DISTANCES TO SPACES OF MEASURABLE AND INTEGRABLE FUNCTIONS

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ABSTRACT. Given a complete probability space (Ω, Σ, μ) and a Banach space X we establish formulas to compute the distance from a function $f \in X^\Omega$ to the spaces of strongly measurable functions and Bochner integrable functions. We study the relationship between these distances and use them to prove some quantitative counterparts of Pettis' measurability theorem. We also give several examples showing that some of our estimates are sharp.

1. INTRODUCTION

A good number of mathematical analysis strategies, classical and modern results deal with distances, optimization, inequalities and the consequences that can be derived from them. The model we follow here is based on the simple idea of computing the distance of an arbitrary function to a space of particular functions, namely continuous, Baire one, integrable, etc. Below there is a beautiful example:

Theorem 1.1 (Benyamini and Lindenstrauss, [6, Proposition 1.18])**.** *Let T be a normal topological space. If $f \in \mathbb{R}^T$ is bounded, then the distance of f to the space $C_b(T)$ of real bounded and continuous functions is given by the equality*

$$d(f, C_b(T)) = \frac{1}{2}\operatorname{osc}(f)\,, \tag{1.1}$$

where

$$\operatorname{osc}(f) = \sup_{x\in T} \inf \Big\{ \sup_{y,z\in U} |f(y) - f(z)| : U \text{ open neighborhood of } x \Big\}.$$

The formula (1.1) as well as the computation of such kind of distances to other spaces have been successfully used in a good a number of recent papers dealing with measures of non-compactness, interpolation, Dunford-Pettis property, reflexivity, weak sequential completness, etc. (see for instance [2, 3, 8, 9, 15, 20, 21, 22, 24, 27, 28]). Our aim here is to establish a formula analogous to (1.1) but when dealing with measurable and integrable (vector-valued) functions instead of continuous functions. Our hope is that beyond the applications that we present in this paper, the formulas that we establish can be useful for some other purposes as well.

Throughout the paper $(X, \|\cdot\|)$ is a Banach space and (Ω, Σ, μ) is a complete probability space for which we denote $\Sigma^+ = \{B \in \Sigma : \mu(B) > 0\}$ and $\Sigma_A^+ = \{B \in \Sigma^+ : B \subset A\}$ for $A \in \Sigma^+$. μ^* stands for the outer measure induced by μ. We denote by λ the Lebesgue measure on the unit interval $[0, 1]$. B_X stands for the closed unit ball of X, and X^* is the dual space of X. Given any set $W \subset X$ we write $\operatorname{diam}(W) = \sup\{\|x - y\| : x, y \in W\}$.

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For $f \in X^\Omega$ and $S \subset X^\Omega$ we write $d(f,S) = \inf\{d(f,g) : g \in S\}$, where d is the supremum metric (that we allow to take the value ∞) given by

$$d(f,g) = \sup\{\|f(t) - g(t)\| : t \in \Omega\}.$$

We use the convention $\inf \emptyset = \infty$. All unexplained terminology can be found in our standard references [1] and [11].

We write $\mathcal{M}(\mu, X)$ (resp. $\mathcal{L}^1(\mu, X)$) to denote the subspace of X^Ω made up of all strongly measurable (resp. Bochner integrable) functions. Recall that a function $f \in X^\Omega$ is called: (i) *simple* if it can be written as a finite sum of functions of the form $x1_A$, where $x \in X$ and $A \in \Sigma$; (ii) *strongly measurable* if it is the μ-a.e. limit of a sequence of simple functions; (iii) *Bochner integrable* if it is strongly measurable and $\int_\Omega \|f(t)\|\, d\mu(t) < \infty$. Note that here we deal with 'functions' instead of 'equivalence classes'. For simplicity, we write $\mathcal{M}(\mu) := \mathcal{M}(\mu, \mathbb{R})$ and $\mathcal{L}^1(\mu) := \mathcal{L}^1(\mu, \mathbb{R})$.

To estimate the distance of $f \in X^\Omega$ to $\mathcal{M}(\mu, X)$ and $\mathcal{L}^1(\mu, X)$ we introduce the strong measurability index

$$\mathrm{meas}(f) := \inf\{\varepsilon > 0 : \text{ for every } A \in \Sigma^+ \text{ there is } B \in \Sigma_A^+ \text{ with } \mathrm{diam}(f(B)) \le \varepsilon\},$$

and the Bochner integrability index

$$\mathrm{int}(f) := \sup_{A \in \Sigma^+} \frac{\mathrm{a}(f,A)}{\mu(A)}$$

where

$$\mathrm{a}(f,A) := \inf\Big\{ \sum_{B \in \Gamma} \mu(B)\, \mathrm{diam}(f(B)) : \Gamma \in \Pi_A(f) \Big\}$$

and $\Pi_A(f)$ is the set of all partitions Γ of A into countably many elements of Σ such that:

(i) $f(B)$ is bounded for every $B \in \Gamma \cap \Sigma^+$;

(ii) the series $\sum_{B \in \Gamma} \mu(B)\|f(t_B)\|$ is convergent for every choice $t_B \in B$, $B \in \Gamma$.

Sections 2 and 3 are devoted to prove that for $f \in X^\Omega$ we have

$$\frac{1}{2}\mathrm{meas}(f) \le d(f, \mathcal{M}(\mu, X)) \le \mathrm{meas}(f) \quad \text{and} \quad \frac{1}{2}\mathrm{int}(f) \le d(f, \mathcal{L}^1(\mu, X)) \le \mathrm{int}(f)$$

and that, when $X = \mathbb{R}$, we actually have the following equalities:

$$d(f, \mathcal{M}(\mu)) = \frac{1}{2}\mathrm{meas}(f) \quad \text{and} \quad d(f, \mathcal{L}^1(\mu)) = \frac{1}{2}\mathrm{int}(f).$$

Section 4 illustrates how the above indexes and distances can be used to offer a *quantitative* version of Pettis' measurability theorem, see [11, p. 42, Theorem 2]. The paper is completed with several examples that show that some of the estimates for the distances that we offer in the vector-valued case are sharp.

2. Measurability index and distances to measurable functions

It is well-known that a function $f : \Omega \to X$ is strongly measurable (i.e. $f \in \mathcal{M}(\mu, X)$) if and only if it satisfies the following property:

($\star$) for every $\varepsilon > 0$ and every $A \in \Sigma^+$ there is $B \in \Sigma_A^+$ such that $\mathrm{diam}(f(B)) \le \varepsilon$.

This fact is stated without proof in [18, Lemma 2.7] and implicitly can be found in [11, p. 42, Corollary 3]. Inspired by this equivalence, we introduce the following definition.

Definition 2.1. *The strong measurability index of a function $f : \Omega \to X$ is defined as*

$$\mathrm{meas}(f) := \inf\{\varepsilon > 0 : \textit{ for every } A \in \Sigma^+ \textit{ there is } B \in \Sigma_A^+ \textit{ such that } \mathrm{diam}(f(B)) \le \varepsilon\}.$$

Note that, for any function $f \in X^\Omega$, we have $\mathrm{meas}(f) = 0$ if and only if f satisfies property ($\star$) above, which is equivalent to saying that $f \in \mathcal{M}(\mu, X)$. On the other hand, by using a standard exhaustion process the index $\mathrm{meas}(f)$ can be rewritten as follows.

Remark 2.2. *Let $f : \Omega \to X$ be a function. Then*

$$\mathrm{meas}(f) = \inf\{\varepsilon > 0 : \text{ there is a countable collection } \mathcal{B} \text{ of pairwise disjoint elements of } \Sigma^+ \text{ such that } \mu\big(\Omega \setminus \bigcup \mathcal{B}\big) = 0 \text{ and } \mathrm{diam}(f(B)) \leq \varepsilon \text{ for all } B \in \mathcal{B}\}.$$

The next result provides a quantitative version of the equivalence between ($\star$) and strong measurability.

