radon-Nikodým derivative $f = \frac{d\nu}{d\mu}$ satisfies $0 \le f \le \mathbb{1}_A$.

To prove "$\supset$" in (12), take any $f \in L^1[0,1]$ with $0 \le f \le \mathbb{1}_A$ and consider the associated measure $\nu \in X^*$ defined by $\nu(B) := \int_B f\,d\mu$ for every Borel set $B \subset [0,1]$. Fix $x \in X$. Since $-x^-\mathbb{1}_A \le xf \le x^+\mathbb{1}_A$ μ-a.e. and $\langle \nu, x\rangle = \int_{[0,1]} xf\,d\mu$, we have

$$\int_A \delta_*(x,F)\,d\mu \le \langle \nu, x\rangle \le \int_A \delta^*(x,F)\,d\mu.$$

As $x \in X$ is arbitrary, we get $\nu \in \int_A F\,d\mu$. This finishes the proof of equality (12) .

Since $\{f \in L^1[0,1] : 0 \le f \le \mathbb{1}_A\}$ is not norm compact whenever $\mu(A) > 0$, statement (iii) follows from equality (12). □

Next definition and lemmata are intended to split into several parts and simplify the proof of Theorem 6.8.

Definition 6.3. *Let $F, G : \Omega \to 2^{X^*}$ be two multi-functions and $\varepsilon > 0$. We say that G is an almost ε-net for F if:*

(i) *$G(t) \subset F(t)$ for every $t \in \Omega$;*
(ii) *for each $x \in S_X$ we have $\delta^*(x,F) \le \delta^*(x,G) + \varepsilon$ μ-a.e.*

Lemma 6.4. *Let $F : \Omega \to k(X^*)$ be a w^*-scalarly measurable multi-function and let $\varepsilon > 0$. Then there exist $A \in \Sigma^+$ and finitely many w^*-scalarly measurable selectors $g_1, \dots, g_n$ of $F|_A$ such that the multi-function $G : A \to k(X^*)$ given by*

$$G(t) := \{g_1(t), \dots, g_n(t)\}, \quad t \in \Omega,$$

is an almost ε-net for $F|_A$.

Proof. Our proof is by contradiction. Assume there is $\varepsilon > 0$ such that:

(♡) For every $A \in \Sigma^+$ and every finite collection $g_1, \dots, g_n$ of w^*-scalarly measurable selectors of $F|_A$, the multi-function $t \mapsto \{g_1(t), \dots, g_n(t)\}$ is not an almost ε-net for $F|_A$.

CLAIM.- There is a sequence $f_n : \Omega \to X^*$ of w^*-scalarly measurable selectors of F such that $\|f_i(t) - f_j(t)\| > \varepsilon$ for μ-a.e. $t \in \Omega$ whenever $i \neq j$.

To show this we proceed by induction. Assume that $f_1, \dots, f_n$ have been already constructed. Define the multi-function $F_n : \Omega \to k(X^*)$ by $F_n(t) := \{f_1(t), \dots, f_n(t)\}$. According to condition (♡), for each $A \in \Sigma^+$ there exist $x \in S_X$ and $B \in \Sigma_A^+$ such that $\delta^*(x, F(t)) > \delta^*(x, F_n(t)) + \varepsilon$ for all $t \in B$. A standard exhaustion argument ensures the existence of countably many pairwise disjoint measurable sets $B_1, B_2, \dots$ with $\mu(\Omega \setminus \bigcup_k B_k) = 0$ and vectors $x_k \in S_X$ such that

$$\delta^*(x_k, F(t)) > \delta^*(x_k, F_n(t)) + \varepsilon \quad \text{for every } t \in B_k \text{ and } k \in \mathbb{N}.$$

