The Lindelöf property and σ-fragmentability

B. Cascales and I. Namioka

Universidad de Murcia/University of Washington.

10th September 2004

1. The Proto-Idea

The Proto-Idea

- Let (X,τ) be a Tychonoff (completely regular and T_1) space, and let $C(X,I)$ be the space of all continuous functions $f : X \to I = [0,1]$. Then the map $\Phi : X \to I^{C(X,I)}$, given by $\Phi(x)(f) = f(x)$ for $x \in X, f \in C(X,I)$, embeds X topologically in $(I^{C(X,I)},\tau_p)$ (see *e.g.* [Kel75]).

The Proto-Idea

- Let (X,τ) be a Tychonoff (completely regular and T_1) space, and let $C(X,I)$ be the space of all continuous functions $f : X \to I = [0,1]$. Then the map $\Phi : X \to I^{C(X,I)}$, given by $\Phi(x)(f) = f(x)$ for $x \in X, f \in C(X,I)$, embeds X topologically in $(I^{C(X,I)},\tau_p)$ (see *e.g.* [Kel75]).
- Let (M,ρ) be a metric space with the metric ρ bounded, and let D be an index set. We consider various topologies, pseudometrics, metrics, etc. on the product space M^D and study their relationship between them in subspaces $X \subset M^D$, namely,
    - the product (= pointwise) topology τ_p
    - the topology $\gamma(D)$ of uniforme convergence on the family of all countable subsets $\mathscr{C}$ of D.
    - the metric d of uniform convergence on D.

Theorem

Let X be a K-analytic subspace of M^D where (M,ρ) is a metric space with ρ bounded. Then the following statements are equivalent.

(a) *The space (X,τ_p) is σ-fragmented by d.*

(b) *For each compact set $K \subset X$, (K,τ_p) is fragmented by d.*

(c) *For each $A \in \mathscr{C}$, the pseudo-metric space (X,d_A) is separable.*

(d) *$(X,\gamma(D))$ is Lindelöf.*

(e) *$(X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.*

Theorem

Let X be a K-analytic subspace of M^D where (M,ρ) is a metric space with ρ bounded. Then the following statements are equivalent.

- *(a) The space (X,τ_p) is σ-fragmented by d.*
- *(b) For each compact set $K \subset X$, (K,τ_p) is fragmented by d.*
- *(c) For each $A \in \mathscr{C}$, the pseudo-metric space (X,d_A) is separable.*
- *(d) $(X,\gamma(D))$ is Lindelöf.*
- *(e) $(X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.*

Known and easy parts: (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c) $\Leftarrow$ (d) $\Leftarrow$ (e) [JNR93] (a simpler proof [NP96]) and [CNO03].
(a) $\Rightarrow$ (c) needs the following simple lemma.

Theorem

Let X be a K-analytic subspace of M^D where (M,ρ) is a metric space with ρ bounded. Then the following statements are equivalent.

- *(a) The space (X,τ_p) is σ-fragmented by d.*
- *(b) For each compact set $K \subset X$, (K,τ_p) is fragmented by d.*
- *(c) For each $A \in \mathscr{C}$, the pseudo-metric space (X,d_A) is separable.*
- *(d) $(X,\gamma(D))$ is Lindelöf.*
- *(e) $(X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.*

Known and easy parts: (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c) $\Leftarrow$ (d) $\Leftarrow$ (e) [JNR93] (a simpler proof [NP96]) and [CNO03].
(a) $\Rightarrow$ (c) needs the following simple lemma.

Lemma

Let (T,τ) be metrizable and separable and let δ be a metric on T. Then (T,τ) is σ-fragmented by δ if and only if (T,δ) is separable.

Difficult part: (c) $\Rightarrow$ (d) (by contradiction)

Useful facts about Baire sets:

Difficult part: (c) $\Rightarrow$ (d) (by contradiction)

Useful facts about Baire sets:

- A subset Z of T is called a *zero-set* (in T) if $Z = f^{-1}(0)$ for some continuous function $f : T \to \mathbb{R}$.