Theorem 2.3. *Let $f : \Omega \to X$ be a function. Then*

$$\frac{1}{2}\,\mathrm{meas}(f) \leq d(f, \mathcal{M}(\mu, X)) \leq \mathrm{meas}(f). \tag{2.1}$$

Moreover, if $X = \mathbb{R}$ then

$$d(f, \mathcal{M}(\mu)) = \frac{1}{2}\,\mathrm{meas}(f). \tag{2.2}$$

Proof. We begin by proving the first inequality in (2.1). Suppose $d(f, \mathcal{M}(\mu, X)) < \infty$. Fix $d(f, \mathcal{M}(\mu, X)) < \varepsilon$ and $\eta > 0$. Take $g \in \mathcal{M}(\mu, X)$ such that

$$\|f(t) - g(t)\| < \varepsilon \quad \text{for all } t \in \Omega. \tag{2.3}$$

Take any $A \in \Sigma^+$. Since g is strongly measurable, there is a sequence of simple functions $g_n : \Omega \to X$ such that $\lim_n g_n = g$ μ-a.e. By Egorov's theorem (see e.g. [12, p. 94, Theorem 1]), there exist $D \in \Sigma$ with $\mu(\Omega \setminus D) < \mu(A)$ and $n \in \mathbb{N}$ such that

$$\|g_n(t) - g(t)\| \leq \eta \quad \text{for all } t \in D. \tag{2.4}$$

Clearly, $A \cap D \in \Sigma^+$. Since g_n is a simple function, there is $B \in \Sigma^+_{A\cap D}$ such that the restriction $g_n|_B$ is constant. For every $t_1, t_2 \in B$ we have

$$\begin{aligned}\|f(t_1) - f(t_2)\| \leq \|f(t_1) - g(t_1)\| + \|g(t_1) - g_n(t_1)\| + \|g_n(t_1) - g_n(t_2)\| \\ + \|g_n(t_2) - g(t_2)\| + \|g(t_2) - f(t_2)\| < 2\varepsilon + 2\eta\end{aligned}$$

thanks to (2.3) and (2.4). Hence $\mathrm{diam}(f(B)) \leq 2\varepsilon + 2\eta$. This shows that

$$\mathrm{meas}(f) \leq 2\varepsilon + 2\eta.$$

Since $\eta > 0$ and $\varepsilon > d(f, \mathcal{M}(\mu, X))$ are arbitrary, $\mathrm{meas}(f) \leq 2d(f, \mathcal{M}(\mu, X))$.

We now prove the second inequality in (2.1). Assume that $\mathrm{meas}(f) < \infty$. We shall check that for every $\varepsilon > \mathrm{meas}(f)$ there is some $g \in \mathcal{M}(\mu, X)$ with $d(f, g) \leq \varepsilon$. Indeed, by Remark 2.2 we can find a countable collection $\mathcal{B}$ of pairwise disjoint elements of Σ^+ such that $\mu(\Omega \setminus \bigcup \mathcal{B}) = 0$ and $\mathrm{diam}(f(B)) \leq \varepsilon$ for all $B \in \mathcal{B}$. We pick $x_B \in f(B)$ for every $B \in \mathcal{B}$. Define $g \in \mathcal{M}(\mu, X)$ as

$$g(t) := \begin{cases} x_B & \text{if } t \in B,\ B \in \mathcal{B}, \\ f(t) & \text{if } t \in \Omega \setminus \bigcup \mathcal{B}. \end{cases} \tag{2.5}$$

Since $\mathrm{diam}(f(B)) \leq \varepsilon$ and $x_B \in f(B)$ for every $B \in \mathcal{B}$, we have $d(f, g) \leq \varepsilon$. Hence $d(f, \mathcal{M}(\mu, X)) \leq \varepsilon$. Since $\varepsilon > \mathrm{meas}(f)$ is arbitrary, $d(f, \mathcal{M}(\mu, X)) \leq \mathrm{meas}(f)$.

In the case $X = \mathbb{R}$ we can prove equality (2.2) by using a similar argument, but now x_B is chosen as the middle point of $\mathrm{conv}(f(B))$ (which is an interval of length less than or equal to ε). Then $|f(t) - x_B| \leq \varepsilon/2$ for every $t \in B$ and every $B \in \mathcal{B}$, so the function $g \in \mathcal{M}(\mu)$ defined as in (2.5) using the new x_B's satisfies $d(f, g) \leq \varepsilon/2$. ☐

The inequality $\frac{1}{2}\operatorname{meas}(f) \leq d(f, \mathcal{M}(\mu, X))$ in (2.1) is sharp because it becomes an equality when $X = \mathbb{R}$. On the other hand, Example 2.5 below shows that the inequality $d(f, \mathcal{M}(\mu, X)) \leq \operatorname{meas}(f)$ in (2.1) is also sharp. The example is based on the general construction isolated in the following lemma. Recall first that a biorthogonal system $\{(x_i, x_i^*)\}_{i\in I} \subset X \times X^*$ is called a *Markushevich basis* (shortly *M-basis*) of X if $X = \overline{\operatorname{span}}\{x_i\}_{i\in I}$ and $\{x_i^*\}_{i\in I}$ separates the points of X.

Lemma 2.4. *Suppose μ is atomless and $\{(x_i, x_i^*)\}_{i\in I} \subset X \times X^*$ is a biorthogonal system for which there is a set $W \subset X$ such that*

$$X = \overline{\operatorname{span}}\big(W \cup \{x_i\}_{i\in I}\big) \qquad \text{and} \qquad x_i^*|_W = 0 \text{ for every } i \in I.$$

If $f : \Omega \to X$ is a one-to-one function taking values in $\{x_i\}_{i\in I}$, then:

(i) $d(f, \mathcal{M}(\mu, X)) \geq 1$ *if* $\|x_i^*\| \leq 1$ *for all* $i \in I$.
(ii) $d(f, \mathcal{M}(\mu, X)) = 1$ *if* $\|x_i\| = \|x_i^*\| = 1$ *for all* $i \in I$.

Proof. (i) Let $g \in \mathcal{M}(\mu, X)$. Then there is $A \in \Sigma$ with $\mu(\Omega \setminus A) = 0$ such that $g(A)$ is separable. Since $X = \overline{\operatorname{span}}\big(W \cup \{x_i\}_{i\in I}\big)$, we can find a countable set $I_0 \subset I$ such that $g(A) \subset Y := \overline{\operatorname{span}}\big(W \cup \{x_i\}_{i\in I_0}\big)$. Since f is one-to-one, the set $D := f^{-1}(\{x_i\}_{i\in I_0})$ is countable and the fact that μ is atomless guarantees that $D \in \Sigma$ and $\mu(D) = 0$. In particular, we can choose $t \in A \setminus D$. Write $f(t) = x_i$ for some $i \in I \setminus I_0$ and observe that

$$\|f(t) - g(t)\| \geq |x_i^*(f(t) - g(t))| = |x_i^*(x_i) - x_i^*(g(t))| = 1,$$

because x_i^* vanishes on $Y \ni g(t)$. Hence $d(f, g) \geq 1$. Since $g \in \mathcal{M}(\mu, X)$ is arbitrary, we get $d(f, \mathcal{M}(\mu, X)) \geq 1$. Part (ii) now follows at once: if $\|x_i\| = 1$ for all $i \in I$ then $f(\Omega) \subset B_X$ and so $d(f, 0) \leq 1$. □

Example 2.5. *Let $\{(e_t, e_t^*)\}_{t\in[0,1]}$ be the usual M-basis of $X := c_0([0, 1])$ and define*

$$f : [0, 1] \to X, \quad f(t) := e_t.$$

Then $d(f, \mathcal{M}(\lambda, X)) = \operatorname{meas}(f) = 1$.

Proof. Since $\|f(t) - f(s)\| = 1$ whenever $t \neq s$, we have $\operatorname{diam}(f(B)) = 1$ for every measurable set $B \subset [0, 1]$ with $\lambda(B) > 0$ and so $\operatorname{meas}(f) = 1$. The equality $d(f, \mathcal{M}(\lambda, X)) = 1$ follows from Lemma 2.4(ii). □

Remark 2.6. *Recall that X is said to have uniform normal structure if there is a constant $0 < K < 1$ such that the inequality*

$$\inf_{x\in C} \sup_{y\in C} \|x - y\| \leq K \operatorname{diam}(C)$$

holds for every bounded convex closed set $C \subset X$. For instance, every uniformly convex Banach space has this property (see e.g. [19, Chapter 5]*). When X has uniform normal structure, the proof of Theorem 2.3 can be modified easily to obtain the inequality*

$$d(f, \mathcal{M}(\mu, X)) \leq K \operatorname{meas}(f) \tag{2.6}$$

for every function $f : \Omega \to X$. Of course, for $(\mathbb{R}, |\cdot|)$ we have that $K = \frac{1}{2}$ and therefore equality (2.2) *is again explained by* (2.6).

3. Integrability index and distances to integrable functions

In this section we discuss a Bochner integrability index which is inspired by the construction of some vector integrals via Riemann-type sums, like the Birkhoff integral [7, 10] and the so-called Riemann-Lebesgue integral studied in [25, 26].