Define $\widetilde{F} : \Omega \to k(X^*)$ by

$$\widetilde{F}(t) := \big\{x^* \in F(t) : \ x^*(x_k) = \delta^*(x_k, F(t))\big\} \quad \text{for } t \in B_k, \ k \in \mathbb{N},$$

and $\widetilde{F}(t) := \{0\}$ for $t \in \Omega \setminus \bigcup_k B_k$. Then $\widetilde{F}$ is w^*-scalarly measurable (see Remark 4.4). We use now Corollary 3.12 to find a w^*-scalarly measurable selector f_{n+1} of $\widetilde{F}$. Fix $t \in B_k$, $k \in \mathbb{N}$. Then, for each $i = 1, \dots, n$, we have

$$\langle f_{n+1}(t), x_k \rangle = \delta^*(x_k, F(t)) > \delta^*(x_k, F_n(t)) + \varepsilon \geq \langle f_i(t), x_k \rangle + \varepsilon,$$

and so $\|f_{n+1}(t) - f_i(t)\| > \varepsilon$. The proof of the *CLAIM* is over.

Finally, observe that the *CLAIM* ensures the existence of $B \in \Sigma$ with $\mu(\Omega \setminus B) = 0$ such that, for each $t \in B$, we have $\|f_i(t) - f_j(t)\| > \varepsilon$ whenever $i \neq j$. This contradicts the norm compactness of $F(t)$, because $f_n(t) \in F(t)$ for all $n \in \mathbb{N}$. □

Lemma 6.5. *Let $F, G : \Omega \to cw^*k(X^*)$ be Gelfand integrable multi-functions. Then $F + G$ is Gelfand integrable and $\int_A (F+G)\, d\mu = \int_A F\, d\mu + \int_A G\, d\mu$ for every $A \in \Sigma$.*

Proof. Clearly $F + G$ takes values in $cw^*k(X^*)$ and for each $x \in X$ we have

$$\delta^*(x, F+G) = \delta^*(x, F) + \delta^*(x, G),$$

hence $\delta^*(x, F+G)$ is integrable. Fix $A \in \Sigma$. Then the set $L := \int_A F\, d\mu + \int_A G\, d\mu$ belongs to $cw^*k(X^*)$ and Theorem 4.5 (ii) ensures that

$$\int_A \delta^*(x, F+G)\, d\mu = \int_A \delta^*(x, F)\, d\mu + \int_A \delta^*(x, G)\, d\mu = \\ = \delta^*\Big(x, \int_A F\, d\mu\Big) + \delta^*\Big(x, \int_A G\, d\mu\Big) = \delta^*(x, L).$$

An appeal to the separation Hahn-Banach theorem and again to Theorem 4.5 (ii) allows us to obtain $\int_A (F+G)\, d\mu = L$, as claimed. □

Lemma 6.6. *Let $g_1, \dots, g_n : \Omega \to X^*$ be bounded Gelfand integrable functions. Then the multi-function $G : \Omega \to cw^*k(X^*)$ given by*

$$G(t) := \operatorname{co}\{g_1(t), \dots, g_n(t)\}, \quad t \in \Omega$$

is Gelfand integrable and $\int_A G\, d\mu$ is weakly compact for every $A \in \Sigma$.

Proof. For each $x \in X$ we have $\delta^*(x, G) = \max\{\langle g_i, x\rangle : i = 1, \dots, n\}$ pointwise, and so $\delta^*(x, G)$ is integrable. Thus G is Gelfand integrable.

We prove that $\int_A G\, d\mu$ is weakly compact for $A = \Omega$: same ideas work for an arbitrary $A \in \Sigma$. Fix $i \in \{1, \dots, n\}$. Since g_i is bounded, its indefinite Gelfand integral given by $C \mapsto \int_C g_i\, d\mu$ is a finitely additive X^*-valued measure satisfying

$$\Big\| \int_C g_i\, d\mu \Big\| \leq \mu(C) \cdot \sup_{t \in \Omega} \|g_i(t)\| \quad \text{for all } C \in \Sigma,$$