Difficult part: (c) $\Rightarrow$ (d) (by contradiction)

Useful facts about Baire sets:

- A subset Z of T is called a *zero-set* (in T) if $Z = f^{-1}(0)$ for some continuous function $f : T \to \mathbb{R}$.
- Let $\mathscr{Z}$ (or $\mathscr{Z}(T)$) denote the family of all zero-sets in T.Then $\mathscr{Z}$ is closed under finite unions and countable intersections.

Difficult part: (c) $\Rightarrow$ (d) (by contradiction)

Useful facts about Baire sets:

- A subset Z of T is called a *zero-set* (in T) if $Z = f^{-1}(0)$ for some continuous function $f : T \to \mathbb{R}$.
- Let $\mathscr{Z}$ (or $\mathscr{Z}(T)$) denote the family of all zero-sets in T.Then $\mathscr{Z}$ is closed under finite unions and countable intersections.
- The σ-algebra generated by $\mathscr{Z}$ is denoted by Baire(T) (*Baire sets* in T).

Difficult part: (c) $\Rightarrow$ (d) (by contradiction)

Useful facts about Baire sets:

- A subset Z of T is called a *zero-set* (in T) if $Z = f^{-1}(0)$ for some continuous function $f : T \to \mathbb{R}$.
- Let $\mathscr{Z}$ (or $\mathscr{Z}(T)$) denote the family of all zero-sets in T.Then $\mathscr{Z}$ is closed under finite unions and countable intersections.
- The σ-algebra generated by $\mathscr{Z}$ is denoted by $\mathrm{Baire}(T)$ (*Baire sets* in T).
- $\mathrm{Baire}(T) \subset \mathrm{Souslin}(\mathscr{Z})$.

Difficult part: (c) $\Rightarrow$ (d) (by contradiction)

Useful facts about Baire sets:

- A subset Z of T is called a *zero-set* (in T) if $Z = f^{-1}(0)$ for some continuous function $f : T \to \mathbb{R}$.
- Let $\mathscr{Z}$ (or $\mathscr{Z}(T)$) denote the family of all zero-sets in T.Then $\mathscr{Z}$ is closed under finite unions and countable intersections.
- The σ-algebra generated by $\mathscr{Z}$ is denoted by Baire(T) (*Baire sets* in T).
- $\mathrm{Baire}(T) \subset \mathrm{Souslin}(\mathscr{Z})$.
- If X is K-analytic subset of M^D, then each zero-set in X, being closed, is K-analytic and therefore each member of $\mathrm{Souslin}(\mathscr{Z})$ is K-analytic. Since $\mathrm{Baire}(T) \subset \mathrm{Souslin}(\mathscr{Z})$, each Baire set in X is K-analytic hence Lindelöf relative to τ_p.

Preparatory things

Notation: $x \in X$, $S \subset D$ and $\varepsilon > 0$.

- $U(x,S,\varepsilon) := \{y \in X : d_S(y,x) < \varepsilon\}$.

Preparatory things

Notation: $x \in X$, $S \subset D$ and $\varepsilon > 0$.

- $U(x,S,\varepsilon) := \{y \in X : d_S(y,x) < \varepsilon\}$.
- $\mathscr{U} = \{U_j : j \in J\}$ is a family of $\gamma(D)$-open sets in X that covers X without a countable subcover. We may assume that each U_j is of the form

$$U_j = U(x_j,A_j,\varepsilon_j) = \{y \in X : d_{A_j}(y,x_j) < \varepsilon_j\},$$

where $x_j \in X$, $A_j \in \mathscr{C}$, $\varepsilon_j > 0$ for each $j \in J$.

Preparatory things

Notation: $x \in X$, $S \subset D$ and $\varepsilon > 0$.

- $U(x,S,\varepsilon) := \{y \in X : d_S(y,x) < \varepsilon\}$.
- $\mathscr{U} = \{U_j : j \in J\}$ is a family of $\gamma(D)$-open sets in X that covers X without a countable subcover. We may assume that each U_j is of the form

$$U_j = U(x_j,A_j,\varepsilon_j) = \{y \in X : d_{A_j}(y,x_j) < \varepsilon_j\},$$

where $x_j \in X$, $A_j \in \mathscr{C}$, $\varepsilon_j > 0$ for each $j \in J$.