Given a function $f : \Omega \to X$ and $A \in \Sigma$, we denote by $\Pi_A(f)$ the set made up of all partitions Γ of A into countably many elements of Σ such that:

(i) $f(B)$ is bounded for every $B \in \Gamma \cap \Sigma^+$;
(ii) the series $\sum_{B\in\Gamma} \mu(B)\|f(t_B)\|$ is convergent for every choice $t_B \in B$, $B \in \Gamma$.

Definition 3.1. *Let $f : \Omega \to X$ be a function. For every $A \in \Sigma$ we define*

$$\mathrm{a}(f, A) := \inf \Big\{ \sum_{B\in\Gamma} \mu(B) \operatorname{diam}(f(B)) : \ \Gamma \in \Pi_A(f) \Big\}.$$

The integrability index of f is defined as

$$\operatorname{int}(f) := \sup_{A\in\Sigma^+} \frac{\mathrm{a}(f, A)}{\mu(A)}.$$

It turns out that a function $f : \Omega \to X$ is Bochner integrable (i.e. $f \in \mathcal{L}^1(\mu, X)$) if and only if $\operatorname{int}(f) = 0$. In fact, our Theorem 3.6 below provides a quantitative and more general version of such equivalence. We first need several auxiliary results. The next proposition gives us the relationship between $\operatorname{meas}(f)$ and $\operatorname{int}(f)$.

Proposition 3.2. *Let $f : \Omega \to X$ be a function. Then*

$$\operatorname{meas}(f) \le \operatorname{int}(f).$$

Proof. Assume that $\operatorname{int}(f) < \infty$. Fix $\alpha > \operatorname{int}(f)$ and $A \in \Sigma^+$. Since

$$\mathrm{a}(f, A) \le \mu(A) \operatorname{int}(f) < \mu(A)\alpha,$$

we can choose $\Gamma \in \Pi_A(f)$ such that

$$\sum_{C\in\Gamma} \mu(C) \operatorname{diam}(f(C)) < \mu(A)\alpha.$$

Define

$$\Gamma' := \{C \in \Gamma : \ \operatorname{diam}(f(C)) \ge \alpha\} \quad \text{and} \quad D := \bigcup_{C\in\Gamma'} C.$$

Note that $D \subset A$, $D \in \Sigma$ and

$$\mu(D)\alpha = \sum_{C\in\Gamma'} \mu(C)\alpha \le \sum_{C\in\Gamma'} \mu(C) \operatorname{diam}(f(C)) \le \sum_{C\in\Gamma} \mu(C) \operatorname{diam}(f(C)) < \mu(A)\alpha.$$

Therefore $\mu(A \setminus D) > 0$, so there is $C \in \Gamma \setminus \Gamma'$ such that $\mu(C) > 0$. Then $C \in \Sigma_A^+$ and $\operatorname{diam}(f(C)) < \alpha$. Since $A \in \Sigma^+$ and $\alpha > \operatorname{int}(f)$ are arbitrary, we conclude that $\operatorname{meas}(f) \le \operatorname{int}(f)$. □

Lemma 3.3. *If $f \in \mathcal{L}^1(\mu, X)$ then* $\operatorname{int}(f) = 0$.

Proof. Fix $A \in \Sigma^+$ and $\varepsilon > 0$. We shall check that $\mathrm{a}(f, A) \le \varepsilon$. Since f is strongly measurable, we can find a partition $\mathcal{B}$ of Ω into countably many elements of Σ such that

$$\operatorname{diam}(f(B)) \le \frac{\varepsilon}{\mu(A)} \quad \text{for every } B \in \mathcal{B} \cap \Sigma^+$$

(see Remark 2.2 and the comments preceeding it). We claim that

$$\Gamma := \{B \cap A : \ B \in \mathcal{B}\} \in \Pi_A(f).$$

Indeed, note that $\operatorname{diam}(f(B\cap A)) \le \operatorname{diam}(f(B)) \le \frac{\varepsilon}{\mu(A)}$ (and so $f(B\cap A)$ is bounded) for every $B \in \mathcal{B}$ with $B\cap A \in \Sigma^+$. On the other hand, pick $t_B \in B \cap A$ for every $B \in \mathcal{B}$ with $B \cap A \neq \emptyset$. Define $g \in \mathcal{M}(\mu, X)$ by the formula

$$g(t) := \begin{cases} f(t_B) & \text{if } t \in B\cap A,\ B \in \mathcal{B}, \\ f(t) & \text{if } t \in \Omega \setminus A. \end{cases}$$

Since $\|f(t) - g(t)\| \le \frac{\varepsilon}{\mu(A)}$ for μ-a.e. $t \in \Omega$, we have

$$\sum_{\substack{B\in\mathcal{B}\\ B\cap A\neq\emptyset}} \mu(B\cap A)\|f(t_B)\| = \int_A \|g(t)\|\, d\mu(t) \le \varepsilon + \int_A \|f(t)\|\, d\mu(t) < \infty.$$

This shows that $\Gamma \in \Pi_A(f)$, as claimed. Moreover, since $\operatorname{diam}(f(B\cap A)) \le \frac{\varepsilon}{\mu(A)}$ for every $B \in \mathcal{B}$ with $B \cap A \in \Sigma^+$, we also have

$$\sum_{B\in\mathcal{B}} \mu(B\cap A)\operatorname{diam}(f(B\cap A)) \le \varepsilon.$$

Therefore $\mathrm{a}(f, A) \le \varepsilon$. Since $A \in \Sigma^+$ and $\varepsilon > 0$ are arbitrary, we get $\operatorname{int}(f) = 0$. □

Lemma 3.4. *Let $f, g : \Omega \to X$ be two functions. Then*

$$\operatorname{int}(f) \le \operatorname{int}(g) + 2d(f, g).$$

Proof. Assume that $d(f, g) < \infty$. It suffices to check that

$$\mathrm{a}(f, A) \le \mathrm{a}(g, A) + 2d(f, g)\mu(A) \quad \text{for every } A \in \Sigma^+. \tag{3.1}$$

To this end, fix $A \in \Sigma^+$ with $\mathrm{a}(g, A) < \infty$. Given any $\varepsilon > 0$, we can find $\Gamma \in \Pi_A(g)$ such that

$$\sum_{B\in\Gamma} \mu(B)\operatorname{diam}(g(B)) < \mathrm{a}(g, A) + \varepsilon.$$

Clearly, the fact that $d(f, g) < \infty$ ensures that $\Gamma \in \Pi_A(f)$ as well. Moreover, since $\operatorname{diam}(f(B)) \le \operatorname{diam}(g(B)) + 2d(f, g)$ for every $B \in \Gamma$, it follows that

$$\begin{aligned}\mathrm{a}(f, A) &\le \sum_{B\in\Gamma} \mu(B)\operatorname{diam}(f(B)) \le \sum_{B\in\Gamma} \mu(B)\big(\operatorname{diam}(g(B)) + 2d(f, g)\big) = \\ &= \sum_{B\in\Gamma} \mu(B)\operatorname{diam}(g(B)) + 2d(f, g)\mu(A) < \mathrm{a}(g, A) + \varepsilon + 2d(f, g)\mu(A).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the inequality in (3.1) holds true and the proof is over. □

The next lemma is similar to [7, Theorem 1] (cf. [10, Lemma 2.1]), but here we deal with absolute convergence instead of unconditional convergence of series in Banach spaces. We include a proof for the convenience of the reader.

Lemma 3.5. *Let $f : \Omega \to X$ be a function and let Γ, Γ' be two partitions of Ω into countably many elements of Σ. If Γ' iffiner than Γ and $\Gamma \in \Pi_\Omega(f)$, then $\Gamma' \in \Pi_\Omega(f)$.*

Proof. Clearly, $f(C)$ is bounded whenever $C \in \Gamma' \cap \Sigma^+$ (because any such C is contained in some $B \in \Gamma \cap \Sigma^+$). Write

$$\alpha_B := \sup\big\{\|f(t)\| : t \in B\big\}$$

for every $B \in \Gamma \cap \Sigma^+$. It is easy to check that

$$\sum_{B\in\Gamma\cap\Sigma^+} \mu(B)\alpha_B < \infty.$$

Pick $t_C \in C$ for every (non-empty) $C \in \Gamma'$. Then

$$\sum_{C\in\Gamma'} \mu(C)\|f(t_C)\| = \sum_{B\in\Gamma\cap\Sigma^+} \sum_{\substack{C\in\Gamma' \\ C\subset B}} \mu(C)\|f(t_C)\| \leq$$
$$\leq \sum_{B\in\Gamma\cap\Sigma^+} \Big(\sum_{\substack{C\in\Gamma' \\ C\subset B}} \mu(C)\Big)\alpha_B = \sum_{B\in\Gamma\cap\Sigma^+} \mu(B)\alpha_B < \infty.$$