hence it is countably additive. Consequently the set $K_i := \{\int_C g_i\, d\mu : C \in \Sigma\}$ is weakly relatively compact in X^*, *cf.* [13, Corollary 7, p. 14]. The Krein-Smulyan theorem (*cf.* [13, Theorem 11, p. 51]) ensures us that $\overline{\mathrm{co}}(K_i)$ is weakly compact in X^*. The multi-function $H_i : \Omega \to cw^*k(X^*)$ given by

$$H_i(t) := \{\lambda g_i(t) : \, 0 \leq \lambda \leq 1\},$$

is Gelfand integrable, since $\delta^*(x, H_i) = \langle g_i, x\rangle^+$ for every $x \in X$. Moreover, we have $\int_\Omega H_i \, d\mu \subset \overline{\mathrm{co}}(K_i)$. Indeed, to prove this it suffices to apply the separation Hahn-Banach theorem, taking into account that $\overline{\mathrm{co}}(K_i)$ is convex w^*-closed (it is weakly compact) and that, for each $x \in X$, we have

$$\delta^*\Big(x, \int_\Omega H_i \, d\mu\Big) = \int_\Omega \delta^*(x, H_i) \, d\mu = \Big\langle \int_{\Omega_x} g_i \, d\mu, x \Big\rangle \leq \delta^*(x, \overline{\mathrm{co}}(K_i)),$$

where $\Omega_x := \{t \in \Omega : \langle g_i, x\rangle \geq 0\} \in \Sigma$.

Let $H : \Omega \to cw^*k(X^*)$ be the multi-function defined by $H(t) := \sum_{i=1}^n H_i(t)$. By Lemma 6.5, H is Gelfand integrable and

$$\int_\Omega H \, d\mu = \sum_{i=1}^n \int_\Omega H_i \, d\mu \subset \sum_{i=1}^n \overline{\mathrm{co}}(K_i).$$

Observe that $K := \sum_{i=1}^n \overline{\mathrm{co}}(K_i)$ is weakly compact. Since $G(t) \subset H(t)$ for every $t \in \Omega$, it follows that $\int_\Omega G \, d\mu \subset \int_\Omega H \, d\mu \subset K$ and so $\int_\Omega G \, d\mu$ is weakly compact. □

Lemma 6.7. *Let $F : \Omega \to cw^*k(X^*)$ be a bounded Gelfand integrable multi-function. Then the set function*

$$\nu_F : \Sigma \to \ell^\infty(B_X), \quad \nu_F(A)(x) := \delta^*\Big(x, \int_A F \, d\mu\Big),$$

is a countably additive vector measure.

Proof. We have $\nu_F(A)(x) = \int_A \delta^*(x, F) \, d\mu$ for every $A \in \Sigma$ and $x \in B_X$ by Theorem 4.5 (ii), hence ν_F is finitely additive. Fix $C > 0$ large enough such that $\|x^*\| \leq C$ for every $x^* \in \bigcup_{t\in\Omega} F(t)$. Then $|\delta^*(x, F)| \leq C$ pointwise for every $x \in B_X$ and so $\|\nu_F(A)\|_{\ell^\infty(B_X)} \leq C\mu(A)$ for every $A \in \Sigma$. Thus ν_F is countably additive. □

We arrive at the main result of this section:

Theorem 6.8. *Let $F : \Omega \to cw^*k(X^*)$ be a bounded Gelfand integrable multi-function having norm compact values. Then $\int_C F \, d\mu$ is weakly compact for all $C \in \Sigma$.*