- For each $A \in \mathscr{C}$, let $U(A) = \bigcup\{U_j : j \in J, A_j \subset A\}$.

Preparatory things

Notation: $x \in X$, $S \subset D$ and $\varepsilon > 0$.

- $U(x,S,\varepsilon) := \{y \in X : d_S(y,x) < \varepsilon\}$.
- $\mathscr{U} = \{U_j : j \in J\}$ is a family of $\gamma(D)$-open sets in X that covers X without a countable subcover. We may assume that each U_j is of the form

$$U_j = U(x_j,A_j,\varepsilon_j) = \{y \in X : d_{A_j}(y,x_j) < \varepsilon_j\},$$

  where $x_j \in X$, $A_j \in \mathscr{C}$, $\varepsilon_j > 0$ for each $j \in J$.
- For each $A \in \mathscr{C}$, let $U(A) = \bigcup\{U_j : j \in J, A_j \subset A\}$.

Remark

If $A \subset A'$, then $U(A) \subset U(A')$ and $X = \bigcup\{U(A) : A \in \mathscr{C}\}$.

Assume (c) holds and (d) doesn't

Lemma (a tool!!!)

(i) $U(x,A,\varepsilon) \in \mathrm{Baire}(X)$ whenever $x \in X, A \in \mathscr{C}, \varepsilon > 0$.

(ii) $U(A) \in \mathrm{Baire}(X)$: $U(A)$ is K-analytic and Lindelöf, $A \in \mathscr{C}$.

(iii) $S \subset X$ is covered by a count. subfamily of $\mathscr{U}$ iff $S \subset U(A)$.

Assume (c) holds and (d) doesn't

Lemma (a tool!!!)

(i) $U(x,A,\varepsilon) \in \mathrm{Baire}(X)$ whenever $x \in X, A \in \mathscr{C}, \varepsilon > 0$.

(ii) $U(A) \in \mathrm{Baire}(X)$: $U(A)$ is K-analytic and Lindelöf, $A \in \mathscr{C}$.

(iii) $S \subset X$ is covered by a count. subfamily of $\mathscr{U}$ iff $S \subset U(A)$.

Let $\mathscr{Y}$ be the family of all K-analytic subsets Y of (X,τ_p) such that there is no countable subfamily of $\mathscr{U}$ that covers Y.

Assume (c) holds and (d) doesn't

Lemma (a tool!!!)

(i) $U(x,A,\varepsilon) \in \mathrm{Baire}(X)$ whenever $x \in X, A \in \mathscr{C}, \varepsilon > 0$.

(ii) $U(A) \in \mathrm{Baire}(X)$: $U(A)$ is K-analytic and Lindelöf, $A \in \mathscr{C}$.

(iii) $S \subset X$ is covered by a count. subfamily of $\mathscr{U}$ iff $S \subset U(A)$.

Let $\mathscr{Y}$ be the family of all K-analytic subsets Y of (X,τ_p) such that there is no countable subfamily of $\mathscr{U}$ that covers Y.

Two possibilities:

Assume (c) holds and (d) doesn't

Lemma (a tool!!!)

(i) $U(x,A,\varepsilon) \in \mathrm{Baire}(X)$ whenever $x \in X, A \in \mathscr{C}, \varepsilon > 0$.

(ii) $U(A) \in \mathrm{Baire}(X)$: $U(A)$ is K-analytic and Lindelöf, $A \in \mathscr{C}$.

(iii) $S \subset X$ is covered by a count. subfamily of $\mathscr{U}$ iff $S \subset U(A)$.

Let $\mathscr{Y}$ be the family of all K-analytic subsets Y of (X,τ_p) such that there is no countable subfamily of $\mathscr{U}$ that covers Y.