It follows that $\Gamma' \in \Pi_\Omega(f)$. □

We have already gathered all the tools needed to prove the main result of this section:

Theorem 3.6. *Let $f : \Omega \to X$ be a function. Then*

$$\frac{1}{2}\,\mathrm{int}(f) \leq d(f, \mathcal{L}^1(\mu, X)) \leq \mathrm{int}(f). \tag{3.2}$$

Moreover, if $X = \mathbb{R}$ then

$$d(f, \mathcal{L}^1(\mu)) = \frac{1}{2}\,\mathrm{int}(f). \tag{3.3}$$

Proof. We first prove that $\mathrm{int}(f) \leq 2d(f, \mathcal{L}^1(\mu, X))$. Assume that $d(f, \mathcal{L}^1(\mu, X)) < \infty$ and fix $\varepsilon > d(f, \mathcal{L}^1(\mu, X))$. Take $g \in \mathcal{L}^1(\mu, X)$ such that $d(f, g) < \varepsilon$. By Lemmas 3.3 and 3.4, we have $\mathrm{int}(f) \leq 2\varepsilon$. Since $\varepsilon > d(f, \mathcal{L}^1(\mu, X))$ is arbitrary, we conclude that $\mathrm{int}(f) \leq 2d(f, \mathcal{L}^1(\mu, X))$.

We now prove that $d(f, \mathcal{L}^1(\mu, X)) \leq \mathrm{int}(f)$. Assume that $\mathrm{int}(f) < \infty$, fix $\alpha > \mathrm{int}(f)$ and fix a partition $\Gamma \in \Pi_\Omega(f)$ (the collection $\Pi_\Omega(f)$ is non-empty because $a(f, \Omega) < \infty$ and $\inf \emptyset = \infty$). Since $\mathrm{meas}(f) \leq \mathrm{int}(f) < \alpha$ (by Proposition 3.2), there is a countable collection $\mathcal{B}$ of pairwise disjoint elements of Σ^+ such that

$$\mu(\Omega \setminus \bigcup \mathcal{B}) = 0 \quad \text{and} \quad \mathrm{diam}(f(B)) \leq \alpha \text{ for all } B \in \mathcal{B}$$

(see Remark 2.2). Let Γ' be any partition of Ω into countably many elements of Σ which is finer than both Γ and $\mathcal{B} \cup \{\Omega \setminus \bigcup \mathcal{B}\}$. Let Γ'' be the collection of all non-empty $D \in \Gamma'$ which are contained in $\bigcup \mathcal{B}$. Select $x_D \in f(D)$ for every $D \in \Gamma''$. Let g be the function which coincides with f on $\Omega \setminus \bigcup \mathcal{B}$ and satisfies $g(t) = x_D$ for every $t \in D$ and every $D \in \Gamma''$. Clearly, we have $\|g(t) - f(t)\| \leq \alpha$ for all $t \in \Omega$ and so $d(f, g) \leq \alpha$.

We now check that $g \in \mathcal{L}^1(\mu, X)$. We have $g \in \mathcal{M}(\mu, X)$ by construction. Since $\Gamma'' \subset \Gamma' \in \Pi_\Omega(f)$ (by Lemma 3.5), we have

$$\int_\Omega \|g(t)\|\, d\mu(t) = \sum_{D\in\Gamma''} \mu(D)\|x_D\| < \infty,$$

hence $g \in \mathcal{L}^1(\mu, X)$. Therefore, $d(f, \mathcal{L}^1(\mu, X)) \leq \alpha$. Since $\alpha > \mathrm{int}(f)$ is arbitrary, it follows that $d(f, \mathcal{L}^1(\mu, X)) \leq \mathrm{int}(f)$. This finishes the proof of (3.2).

Finally, we can prove (3.3) when $X = \mathbb{R}$ by using a similar argument, but now x_D is chosen as the middle point of $\mathrm{conv}(f(D))$ (which is an interval of length less than or equal to α). Since $|f(t) - x_D| \leq \alpha/2$ for every $t \in D$ and every $D \in \Gamma''$, the function g defined as above using the new x_D's satisfies $d(f, g) \leq \alpha/2$. □

Remark 3.7. *The proof of Theorem 3.6 actually shows that*

$$d(f, \mathcal{L}^1(\mu, X)) \leq \mathrm{meas}(f)$$

for any function $f : \Omega \to X$ such that $\Pi_\Omega(f) \neq \emptyset$.

Both inequalities in (3.2) are sharp. Indeed, for the second one, take $X := c_0([0,1])$ and note that the function $f : [0,1] \to X$ given by $f(t) := e_t$ (see Example 2.5) satisfies $\mathrm{a}(f, A) = 1$ for every measurable set $A \subset [0,1]$ with $\lambda(A) > 0$, hence $\mathrm{int}(f) = 1$. On the other hand, since

$$d(f, \mathcal{L}^1(\lambda, X)) \geq d(f, \mathcal{M}(\lambda, X)) = 1,$$

from Theorem 3.6 we conclude that $d(f, \mathcal{L}^1(\lambda, X)) = \mathrm{int}(f) = 1$.

Remark 3.8. *If X has uniform normal structure (see Remark 2.6 for the definition), then there is a constant $0 < K < 1$ such that*

$$d(f, \mathcal{L}^1(\mu, X)) \leq K \,\mathrm{int}(f)$$

for every function $f : \Omega \to X$.

4. A QUANTITATIVE VERSION OF PETTIS' MEASURABILITY THEOREM

The celebrated Pettis' measurability theorem (see e.g. [11, p. 42, Theorem 2]) states that a function $f : \Omega \to X$ is strongly measurable if and only if it is *scalarly measurable* (i.e. the composition $x^* \circ f$ is measurable for every $x^* \in X^*$) and *essentially separably valued* (i.e. there is $A \in \Sigma$ with $\mu(A) = 0$ such that $f(\Omega \setminus A)$ is separable). In fact, for the "if" part it suffices to assume that $x^* \circ f$ is measurable for every x^* belonging to some *norming* set $\Upsilon \subset B_{X^*}$ (i.e. $\|x\| = \sup_{x^* \in \Upsilon} |x^*(x)|$ for every $x \in X$).

In order to provide a quantitative version of Pettis' result (see Theorem 4.5 below) we introduce the following definition.

Definition 4.1. *Let $f : \Omega \to X$ be a function. Given $\varepsilon > 0$, we say that f is ε-essentially separably valued if there exist $A \in \Sigma$ with $\mu(A) = 0$ and a countable set $C \subset f(\Omega \setminus A)$ such that*

$$f(\Omega \setminus A) \subset C + \varepsilon B_X.$$

We define

$$\mathrm{sep}(f) = \inf\{\varepsilon > 0 : \ f \textit{ is } \varepsilon\textit{-essentially separably valued}\}.$$

It is easy to check that a function $f : \Omega \to X$ is essentially separably valued if and only if $\mathrm{sep}(f) = 0$. On the other hand, $\mathrm{sep}(f)$ and $d(f, \mathcal{M}(\mu, X))$ are related as follows:

Lemma 4.2. *Let $f : \Omega \to X$ be a function. Then*

$$\frac{1}{2}\,\mathrm{sep}(f) \leq d(f, \mathcal{M}(\mu, X)).$$

Proof. Assume that $d(f, \mathcal{M}(\mu, X)) < \infty$, fix $\varepsilon > d(f, \mathcal{M}(\mu, X))$ and $\eta > 0$. Take $g \in \mathcal{M}(\mu, X)$ such that $d(f, g) \leq \varepsilon$. There is $A \in \Sigma$ with $\mu(A) = 0$ such that $g(\Omega \setminus A)$ is separable. Fix a countable set $D \subset \Omega \setminus A$ such that $g(D)$ is dense in $g(\Omega \setminus A)$. Then

$$f(\Omega \setminus A) \subset g(\Omega \setminus A) + \varepsilon B_X \subset g(D) + (\varepsilon + \eta) B_X \subset f(D) + (2\varepsilon + \eta) B_X.$$

Since $\varepsilon > d(f, \mathcal{M}(\mu, X))$ and $\eta > 0$ are arbitrary, we get $\mathrm{sep}(f) \leq 2d(f, \mathcal{M}(\mu, X))$. □

The next example (for $p = 1$) shows that the constant $\frac{1}{2}$ appearing in Lemma 4.2 is sharp.