Proof. We prove that $\int_C G \, d\mu$ is weakly compact for $C = \Omega$: same ideas work for an arbitrary $C \in \Sigma$. Fix $\varepsilon > 0$. By Lemma 6.4 there exist $A \in \Sigma^+$ and finitely many w^*-scalarly measurable selectors $g_1, \dots, g_n$ of $F|_A$ such that the multi-function $t \mapsto \{g_1(t), \dots, g_n(t)\}$ is an almost ε-net for $F|_A$. Clearly, the multi-function

$$G : A \to cw^*k(X^*), \quad G(t) := \mathrm{co}\{g_1(t), \dots, g_n(t)\},$$

is also an almost ε-net for $F|_A$. By Lemma 6.6, G is Gelfand integrable and its integral $\int_A G \, d\mu$ is weakly compact. Since G is an almost ε-net for $F|_A$, for each $x \in S_X$ we have $\delta^*(x, F) \leq \delta^*(x, G) + \varepsilon$ μ-a.e. in A and Theorem 4.5 (ii) ensures us that

$$\delta^*\Big(x, \int_A F \, d\mu\Big) = \int_A \delta^*(x, F) \, d\mu \leq \int_A \delta^*(x, G) \, d\mu + \varepsilon\mu(A) =$$
$$= \delta^*\Big(x, \int_A G \, d\mu\Big) + \varepsilon\mu(A) = \delta^*\Big(x, \int_A G \, d\mu + \varepsilon\mu(A)B_{X^*}\Big).$$

It follows that $\int_A F \, d\mu \subset \int_A G \, d\mu + \varepsilon\mu(A)B_{X^*}$.

A standard exhaustion argument guarantees the existence of countably many pairwise disjoint measurable sets $A_1, A_2, \dots$ with $\mu(\Omega \setminus \bigcup_k A_k) = 0$ and countably many Gelfand integrable multi-functions $G_k : A_k \to cw^*k(X^*)$, with $\int_{A_k} G_k \, d\mu$ being weakly compact, such that

$$\int_{A_k} F \, d\mu \subset \int_{A_k} G_k \, d\mu + \varepsilon\mu(A_k) B_{X^*} \quad \text{for all } k \in \mathbb{N}. \tag{13}$$

By Lemma 6.7, there is $K \in \mathbb{N}$ such that $\|\nu_F(\bigcup_{k>K} A_k)\|_{\ell^\infty(B_X)} \leq \varepsilon$, that is,

$$\delta^*\Big(x, \int_{\bigcup_{k>K} A_k} F \, d\mu\Big) \leq \varepsilon \quad \text{for all } x \in B_X.$$

Bearing in mind Theorem 4.5 (ii), it follows that

$$\begin{aligned}
\delta^*\Big(x, \int_\Omega F \, d\mu\Big) = \int_\Omega \delta^*(x, F) \, d\mu = \sum_{k=1}^K \int_{A_k} \delta^*(x, F) \, d\mu + \int_{\bigcup_{k>K} A_k} \delta^*(x, F) \, d\mu = \\
= \sum_{k=1}^K \delta^*\Big(x, \int_{A_k} F \, d\mu\Big) + \delta^*\Big(x, \int_{\bigcup_{k>K} A_k} F \, d\mu\Big) = \\
= \delta^*\Big(x, \sum_{k=1}^K \int_{A_k} F \, d\mu\Big) + \delta^*\Big(x, \int_{\bigcup_{k>K} A_k} F \, d\mu\Big) \leq \\
\leq \delta^*\Big(x, \sum_{k=1}^K \int_{A_k} F \, d\mu\Big) + \varepsilon \;=\; \delta^*\Big(x, \sum_{k=1}^K \int_{A_k} F \, d\mu + \varepsilon B_{X^*}\Big),
\end{aligned}$$

for all $x \in S_X$, hence

$$\int_\Omega F \, d\mu \subset \sum_{k=1}^K \int_{A_k} F \, d\mu + \varepsilon B_{X^*} \overset{(13)}{\subset} \sum_{k=1}^K \int_{A_k} G_k \, d\mu + 2\varepsilon B_{X^*},$$

and the set $\sum_{k=1}^K \int_{A_k} G_k \, d\mu$ is weakly compact. As $\varepsilon > 0$ is arbitrary, Grothendieck's test (*cf.* [12, Lemma 2, p. 227]) guarantees that $\int_\Omega F \, d\mu$ is weakly compact. □

Remark 6.9. The previous Theorem also works when F is *integrably bounded*, that is, there is $h \in \mathcal{L}^1(\mu)$ such that $\sup\{\|x^*\| : x^* \in F(t)\} \leq h(t)$ for μ-a.e. $t \in \Omega$. This can be deduced easily from the proofs of Lemmas 6.6 and 6.7.