Two possibilities:

- For each $Y \in \mathscr{Y}$ and each $\varepsilon > 0$, there is a $Z \in \mathscr{Y}$ such that $Z \subset Y$ and d-diam $(Z) \leq \varepsilon$.

Assume (c) holds and (d) doesn't

Lemma (a tool!!!)

(i) $U(x,A,\varepsilon) \in \mathrm{Baire}(X)$ whenever $x \in X, A \in \mathscr{C}, \varepsilon > 0$.

(ii) $U(A) \in \mathrm{Baire}(X)$: $U(A)$ is K-analytic and Lindelöf, $A \in \mathscr{C}$.

(iii) $S \subset X$ is covered by a count. subfamily of $\mathscr{U}$ iff $S \subset U(A)$.

Let $\mathscr{Y}$ be the family of all K-analytic subsets Y of (X, τ_p) such that there is no countable subfamily of $\mathscr{U}$ that covers Y.

Two possibilities:

- For each $Y \in \mathscr{Y}$ and each $\varepsilon > 0$, there is a $Z \in \mathscr{Y}$ such that $Z \subset Y$ and d-diam $(Z) \leq \varepsilon$.
- The negation of the previous case.

Assume (c) holds and (d) doesn't

Lemma (a tool!!!)

(i) $U(x,A,\varepsilon) \in \mathrm{Baire}(X)$ whenever $x \in X, A \in \mathscr{C}, \varepsilon > 0$.

(ii) $U(A) \in \mathrm{Baire}(X)$: $U(A)$ is K-analytic and Lindelöf, $A \in \mathscr{C}$.

(iii) $S \subset X$ is covered by a count. subfamily of $\mathscr{U}$ iff $S \subset U(A)$.

Let $\mathscr{Y}$ be the family of all K-analytic subsets Y of (X,τ_p) such that there is no countable subfamily of $\mathscr{U}$ that covers Y.

Two possibilities:

- For each $Y \in \mathscr{Y}$ and each $\varepsilon > 0$, there is a $Z \in \mathscr{Y}$ such that $Z \subset Y$ and d-diam $(Z) \leq \varepsilon$.
- The negation of the previous case.

The proof

We show that each case leads to a contradiction.

(d)⇒(e)

Corollary

Let X be a K-analytic subspace of M^D where (M,ρ) is a metric space with ρ bounded. If $(X,\gamma(D))$ is Lindelöf, then $(X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.

(d)⇒(e)

Corollary

Let X be a K-analytic subspace of M^D where (M,ρ) is a metric space with ρ bounded. If $(X,\gamma(D))$ is Lindelöf, then $(X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.

- $\varphi : (M^D)^{\mathbb{N}} \to (M^{\mathbb{N}})^D$ is τ_p and γ-homeomorphism.
$$\varphi(\xi)(p)(j) = \xi(j)(p) \text{ for all } \xi \in (M^D)^{\mathbb{N}}, p \in D, j \in \mathbb{N}$$
- $M^{\mathbb{N}}$ is metrizable $(\rho_\infty(m,m'))$. Consider
$$d_\infty(x,x') = \sup\{\rho_\infty(x(p),x'(p)) : p \in D\} \quad \text{for} \quad x,x' \in (M^{\mathbb{N}})^D.$$

(d)⇒(e)

Corollary

Let X be a K-analytic subspace of M^D where (M,ρ) is a metric space with ρ bounded. If $(X,\gamma(D))$ is Lindelöf, then $(X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.

- $\varphi : (M^D)^{\mathbb{N}} \to (M^{\mathbb{N}})^D$ is τ_p and γ-homeomorphism.

$$\varphi(\xi)(p)(j) = \xi(j)(p) \text{ for all } \xi \in (M^D)^{\mathbb{N}}, p \in D, j \in \mathbb{N}$$

- $M^{\mathbb{N}}$ is metrizable $(\rho_\infty(m,m'))$. Consider

$$d_\infty(x,x') = \sup\{\rho_\infty(x(p),x'(p)) : p \in D\} \quad \text{for} \quad x,x' \in (M^{\mathbb{N}})^D.$$

- $X^{\mathbb{N}}$ is K-analytic, hence so is $\varphi(X^{\mathbb{N}})$ and each compact subset of $\varphi(X^{\mathbb{N}})$ is fragmented by d_∞.