Example 4.3. *Let $1 \leq p \leq \infty$ and take $X_p := \ell_p([0,1])$ if $p < \infty$ and $X_p := c_0([0,1])$ if $p = \infty$. Define $f : [0,1] \to X_p$ by $f(t) := e_t$, where $\{(e_t, e_t^*)\}_{t \in [0,1]}$ is the usual M-basis of X_p. Then $d(f, \mathcal{M}(\lambda, X_p)) = 1$ and $\mathrm{sep}(f) = 2^{1/p}$.*

Proof. The equality $d(f, \mathcal{M}(\lambda, X_p)) = 1$ follows from Lemma 2.4 (ii). On the other hand, the fact that

$$\|f(t) - f(s)\| = \|e_t - e_s\| = 2^{1/p} \quad \text{whenever } t \neq s$$

clearly implies that $\operatorname{sep}(f) \leq 2^{1/p}$. Conversely, observe that for every measurable set $B \subset [0, 1]$ with $\lambda(B) = 1$ and every countable set $D \subset B$, we can pick $t \in B \setminus D$ and so $\|f(t) - f(s)\| = 2^{1/p}$ for every $s \in D$; hence $\operatorname{sep}(f) \geq 2^{1/p}$. □

The following easy lemma will be used in the proof of Theorem 4.5.

Lemma 4.4. *Let $\mathcal{F} \subset \mathbb{R}^\Omega$ be a countable pointwise bounded family and define*

$$h(t) := \sup_{f \in \mathcal{F}} |f(t)| < \infty \quad \text{for every } t \in \Omega.$$

Then

$$d(h, \mathcal{M}(\mu)) \leq \sup_{f \in \mathcal{F}} d(f, \mathcal{M}(\mu)).$$

Proof. Assume that $\alpha := \sup_{f \in \mathcal{F}} d(f, \mathcal{M}(\mu)) < \infty$ and fix $\beta > \alpha$. Then for every $f \in \mathcal{F}$ there is $g_f \in \mathcal{M}(\mu)$ such that $d(f, g_f) \leq \beta$. Since $\mathcal{F}$ is pointwise bounded, the same holds for $\{g_f : f \in \mathcal{F}\}$. Define $g(t) := \sup_{f \in \mathcal{F}} |g_f(t)|$ for every $t \in \Omega$, so that $g \in \mathcal{M}(\mu)$. Given any $t \in \Omega$, we have

$$|g_f(t)| - \beta \leq |f(t)| \leq |g_f(t)| + \beta \quad \text{for every } f \in \mathcal{F},$$

hence $g(t) - \beta \leq h(t) \leq g(t) - \beta$. Thus $d(h, g) \leq \beta$ and so $d(h, \mathcal{M}(\mu)) \leq \beta$. Since $\beta > \alpha$ is arbitrary, we get $d(h, \mathcal{M}(\mu)) \leq \alpha$. □

We are now ready to prove a quantitative version of Pettis' measurability theorem.

Theorem 4.5. *Let $f : \Omega \to X$ be a function and $\Upsilon \subset B_{X^*}$ a norming set. Then*

$$d(f, \mathcal{M}(\mu, X)) \leq 2 \sup_{x^* \in \Upsilon} d(x^* \circ f, \mathcal{M}(\mu)) + 3 \operatorname{sep}(f). \tag{4.1}$$

Proof. We can assume without loss of generality that Υ is symmetric, that is, $-x^* \in \Upsilon$ whenever $x^* \in \Upsilon$. Assume that $\alpha := \sup_{x^* \in \Upsilon} d(x^* \circ f, \mathcal{M}(\mu))$ and $\operatorname{sep}(f)$ are finite. Fix

$$u > \alpha, \quad w > \operatorname{sep}(f) \quad \text{and} \quad \varepsilon > 0.$$

Choose $A \in \Sigma$ with $\mu(A) = 0$ and a countable set $D \subset \Omega \setminus A$ such that

$$f(\Omega \setminus A) \subset f(D) + w B_X. \tag{4.2}$$

Since Υ is norming and symmetric, for every pair $(d_1, d_2) \in D \times D$ we can select $x^*_{(d_1,d_2)} \in \Upsilon$ such that

$$x^*_{(d_1,d_2)}(f(d_1) - f(d_2)) > \|f(d_1) - f(d_2)\| - \varepsilon. \tag{4.3}$$

For every $d \in D$ we define a function $g_d : \Omega \to \mathbb{R}$ by

$$g_d(t) := \sup_{(d_1,d_2) \in D \times D} \left| h_{d,d_1,d_2}(t) \right|$$

where $h_{d,d_1,d_2}(t) := x^*_{(d_1,d_2)}(f(t) - f(d))$. We have $d(g_d, \mathcal{M}(\mu)) < u$, thanks to Lemma 4.4 and the fact that

$$d\big(h_{d,d_1,d_2}, \mathcal{M}(\mu)\big) = d\big(x^*_{(d_1,d_2)} \circ f, \mathcal{M}(\mu)\big) \leq \alpha < u \quad \text{for every } (d_1, d_2) \in D \times D.$$

For every $d \in D$ we fix $\tilde{g}_d \in \mathcal{M}(\mu)$ such that $d(g_d, \tilde{g}_d) < u$ and we define

$$A_d := \{t \in \Omega : \tilde{g}_d(t) < u + w\} \in \Sigma.$$

For every $t \in \Omega \setminus A$ we have $\|f(t) - f(d)\| \leq w$ for some $d \in D$ (by (4.2)), hence $g_d(t) \leq w$ and so $t \in A_d$ (bear in mind that $d(g_d, \tilde{g}_d) < u$). This shows that

$$\Omega \setminus A \subset \bigcup_{d \in D} A_d.$$

Since D is countable, we can find a partition $\Omega \setminus A = \bigcup_{d \in D} B_d$ such that $B_d \in \Sigma$ and $B_d \subset A_d$ for every $d \in D$. Define $g \in \mathcal{M}(\mu, X)$ by the formula

$$g(t) := \begin{cases} f(d) & \text{if } t \in B_d,\ d \in D, \\ f(t) & \text{otherwise.} \end{cases}$$

CLAIM. $\|f(t) - g(t)\| < 2u + 3w + \varepsilon$ for every $t \in \Omega$.

Indeed, the claim is obvious if $t \in A$. Suppose $t \in \Omega \setminus A$ and take $d \in D$ such that $t \in B_d \subset A_d$. Since $\tilde{g}_d(t) < u + w$ and $d(g_d, \tilde{g}_d) < u$, we have $g_d(t) < 2u + w$ and therefore

$$\left|x^*_{(d_1,d_2)}(f(t) - f(d))\right| < 2u + w \quad \text{for all } (d_1, d_2) \in D \times D. \tag{4.4}$$

By (4.2) there is some $d' \in D$ such that $\|f(t) - f(d')\| \leq w$. Then

$$\begin{aligned} \|g(t) - f(t)\| &\overset{t \in B_d}{=} \|f(d) - f(t)\| \leq \|f(d) - f(d')\| + \|f(d') - f(t)\| \leq \\ &\leq \|f(d) - f(d')\| + w \overset{(4.3)}{<} x^*_{(d,d')}(f(d) - f(d')) + \varepsilon + w = \\ &= x^*_{(d,d')}(f(d) - f(t)) + x^*_{(d,d')}(f(t) - f(d')) + \varepsilon + w < \\ &\overset{(4.4)}{<} 2u + 2w + x^*_{(d,d')}(f(t) - f(d')) + \varepsilon \leq 2u + 2w + \|f(t) - f(d')\| + \varepsilon \leq 2u + 3w + \varepsilon. \end{aligned}$$

The CLAIM is proved.

It follows that $d(f, \mathcal{M}(\mu, X)) \leq 2u + 3w + \varepsilon$. Since $u > \alpha$, $w > \operatorname{sep}(f)$ and $\varepsilon > 0$ are arbitrary, inequality (4.1) holds and the proof is finished. $\square$

Corollary 4.6. *For any function $f : \Omega \to X$ we have*

$$d(f, \mathcal{M}(\mu, X)) \leq 2 \sup_{x^* \in B_{X^*}} d(x^* \circ f, \mathcal{M}(\mu)) + 3 \operatorname{sep}(f).$$

The next proposition shows that the constant 2 in inequality (4.1) of Theorem 4.5 is sharp, even if the set $\Upsilon \subset B_{X^*}$ is assumed to be a *boundary* (i.e. for every $x \in X$ there is $x^* \in \Upsilon$ such that $\|x\| = x^*(x)$). The construction can be done in any Banach space containing a *complemented* subspace isomorphic to c_0. In particular, it works for Banach spaces containing a subspace isomorphic to c_0 which, in addition, are separable or have the Separable Complementation Property (thanks to Sobczyk's theorem, see e.g. [1, Corollary 2.5.9]), like all (infinite-dimensional) weakly Lindelöf determined $C(K)$ spaces.