7. Some open problems

(A) Let $F : \Omega \to w^*k(X^*)$ be a w^*-scalarly measurable multi-function. Does F admit a w^*-scalarly measurable selector? What about $cw^*k(X^*)$-valued F?

(B) Let K be a compact space equipped with a Radon probability and $X := \ell^\infty(K)$. Let $F : K \to cw^*k(X^*)$ be the multi-function given by

$$F(t) := \{x^* \in B_{X^*} : \; x^*|_{C(K)} = \delta_t\}, \quad t \in K.$$

Does F admit a w^*-scalarly measurable selector?

(C) Let $F : \Omega \to cwk(X)$ be a Dunford (resp. Pettis) integrable multi-function. Does

$$\int_\Omega F \, d\mu = \Big\{\int_\Omega f \, d\mu : \; f \text{ is a Dunford (resp. Pettis) integrable selector of } F\Big\} \quad ?$$

(D) Let $F : \Omega \to cw^*k(X^*)$ be a bounded Gelfand integrable multi-function having weakly compact values. Is $\int_\Omega F \, d\mu$ weakly compact?

REFERENCES

[1] F. Albiac and N. J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006. MR 2192298 (2006h:46005)

[2] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*, third ed., Springer, Berlin, 2006, A hitchhiker's guide. MR 2378491 (2008m:46001)

[3] R. D. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodým property*, Lecture Notes in Mathematics, vol. 993, Springer-Verlag, Berlin, 1983. MR 704815 (85d:46023)

[4] B. Cascales, V. Kadets, and J. Rodríguez, *The Pettis integral for multi-valued functions via single-valued ones*, J. Math. Anal. Appl. **332** (2007), no. 1, 1–10. MR 2319640 (2008e:28025)

[5] B. Cascales, V. Kadets, and J. Rodríguez, *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*, J. Funct. Anal. **256** (2009), no. 3, 673–699. MR 2484932 (2010b:28027)

[6] B. Cascales, V. Kadets, and J. Rodríguez, *Measurability and selections of multi-functions in Banach spaces*, J. Convex Anal. **17** (2010), no. 1, 229–240.

[7] B. Cascales and J. Rodríguez, *Birkhoff integral for multi-valued functions*, J. Math. Anal. Appl. **297** (2004), no. 2, 540–560, Special issue dedicated to John Horváth. MR 2088679 (2005f:26021)

[8] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Springer-Verlag, Berlin, 1977, Lecture Notes in Mathematics, Vol. 580. MR 0467310 (57 #7169)

[9] L. Di Piazza and K. Musiał, *Set-valued Kurzweil-Henstock-Pettis integral*, Set-Valued Anal. **13** (2005), no. 2, 167–179. MR 2148134 (2006a:28013)

[10] L. Di Piazza and K. Musiał, *A decomposition theorem for compact-valued Henstock integral*, Monatsh. Math. **148** (2006), no. 2, 119–126. MR 2235359 (2007i:28015)

[11] L. Di Piazza and K. Musiał, *A decomposition of Henstock-Kurzweil-Pettis integrable multifunctions*, Vector Measures, Integration and Related Topics. Operator Theory: Advances and Applications. Vol. 201, Birkhauser Basel, 2010, pp. 171–182.