(d)⇒(e)

Corollary

Let X be a K-analytic subspace of M^D where (M,ρ) is a metric space with ρ bounded. If $(X,\gamma(D))$ is Lindelöf, then $(X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.

- $\varphi : (M^D)^{\mathbb{N}} \to (M^{\mathbb{N}})^D$ is τ_p and γ-homeomorphism.
$$\varphi(\xi)(p)(j) = \xi(j)(p) \text{ for all } \xi \in (M^D)^{\mathbb{N}}, p \in D, j \in \mathbb{N}$$
- $M^{\mathbb{N}}$ is metrizable $(\rho_\infty(m,m'))$. Consider
$$d_\infty(x,x') = \sup\{\rho_\infty(x(p),x'(p)) : p \in D\} \quad \text{for} \quad x,x' \in (M^{\mathbb{N}})^D.$$
- $X^{\mathbb{N}}$ is K-analytic, hence so is $\varphi(X^{\mathbb{N}})$ and each compact subset of $\varphi(X^{\mathbb{N}})$ is fragmented by d_∞.
- Hence by (a)⇔(b), $(\varphi(X^{\mathbb{N}}),\gamma(D)) = (X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.

Let $I = [-1,1]$ and let Γ be an arbitrary index set. For $x \in I^\Gamma$ we write $\operatorname{supp}(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. Consider

$$\mathscr{F}(\Gamma) = \{x \in [-1,1]^\Gamma : \operatorname{supp}(x) \text{ is finite}\}$$

and

$$\Sigma(\Gamma) = \{x \in [-1,1]^\Gamma : \operatorname{supp}(x) \text{ is countable}\}.$$

Let $I = [-1,1]$ and let Γ be an arbitrary index set. For $x \in I^\Gamma$ we write $\operatorname{supp}(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. Consider

$$\mathscr{F}(\Gamma) = \{x \in [-1,1]^\Gamma : \operatorname{supp}(x) \text{ is finite}\}$$

and

$$\Sigma(\Gamma) = \{x \in [-1,1]^\Gamma : \operatorname{supp}(x) \text{ is countable}\}.$$

Definition

A compact Hausdorff space K is said to be *Corson compact* if K is homeomorphic to a τ_p-compact subset of $\Sigma(\Gamma)$.

Let $I = [-1,1]$ and let Γ be an arbitrary index set. For $x \in I^\Gamma$ we write $\operatorname{supp}(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. Consider

$$\mathscr{F}(\Gamma) = \{x \in [-1,1]^\Gamma : \operatorname{supp}(x) \text{ is finite}\}$$

and

$$\Sigma(\Gamma) = \{x \in [-1,1]^\Gamma : \operatorname{supp}(x) \text{ is countable}\}.$$

Definition

A compact Hausdorff space K is said to be *Corson compact* if K is homeomorphic to a τ_p-compact subset of $\Sigma(\Gamma)$.

Properties:

- if A is a countable subset of a Corson compact space K, then the closure of A is compact and metrizable.

Let $I = [-1,1]$ and let Γ be an arbitrary index set. For $x \in I^\Gamma$ we write $\mathrm{supp}(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. Consider

$$\mathscr{F}(\Gamma) = \{x \in [-1,1]^\Gamma : \mathrm{supp}(x) \text{ is finite}\}$$

and

$$\Sigma(\Gamma) = \{x \in [-1,1]^\Gamma : \mathrm{supp}(x) \text{ is countable}\}.$$

Definition

A compact Hausdorff space K is said to be *Corson compact* if K is homeomorphic to a τ_p-compact subset of $\Sigma(\Gamma)$.

Properties:

- if A is a countable subset of a Corson compact space K, then the closure of A is compact and metrizable.
- $(\Sigma(\Gamma), \tau_p)$ is countably tight. Hence the Corson compact spaces are countable tight.