Proposition 4.7. *Suppose X contains a complemented subspace isomorphic to c_0. Then there exist (up to renorming) a boundary $\Upsilon \subset B_{X^*}$ and a function $f : [0,1] \to X$ having separable range such that*

$$d(f, \mathcal{M}(\lambda, X)) = 2 \sup_{x^* \in \Upsilon} d(x^* \circ f, \mathcal{M}(\lambda)). \tag{4.5}$$

Proof. We divide the proof into several steps.

STEP 1. Write $X = Y \oplus Z$ where Y is isomorphic to c_0. Let $P : X \to Y$ be the corresponding projection onto Y. We can find an equivalent norm $\|\cdot\|$ on X such that

$$\|x\| = \|P(x)\| + \|x - P(x)\| \quad \text{for all } x \in X$$

and $(Y, \|\cdot\|)$ is isometric to $(c_0, \|\cdot\|_\infty)$. Fix an isometry $T : Y \to c_0$. For every $n \in \mathbb{N}$ and every $\varphi \in B_{Z^*}$, let $\varphi_n \in B_{X^*}$ be the functional defined by

$$\langle \varphi_n, x\rangle := \langle e_n^* \circ T, P(x)\rangle + \langle \varphi, x - P(x)\rangle \quad \text{for all } x \in X,$$

where $e_n^* \in B_{c_0^*}$ stands for the n-th coordinate projection. Clearly, the set

$$\Upsilon := \{\varphi_n : n \in \mathbb{N}, \varphi \in B_{Z^*}\} \subset B_{X^*}$$

is a boundary. Fix a countable partition $[0,1] = \bigcup_{n\in\mathbb{N}} A_n$ in such a way that $\lambda^*(A_n) = 1$ for every $n \in \mathbb{N}$ (see e.g. [17, 419I]). Define $g : [0,1] \to c_0$ by the formula

$$g(t) := \big(1_{A_n}(t)\big)_{n\in\mathbb{N}}$$

and set $f := T^{-1} \circ g : [0,1] \to Y \subset X$. Then $f([0,1])$ is separable (because Y is separable). We shall check that f satisfies (4.5) for the boundary Υ defined above.

STEP 2. Fix $n \in \mathbb{N}$ and $\varphi \in B_{Z^*}$. We claim that $f_{n,\varphi} := \varphi_n \circ f$ satisfies

$$d(f_{n,\varphi}, \mathcal{M}(\lambda)) \leq \frac{1}{2}. \tag{4.6}$$

Indeed, observe that for every $t \in [0,1]$ we have

$$\begin{aligned} f_{n,\varphi}(t) = \langle \varphi_n, f(t)\rangle = \big\langle e_n^* \circ T, P(f(t))\big\rangle + \big\langle \varphi, f(t) - P(f(t))\big\rangle &\overset{f(t)\in Y}{=} \\ = \langle e_n^* \circ T, f(t)\rangle = \langle e_n^*, g(t)\rangle &= 1_{A_n}(t), \end{aligned} \tag{4.7}$$

hence $|f_{n,\varphi}(t) - \frac{1}{2}| = |1_{A_n}(t) - \frac{1}{2}| = \frac{1}{2}$. Therefore, $d(f_{n,\varphi}, \mathcal{M}(\lambda)) \leq \frac{1}{2}$.

STEP 3. We shall check that $d(f, \mathcal{M}(\lambda, X)) \geq 1$. Fix any $h \in \mathcal{M}(\lambda, X)$ and set $h_0 \in \mathcal{M}(\lambda, c_0)$ by $h_0 := T \circ P \circ h$. Fix $\varepsilon > 0$. Take a measurable set $A \subset [0,1]$ with $\lambda(A) > 0$ such that $\operatorname{diam}(h_0(A)) \leq \varepsilon$ and pick $t_1 \in A \cap A_1$ (bear in mind that $\lambda^*(A_1) = 1$). Since $h_0(t_1) \in c_0$, we can choose $n \in \mathbb{N}$ large enough such that

$$\big|e_n^*(h_0(t_1))\big| \leq \varepsilon. \tag{4.8}$$

Pick $t_n \in A \cap A_n$ (bear in mind that $\lambda^*(A_n) = 1$) and choose $\varphi \in B_{Z^*}$ arbitrary. Since

$$e_n^* \circ h_0 = e_n^* \circ T \circ P \circ h = \varphi_n \circ P \circ h,$$

we have

$$\begin{aligned} d(f,h) \geq \|f(t_n) - h(t_n)\| &\overset{f(t_n)\in Y}{\geq} \|f(t_n) - P(h(t_n))\| \geq \\ \geq \big|\big\langle \varphi_n, f(t_n) - P(h(t_n))\big\rangle\big| &= \big|f_{n,\varphi}(t_n) - e_n^*(h_0(t_n))\big| \overset{(4.7)}{=} \big|1 - e_n^*(h_0(t_n))\big|. \end{aligned} \tag{4.9}$$

On the other hand,

$$\begin{aligned} 1 - \varepsilon &\overset{(4.8)}{\leq} 1 - \big|e_n^*(h_0(t_1))\big| \leq \\ &\leq \big|1 - e_n^*(h_0(t_n))\big| + \big|e_n^*(h_0(t_n)) - e_n^*(h_0(t_1))\big| \leq \big|1 - e_n^*(h_0(t_n))\big| + \varepsilon, \end{aligned} \tag{4.10}$$

where the last inequality follows from the fact that $\operatorname{diam}(h_0(A)) \leq \varepsilon$. By putting together (4.9) and (4.10), we get $d(f,h) \geq 1 - 2\varepsilon$. Since $h \in \mathcal{M}(\lambda, X)$ and $\varepsilon > 0$ are arbitrary, we conclude that $d(f, \mathcal{M}(\lambda, X)) \geq 1$.

STEP 4. According to Step 3 and Theorem 4.5, we have

$$1 \leq d(f, \mathcal{M}(\lambda, X)) \leq 2 \sup_{x^* \in \Upsilon} d(x^* \circ f, \mathcal{M}(\lambda)).$$

Now, from (4.6) it follows that

$$1 = d(f, \mathcal{M}(\lambda, X)) = 2 \sup_{x^* \in \Upsilon} d(x^* \circ f, \mathcal{M}(\lambda))$$

and the proof is over. □

Remark 4.8. *When $X = c_0$ the construction of Proposition 4.7 can be carried out for the supremum norm $\|\cdot\|_\infty$ and the boundary $\Upsilon = \{e_n^* : n \in \mathbb{N}\}$.*

In view of Theorem 4.5, any *scalarly measurable* function $f : \Omega \to X$ satisfies the inequality $d(f, \mathcal{M}(\mu, X)) \leq 3\,\mathrm{sep}(f)$ and so, bearing in mind Lemma 4.2, we have

$$\frac{1}{2}\,\mathrm{sep}(f) \leq d(f, \mathcal{M}(\mu, X)) \leq 3\,\mathrm{sep}(f), \tag{4.11}$$

and therefore for scalarly measurable functions $\mathrm{meas}(\cdot)$ and $\mathrm{sep}(\cdot)$ are equivalent indexes.

Example 4.3 above makes clear that, when we restrict ourselves to scalarly measurable functions, the constant $\frac{1}{2}$ in (4.11) cannot be replaced by any constant $C > \frac{1}{2}$. However, we do not know an example of a scalarly measurable function f which is not strongly measurable and satisfies the equality $\frac{1}{2}\,\mathrm{sep}(f) = d(f, \mathcal{M}(\mu, X))$ (note that the function given in Example 4.3 is not scalarly measurable when $p = 1$). On the other hand, we also do not know whether the constant 3 in (4.11) is sharp. The rest of the section is devoted to study improvements of that inequality in some particular cases.

Our first result in this direction is valid when the norm function $\|\cdot\| : X \to \mathbb{R}$ is measurable for the Baire σ-algebra of the weak topology, denoted by $\mathrm{Ba}(X, w)$. This is the σ-algebra on X generated by X^*, see [13, Theorem 2.3] (cf. [30, 2-2-4]). Clearly, $\|\cdot\|$ is $\mathrm{Ba}(X, w)$-measurable whenever B_{X^*} is w^*-separable (for instance, this holds for $X = \ell_\infty$ equipped with the supremum norm $\|\cdot\|_\infty$ as well as for any of its subspaces). The connections between the $\mathrm{Ba}(X, w)$-measurability of the norm and the w^*-separability of B_{X^*} and X^* have been studied amongst others in [4, 5, 29].