[12] J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Mathematics, vol. 92, Springer-Verlag, New York, 1984. MR 737004 (85i:46020)

[13] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15. MR 0453964 (56 #12216)

[14] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 8, Springer-Verlag, New York, 2001. MR 1831176 (2002f:46001)

[15] I. M. Gel'fand, *Sur un lemme de la théorie des espaces linéaires*, Comm. Inst. Sci. Math. Méc. Univ. de Kharkoff et Soc. Math. Kharkoff **13** (1936), no. 4, 35–40.

[16] M. Ali Khan, *Equilibrium points of nonatomic games over a Banach space*, Trans. Amer. Math. Soc. **293** (1986), no. 2, 737–749. MR 816322 (87f:90035)

[17] M. Ali Khan and Y. Sun, *Integrals of set-valued functions with a countable range*, Math. Oper. Res. **21** (1996), no. 4, 946–954. MR 1419910 (97k:28022)

[18] M. Ali Khan and Y. Sun, *Nonatomic games on Loeb spaces*, Proc. Nat. Acad. Sci. U.S.A. **93** (1996), no. 26, 15518–15521 (electronic). MR 1626947 (99b:90165)

[19] E. Klein and A. C. Thompson, *Theory of correspondences*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons Inc., New York, 1984, Including applications to mathematical economics, A Wiley-Interscience Publication. MR 752692 (86a:90012)

[20] G. Köthe, *Topological vector spaces. I*, Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159, Springer-Verlag New York Inc., New York, 1969. MR 40 #1750

[21] P. A. Loeb and Y. Sun, *A general Fatou lemma*, Adv. Math. **213** (2007), no. 2, 741–762. MR 2332608 (2009d:28004)

[22] K. Musiał, *Topics in the theory of Pettis integration*, Rend. Istit. Mat. Univ. Trieste **23** (1991), no. 1, 177–262 (1993), School on Measure Theory and Real Analysis (Grado, 1991). MR 1248654 (94k:46084)

[23] K. Podczeck, *On the convexity and compactness of the integral of a Banach space valued correspondence*, J. Math. Econom. **44** (2008), no. 7-8, 836–852. MR 2414904 (2009h:46090)

[24] Y. Sun, *Integration of correspondences on Loeb spaces*, Trans. Amer. Math. Soc. **349** (1997), no. 1, 129–153. MR 1401529 (97f:46065)

[25] Y. Sun and N. C. Yannelis, *Saturation and the integration of Banach valued correspondences*, J. Math. Econom. **44** (2008), no. 7-8, 861–865. MR 2414906 (2009h:46091)

[26] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. **51** (1984), no. 307, ix+224. MR 86j:46042

[27] M. Valadier, *Multi-applications mesurables à valeurs convexes compactes*, J. Math. Pures Appl. (9) **50** (1971), 265–297. MR 0299752 (45 #8800)

[28] D. van Dulst, *Characterizations of Banach spaces not containing l^1*, CWI Tract, vol. 59, Centrum voor Wiskunde en Informatica, Amsterdam, 1989. MR 90h:46037

[29] H. Ziat, *Convergence theorems for Pettis integrable multifunctions*, Bull. Polish Acad. Sci. Math. **45** (1997), no. 2, 123–137. MR 1466838 (98i:46035)

[30] H. Ziat, *On a characterization of Pettis integrable multifunctions*, Bull. Polish Acad. Sci. Math. **48** (2000), no. 3, 227–230. MR 1779005 (2001d:46063)

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30100 ESPINARDO (MURCIA), SPAIN
E-mail address: `beca@um.es`

DEPARTMENT OF MECHANICS AND MATHEMATICS, KHARKOV NATIONAL UNIVERSITY, PL. SVOBODY 4, 61077 KHARKOV, UKRAINE
E-mail address: `vova1kadets@yahoo.com`

DEPARTAMENTO DE MATEMÁTICA APLICADA, FACULTAD DE INFORMÁTICA, UNIVERSIDAD DE MURCIA, 30100 ESPINARDO (MURCIA), SPAIN
E-mail address: `joserr@um.es`