Application 1

If K is Corson compact then $(C(K),\gamma(K))^{\mathbb{N}}$ is Lindelöf.

Application 1

If K is Corson compact then $(C(K),\gamma(K))^{\mathbb{N}}$ is Lindelöf.

In general for K Corson the space $(C(K),\tau_p)$ IS NOT K-analytic.

Application 1

If K is Corson compact then $(C(K),\gamma(K))^{\mathbb{N}}$ is Lindelöf.

In general for K Corson the space $(C(K),\tau_p)$ IS NOT K-analytic.

Lemma

Let Γ be an index set and let H be a norm bounded subset of $\ell^\infty(\Gamma)\subset\mathbb{R}^\Gamma$. If

$$\overline{\operatorname{aco}(H)}^{\tau_p}=\overline{\operatorname{aco}(H)}^{\|\ \|}, \qquad (1)$$

then $X:=\overline{\operatorname{span} H}^{\|\ \|}$ is K-analytic with respect to the pointwise topology τ_p of $\mathbb{R}^\Gamma$. In particular, if H is a norm bounded τ_p-compact subset of $\ell^\infty(\Gamma)$ that is norm-fragmented, then $\overline{\operatorname{span} H}^{\|\ \|}$ is K-analytic relative to τ_p.

Theorem

Let (X,τ) be a K-analytic Tychonoff space. TFAE:

- *(a) The space X is σ-scattered.*
- *(b) The space X does not contain a compact perfect subset.*
- *(c) The space (X,τ_δ) is Lindelöf.*
- *(d) The space $(X,\tau_\delta)^{\mathbb{N}}$ is Lindelöf.*

Theorem

Let (X,τ) be a K-analytic Tychonoff space. TFAE:

- *(a) The space X is σ-scattered.*
- *(b) The space X does not contain a compact perfect subset.*
- *(c) The space (X,τ_δ) is Lindelöf.*
- *(d) The space $(X,\tau_\delta)^{\mathbb{N}}$ is Lindelöf.*
- *(i) For any countable set $A \subset C(X)$, $\overline{A}^{\mathbb{R}^X}$ is τ_p-metrizable.*
- *(ii) $(B_1(X),\tau_p)$ is Fréchet-Urysohn.*
- *(iii) $(C(X),\tau_p)$ is Fréchet-Urysohn.*
- *(iv) $(C(X),\tau_p)$ is sequential.*
- *(v) $(C(X),\tau_p)$ is a k-space.*
- *(vi) $(C(X),\tau_p)$ is a k_R-space.*

Theorem

A dual Banach space X^* has the Radon-Nikodym property (RNP) iff X^* is Lindelöf with the topology $\gamma(X)$ of uniform convergence on bounded sequences of X, [Ori92]

Theorem

A dual Banach space X^* has the Radon-Nikodym property (RNP) iff X^* is Lindelöf with the topology $\gamma(X)$ of uniform convergence on bounded sequences of X, [Ori92]

Theorem

If a dual Banach space X^* is weakly Lindelöf then, $(X^*, w)^{\mathbb{N}}$ is Lindelöf, [Ori92].

Theorem

A dual Banach space X^* has the Radon-Nikodym property (RNP) iff X^* is Lindelöf with the topology $\gamma(X)$ of uniform convergence on bounded sequences of X, [Ori92]

Theorem

If a dual Banach space X^* is weakly Lindelöf then, $(X^*,w)^{\mathbb{N}}$ is Lindelöf, [Ori92].