Proposition 4.9. *Suppose $\|\cdot\|$ is $\mathrm{Ba}(X, w)$-measurable. If $f : \Omega \to X$ is scalarly measurable, then*

$$d(f, \mathcal{M}(\mu, X)) \leq \mathrm{sep}(f). \tag{4.12}$$

Proof. Assume that $\mathrm{sep}(f) < \infty$ and fix $\varepsilon > \mathrm{sep}(f)$. Take $A \in \Sigma$ with $\mu(\Omega \setminus A) = 0$ and a countable set $D \subset A$ such that

$$f(A) \subset f(D) + \varepsilon B_X. \tag{4.13}$$

For each $d \in D$, let $B_d \subset X$ be the closed ball of radius ε centered at $f(d)$. Since $\|\cdot\|$ is $\mathrm{Ba}(X, w)$-measurable and f is Σ-$\mathrm{Ba}(X, w)$-measurable, we have $f^{-1}(B_d) \in \Sigma$ for every $d \in D$. Moreover, (4.13) implies that $A \subset \bigcup_{d \in D} f^{-1}(B_d)$. Since D is countable, we can find a partition $A = \bigcup_{d \in D} A_d$ such that $A_d \subset f^{-1}(B_d)$ and $A_d \in \Sigma$ for every $d \in D$. Define $g \in \mathcal{M}(\mu, X)$ by declaring $g(t) := f(d)$ if $t \in A_d$ and $d \in D$ and $g(t) := f(t)$ otherwise. Clearly, we have $d(f, g) \leq \varepsilon$ and so $d(f, \mathcal{M}(\mu, X)) \leq \varepsilon$. Since $\varepsilon > \mathrm{sep}(f)$ is arbitrary, inequality (4.12) holds. □

The following example shows that the inequality (4.12) in Proposition 4.9 is sharp.

Example 4.10. *There exist a subspace X of $(\ell_\infty, \|\cdot\|_\infty)$ and a scalarly measurable function $f : [0,1] \to X$ such that*

$$d(f, \mathcal{M}(\lambda, X)) = \mathrm{sep}(f) = 1.$$

Proof. Set $T := \bigcup_{n=1}^{\infty}\{0,1\}^n$ and $K := \{0,1\}^{\mathbb{N}}$. For every $u = (u_i)_{i=1}^{\infty} \in K$ we define $B_u := \{u|n : n \in \mathbb{N}\} \subset T$, where $u|n := (u_i)_{i=1}^{n} \in \{0,1\}^n$ for all $n \in \mathbb{N}$. Note that T is countable, $\{B_u : u \in K\}$ has cardinality $\mathfrak{c}$ and is made up of infinite subsets of T such that $B_u \cap B_v$ is finite whenever $u \neq v$ (i.e. $\{B_u : u \in K\}$ is almost disjoint).

Let X be the *Johnson-Lindenstrauss* subspace of $(\ell_\infty(T), \|\cdot\|_\infty)$ defined by the family $\{B_u : u \in K\}$. Namely, X is the closed linear span of $X_0 := c_0(T) \cup \{1_{B_u} : u \in K\}$, see [23, Example 2]. Any $x \in X_0$ can be written in a unique way as $x = y + \sum_{v\in K} a_v 1_{B_v}$, where $y \in c_0(T)$, $a_v \in \mathbb{R}$ and $a_v = 0$ for all but finitely many v's. For each $u \in K$, let φ_u be the linear functional on X_0 which vanishes on $c_0(T)$ and satisfies $\varphi_u(1_{B_v}) = \delta_{u,v}$ (the Kronecker symbol) for all $v \in K$. Since $|\varphi_u(x)| \leq \|x\|_\infty$ for all $x \in X_0$, there is $\tilde{\varphi}_u \in X^*$ extending φ_u, with $\|\tilde{\varphi}_u\| = 1$. Clearly, $\{(1_{B_u}, \tilde{\varphi}_u) : u \in K\} \subset X \times X^*$ is a biorthogonal system.

Define $f : K \to X$ by $f(u) := 1_{B_u}$ (Hagler's function), so that f is scalarly measurable when K is equipped with its standard Radon probability ν (see [11, p. 43, Example 7] or [30, 3-2-4]). Note that (K, ν) is measure space isomorphic to $([0,1], \lambda)$, see e.g. [16, 254K]. In view of Lemma 2.4(ii), we have $d(f, \mathcal{M}(\nu, X)) = 1$. On the other hand, we have $\operatorname{sep}(f) \leq 1$ because

$$\|f(u) - f(v)\|_\infty = \|1_{B_u} - 1_{B_v}\|_\infty \leq 1 \quad \text{whenever } u, v \in K.$$

Bearing in mind Proposition 4.9, we conclude that $1 = d(f, \mathcal{M}(\nu, X)) = \operatorname{sep}(f)$. □

Recall that X is *measure-compact* (in its weak topology) if and only if for every probability measure ν on $\mathrm{Ba}(X, w)$ there is a separable subspace $Y \subset X$ such that $\nu^*(Y) = 1$. For instance, every weakly Lindelöf Banach space is measure-compact. Such a property was considered in connection with Pettis integration, see e.g. [14, 30].

In Corollary 4.13 below we shall prove that the inequality $d(f, \mathcal{M}(\mu, X)) \leq 2\operatorname{sep}(f)$ holds for any scalarly measurable function $f : \Omega \to X$ whenever X is measure-compact. To this end we need the following lemma. Recall that two functions $f, g : \Omega \to X$ are said to be *scalarly equivalent* if for every $x^* \in X^*$ we have $x^* \circ f = x^* \circ g$ μ-a.e. (the exceptional set depending on x^*).

Lemma 4.11. *If $f, g : \Omega \to X$ are scalarly equivalent functions, then*

$$\|f(t) - g(t)\| \leq 2\operatorname{sep}(f) + \operatorname{meas}(g) \quad \textit{for } \mu\textit{-a.e. } t \in \Omega.$$

Proof. Assume that $\operatorname{sep}(f)$ and $\operatorname{meas}(g)$ are finite. Fix $\varepsilon > \operatorname{sep}(f)$ and $\eta > \operatorname{meas}(g)$ arbitrary and define

$$A_{\varepsilon,\eta} := \{t \in \Omega : \|f(t) - g(t)\| > 2\varepsilon + \eta\}.$$

We shall prove that $\mu^*(A_{\varepsilon,\eta}) = 0$ by contradiction. Suppose that $\mu^*(A_{\varepsilon,\eta}) > 0$. Since $\varepsilon > \operatorname{sep}(f)$, we can find $B \in \Sigma$ with $\mu(\Omega \setminus B) = 0$ and a countable set $D \subset B$ such that

$$f(B) \subset f(D) + \varepsilon B_X.$$

For each $t \in D$, set

$$B_t := \{s \in B : \|f(s) - f(t)\| \leq \varepsilon\}, \tag{4.14}$$

so that $B = \bigcup_{t\in D} B_t$. Since D is countable, there is $t_0 \in D$ such that $\mu^*(A_{\varepsilon,\eta} \cap B_{t_0}) > 0$. Fix $W \supset A_{\varepsilon,\eta} \cap B_{t_0}$ with $W \in \Sigma$ and $\mu(W) = \mu^*(A_{\varepsilon,\eta} \cap B_{t_0}) > 0$. The fact that $\eta > \operatorname{meas}(g)$ ensures the existence of $C \in \Sigma_W^+$ for which

$$\operatorname{diam}(g(C)) \leq \eta. \tag{4.15}$$

We have $C \cap A_{\varepsilon,\eta} \cap B_{t_0} \neq \emptyset$ (because $C \in \Sigma_W^+$ and $\mu(W) = \mu^*(A_{\varepsilon,\eta} \cap B_{t_0})$). Pick $s_1 \in C \cap A_{\varepsilon,\eta} \cap B_{t_0}$. In particular, we have $\|f(s_1) - g(s_1)\| > 2\varepsilon + \eta$ and so there is $x^* \in B_{X^*}$ such that

$$\big|x^*(f(s_1)) - x^*(g(s_1))\big| > 2\varepsilon + \eta. \tag{4.16}$$

Set

$$U := \big\{s \in \Omega : \, x^*(f(s)) = x^*(g(s))\big\}. \tag{4.17}$$

Observe that $U \in \Sigma$ and $\mu(\Omega \setminus U) = 0$ (bear in mind that f and g are scalarly equivalent). Since $U \cap C \in \Sigma_W^+$ and $\mu(W) = \mu^*(A_{\varepsilon,\eta} \cap B_{t_0})$, we have $U \cap C \cap A_{\varepsilon,\eta} \cap B_{t_0} \neq \emptyset$, so we can pick $s_2 \in U \cap C \cap A_{\varepsilon,\eta} \cap B_{t_0}$. Therefore

$$\begin{aligned}
2\varepsilon + \eta &\overset{(4.16)}{<} \big|x^*(f(s_1)) - x^*(g(s_1))\big| \leq \\
&\leq \big|x^*(f(s_1) - f(s_2))\big| + \big|x^*(f(s_2)) - x^*(g(s_2))\big| + \big|x^*(g(s_2) - g(s_1))\big| \leq \\
&\leq \|f(s_1) - f(s_2)\| + \big|x^*(f(s_2)) - x^*(g(s_2))\big| + \|g(s_2) - g(s_1)\| \leq \\
&\overset{(4.14)}{\leq} 2\varepsilon + \big|x^*(f(s_2)) - x^*(g(s_2))\big| + \|g(s_2) - g(s_1)\| = \\
&\overset{(4.17)}{=} 2\varepsilon + \|g(s_2) - g(s_1)\| \overset{(4.15)}{\leq} 2\varepsilon + \eta,
\end{aligned}$$

a contradiction.