Theorem

If X is a Banach space and H is a weak-compact subset of X^* which is weak-Lindelöf, then $\overline{co(H)}^{w^*} = \overline{co(H)}^{\| \ \|}$ and this closed convex hull is weakly Lindelöf again; furthermore $Y = \overline{span(H)}^{\| \ \|}$ is weakly Lindelöf (in fact B_{Y^*} is Corson compact), [CNO03].*

Theorem

Let $(X,\|\ \|)$ be a Banach space such that, for some norming subset B of B_{X^*}, $(X,\sigma(X,B))$ is K-analytic. TFAE:

(i) *X has property (C) and $(X,\sigma(X,B))$ is σ-fragmented by $\|\ \|$.*

(ii) *(X,w) is Lindelöf.*

(iii) *(B_{X^*},w^*) countably tight, w^*-separable subsets are metrizable.*

Theorem

Let $(X,\| \|)$ be a Banach space such that, for some norming subset B of B_{X^*}, $(X,\sigma(X,B))$ is K-analytic. TFAE:

(i) *X has property (C) and $(X,\sigma(X,B))$ is σ-fragmented by $\| \|$.*

(ii) *(X,w) is Lindelöf.*

(iii) *(B_{X^*},w^*) countably tight, w^*-separable subsets are metrizable.*

Lemma

Let X be a Banach space and $B \subset X^*$ a norming subset. If X has property (C), then $\gamma(B)$ is stronger than the weak topology of X.

How wide is he class of Banach spaces X for which there is $B \subset B_{X^*}$ norming and $(X,\sigma(X,B))$ is K-analytic?

How wide is he class of Banach spaces X for which there is $B \subset B_{X^*}$ norming and $(X, \sigma(X, B))$ is K-analytic?

Includes

- Weakly K-analytic Banach spaces.

How wide is he class of Banach spaces X for which there is $B \subset B_{X^*}$ norming and $(X,\sigma(X,B))$ is K-analytic?

Includes

- Weakly K-analytic Banach spaces.
- Dual Banach spaces.

How wide is he class of Banach spaces X for which there is $B \subset B_{X^*}$ norming and $(X,\sigma(X,B))$ is K-analytic?

Includes

- Weakly K-analytic Banach spaces.
- Dual Banach spaces.
- Representable Banach spaces, [GT82].

How wide is he class of Banach spaces X for which there is $B \subset B_{X^*}$ norming and $(X,\sigma(X,B))$ is K-analytic?

Includes

- Weakly K-analytic Banach spaces.
- Dual Banach spaces.
- Representable Banach spaces, [GT82].
- Banach spaces generated by a RN-compact, [CNO03].

The central results are sharp

The central results are sharp

- Under CH, there is a Čech-analytic Lindelöf Tychonoff space Y that is σ-scattered and such that (Y,τ_δ) is not Lindelöf.

The central results are sharp

- Under CH, there is a Čech-analytic Lindelöf Tychonoff space Y that is σ-scattered and such that (Y,τ_δ) is not Lindelöf.
- There is a countably K-determined uncountable subspace $Y \subset \mathbb{R}$ such that the compact subsets of Y are countable. Y does not contain perfect compact subsets, Y isn't σ-scattered and (Y,τ_δ) is not Lindelöf.

The central results are sharp

- Under CH, there is a Čech-analytic Lindelöf Tychonoff space Y that is σ-scattered and such that (Y,τ_δ) is not Lindelöf.
- There is a countably K-determined uncountable subspace $Y \subset \mathbb{R}$ such that the compact subsets of Y are countable. Y does not contain perfect compact subsets, Y isn't σ-scattered and (Y,τ_δ) is not Lindelöf.
- There is a compact space K such that $(C(K),\gamma(K))^{\mathbb{N}}$ is Lindelöf and K is not Corson.

References

- B. Cascales, I. Namioka, and J. Orihuela, *The Lindelöf property in Banach spaces*, Studia Math. **154** (2003), no. 2, 165–192. MR 1 949 928
- G. Godefroy and M. Talagrand, *Espaces de Banach représentables*, Israel J. Math. **41** (1982), no. 4, 321–330. MR 84g:46019
- J.E. Jayne, I. Namioka, and C. A. Rogers, *Topological properties of Banach spaces*, Proc. London Math. Soc. (3) **66** (1993), no. 3, 651–672. MR 94c:46041
- J. L. Kelley, *General topology*, Springer-Verlag, New York, 1975, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27. MR 51 #6681
- I. Namioka and R. Pol, *σ-fragmentability and analyticity*,