This shows that $\mu^*(A_{\varepsilon,\eta}) = 0$ for every $\varepsilon > \mathrm{sep}(f)$ and every $\eta > \mathrm{meas}(g)$. Now let (ε_n) and (η_n) be sequences of real numbers with $\varepsilon_n \searrow \mathrm{sep}(f)$ and $\eta_n \searrow \mathrm{meas}(g)$. Since

$$A := \{t \in \Omega : \|f(t) - g(t)\| > 2\,\mathrm{sep}(f) + \mathrm{meas}(g)\} = \bigcup_{n \in \mathbb{N}} A_{\varepsilon_n,\eta_n},$$

we conclude that $\mu^*(A) = 0$ and the proof is over. □

Corollary 4.12. *If $f : \Omega \to X$ is scalarly equivalent to a strongly measurable function, then*

$$d(f, \mathcal{M}(\mu, X)) \leq 2\,\mathrm{sep}(f).$$

Proof. Let $g : \Omega \to X$ be any strongly measurable function such that f and g are scalarly equivalent. By Lemma 4.11, there is $V \in \Sigma$ with $\mu(\Omega \setminus V) = 0$ such that

$$\|f(t) - g(t)\| \leq 2\,\mathrm{sep}(f) \quad \text{for every } t \in V.$$

Define $\tilde{g} \in \mathcal{M}(\mu, X)$ by $\tilde{g}(t) := g(t)$ if $t \in V$ and $\tilde{g}(t) := f(t)$ if $t \in \Omega \setminus V$. Clearly, we have $d(f, \tilde{g}) \leq 2\,\mathrm{sep}(f)$, hence $d(f, \mathcal{M}(\mu, X)) \leq 2\,\mathrm{sep}(f)$. □

A result of Edgar (see [13, Proposition 5.4], cf. [30, 3-4-6]) states that every scalarly measurable function taking values in a measure-compact Banach space is scalarly equivalent to a strongly measurable function. As an immediate consequence we have:

Corollary 4.13. *Suppose X is measure-compact. If $f : \Omega \to X$ is scalarly measurable, then*

$$d(f, \mathcal{M}(\mu, X)) \leq 2\,\mathrm{sep}(f).$$

We do not know whether the inequality above is sharp.

REFERENCES

[1] F. Albiac and N.J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006. MR 2192298 (2006h:46005)

[2] C. Angosto and B. Cascales, *Measures of weak noncompactness in Banach spaces*, Topology Appl. **156** (2009), no. 7, 1412–1421. MR 2502017 (2010c:46055)

[3] C. Angosto, B. Cascales, and I. Namioka, *Distances to spaces of Baire one functions*, Math. Z. **263** (2009), no. 1, 103–124. MR 2529490 (2010h:54055)

[4] A. Avilés, G. Plebanek, and J. Rodríguez, *Measurability in $C(2^\kappa)$ and Kunen cardinals*, to appear in Israel J. Math.

[5] A. Avilés, G. Plebanek, and J. Rodríguez, *A weak* separable $C(K)^*$ space whose unit ball is not weak* separable*, arXiv:1112.5710, 2011.

[6] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000. MR 1727673 (2001b:46001)

[7] G. Birkhoff, *Integration of functions with values in a Banach space*, Trans. Amer. Math. Soc. **38** (1935), no. 2, 357–378. MR 1 501 815

[8] B. Cascales, O.F.K. Kalenda, and J. Spurný, *A quantitative version of James's compactness theorem*, to appear in Proc. Edinburgh Math. Soc.

[9] B. Cascales, W. Marciszewski, and M. Raja, *Distance to spaces of continuous functions*, Topology Appl. **153** (2006), no. 13, 2303–2319. MR 2238732 (2007k:46014)

[10] B. Cascales and J. Rodríguez, *The Birkhoff integral and the property of Bourgain*, Math. Ann. **331** (2005), no. 2, 259–279. MR 2115456 (2006i:28006)

[11] J. Diestel and J.J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15. MR 0453964 (56 #12216)

[12] N. Dinculeanu, *Vector measures*, International Series of Monographs in Pure and Applied Mathematics, Vol. 95, Pergamon Press, Oxford, 1967. MR 34 #6011b

[13] G.A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), no. 4, 663–677. MR 0487448 (58 #7081)

[14] G.A. Edgar, *Measurability in a Banach space. II*, Indiana Univ. Math. J. **28** (1979), no. 4, 559–579. MR 542944 (81d:28016)

[15] M. Fabian, P. Hájek, V. Montesinos, and V. Zizler, *A quantitative version of Krein's theorem*, Rev. Mat. Iberoamericana **21** (2005), no. 1, 237–248. MR 2155020 (2006b:46011)

[16] D.H. Fremlin, *Measure theory. Vol. 2*, Torres Fremlin, Colchester, 2003, Broad foundations, Corrected second printing of the 2001 original. MR 2462280

[17] D.H. Fremlin, *Measure theory. Vol. 4*, Torres Fremlin, Colchester, 2006, Topological measure spaces. Part I, II, Corrected second printing of the 2003 original. MR 2462372

[18] R.F. Geitz, *Geometry and the Pettis integral*, Trans. Amer. Math. Soc. **269** (1982), no. 2, 535–548. MR 637707 (83d:28004)

[19] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990. MR 1074005 (92c:47070)

[20] A.S. Granero, *An extension of the Krein-Šmulian theorem*, Rev. Mat. Iberoam. **22** (2006), no. 1, 93–110. MR 2267314 (2008a:46019)

[21] A.S. Granero, P. Hájek, and V. Montesinos, *Convexity and w^*-compactness in Banach spaces*, Math. Ann. **328** (2004), no. 4, 625–631. MR 2047643 (2005c:46020)

[22] A.S. Granero and M. Sánchez, *The class of universally Krein-Šmulian Banach spaces*, Bull. Lond. Math. Soc. **39** (2007), no. 4, 529–540. MR 2346932 (2008k:46055)

[23] W.B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel J. Math. **17** (1974), 219–230. MR 0417760 (54 #5808)

[24] M. Kačena O.F.K. Kalenda and J. Spurný, *Quantitative Dunford-Pettis property*, arXiv:1110.1243v1, 2011.

[25] V.M. Kadets, B. Shumyatskiy, R. Shvidkoy, L. Tseytlin, and K. Zheltukhin, *Some remarks on vector-valued integration*, Mat. Fiz. Anal. Geom. **9** (2002), no. 1, 48–65. MR 1911073 (2003g:28026)

[26] V.M. Kadets and L. Tseytlin, *On "integration" of non-integrable vector-valued functions*, Mat. Fiz. Anal. Geom. **7** (2000), no. 1, 49–65. MR 1760946 (2001e:28017)

[27] O.F.K. Kalenda, H. Pfitzner, and J. Spurný, *On quantification of weak sequential completeness*, J. Funct. Anal. **260** (2011), no. 10, 2986–2996. MR 2774062 (2012b:46026)

[28] A. Kryczka, S. Prus, and M. Szczepanik, *Measure of weak noncompactness and real interpolation of operators*, Bull. Austral. Math. Soc. **62** (2000), no. 3, 389–401. MR 1799942 (2001i:46116)

[29] J. Rodríguez, *Weak Baire measurability of the balls in a Banach space*, Studia Math. **185** (2008), no. 2, 169–176. MR 2379966 (2009c:28029)

[30] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. **51** (1984), no. 307, ix+224. MR 86j:46042

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