# The Bourgain property and Birkhoff integrability

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# The papers


- B. Cascales and J. Rodríguez, *Birkhoff integral and the property of Bourgain*, Math. Annalen (2005).
- ______, *Birkhoff integral for multi-valued functions*, J. Math. Anal. Appl. **297** (2004), 540-560.

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## Our basic result

We characterize Birkhoff integrability via the property of Bourgain.

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### Boundary problem

If $X$ is a Banach space not containing $\ell^1(\mathbb{R})$ and $B \subset B_{X^*}$ a boundary (i.e. for every $x \in X$ there exists $e^* \in B$ such that $e^*(x) = \| x \|$) then the norm bounded $\sigma(X, B)$- relatively compact subsets of $X$ are relatively weakly compact.

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### Banach spaces without copies of $\ell^1$

For dual Banach spaces WRNP is characterized via Birkhoff integrable Radon-Nikodým derivatives instead of Pettis integrable ones.

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### Birkhoff integrability is rediscovered

Kadets et al. in 2000-2002, [KSS$^+$02, KT00] introduced and studied a notion of integrability that is equivalent to Birkhoff integrability introduced in 1935.

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- Integrals for multifunctions.

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### Aumann and Debreu

Certain integrals for multifuctions used for models in Mathematics for Economy can be computed as limits in the Hausdorff distance of Riemann (Minkowski) sums of sets.

## The property of Bourgain

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for each $x$ in $E$. Since the set $\{\langle f, x\rangle : \|x\| \le 1\}$ contains no copy of the $l_1$-basis in $L_\infty(\Sigma, \mu)$ and the conditional expectation operator $\xi$ is a contraction from $L_\infty(\Sigma, \mu)$ into $L_\infty(\Gamma, \mu)$, we may conclude that $T(B_E)$ contains no copy of the $l_1$-basis in $L_\infty(\Gamma, \mu)$. Consequently $T(B_E)$ is weakly precompact in $L_\infty(\Gamma, \mu)$ and there is a Pettis integrable kernel $g : (\Omega, \Gamma, \mu) \to E^*$ for the operator

$$T^* : L_1(\Gamma, \mu) \to E^*.$$

Then $\langle g, x\rangle = Tx = \xi(\langle f, x\rangle | \Gamma)$ a.e. for every $x$ in $E$. Therefore

$$\int_B \langle g, x\rangle \, d\mu = \int_B \xi(\langle f, x\rangle | \Gamma) \, d\mu = \int_B \langle f, x\rangle \, d\mu$$

for every set $B$ in $\Gamma$ and hence $\int_B g \, d\mu = \int_B f \, d\mu$ for every set $B$ in $\Gamma$. This shows that $g$ is a Pettis conditional expectation of $f$ for the $\sigma$-algebra $\Gamma$.

In view of Theorems 5 and 9, one can ask the following.

*Question.* If, in Theorem 9, we suppose that the set

$$\{\langle f, x\rangle : \|x\| \le 1\}$$

is almost weakly precompact in $L_\infty(\mu)$, does $f$ have a Pettis conditional expectation with respect to all sub-$\sigma$-algebras of $\Sigma$?

If the above were true, then any function satisfying the conditions of Theorem 5 would have a Pettis conditional expectation with respect to all Radon measurers on all sub-$\sigma$-algebras of the Borel $\sigma$-algebra of $K$.

### IV. The Bourgain property

So far we have seen that the family $\{\langle f, x\rangle : \|x\| \le 1\}$ plays a strong role in determining Pettis integrability for a bounded scalarly measurable function $f$ from $\Omega$ into a dual space $E^*$. We continue this approach in this part, but, rather than viewing such families as subsets of $L_\infty(\mu)$, we now consider them simply as families of real-valued functions on $\Omega$. A property of real-valued functions formulated by J. Bourgain [2] is the cornerstore of our discussion.

DEFINITION 10. Let $(\Omega, \Sigma, \mu)$ be a measure space. A family $\Psi$ of real-valued functions on $\Omega$ is said to have the *Bourgain property* if the following condition is satisfied: For each set $A$ of positive measure and for each $\alpha > 0$, there is a finite collection $F$ of subsets of positive measure of $A$ such that for each function $f$ in $\Psi$, the inequality $\sup f(B) - \inf f(B) < \alpha$ holds for some member $B$ of $F$.

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## Definition

We say that a family $\mathscr{F} \subset \mathbb{R}^\Omega$ has **Bourgain property** if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \dots, B_n \subset A$, $B_i \in \Sigma$, with $\mu(B_i) > 0$ such that for every $f \in \mathscr{F}$

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## Remarkable facts

- If $\mathscr{F} = \{f\}$, TFAE:
  - (i) (Bourgain property) For every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there is $B \in \Sigma$, $B \subset A$ with $\mu(B) > 0$ and $|\cdot|$-diam $f(B) < \varepsilon$.
  - (ii) $f$ is measurable.

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- $\mathscr{F}$ has Bourgain property $\Rightarrow \overline{\mathscr{F}}$ has too.
- $\mathscr{F}$ has Bourgain property and $f \in \overline{\mathscr{F}}$, then there is a sequence $(f_n)$ in $\mathscr{F}$ that converges to $f$, $\mu$-almost everywhere.
- $\mathscr{F} \subset \mathbb{R}^\Omega$ has Bourgain property $\Rightarrow \mathscr{F}$ is stable, [Tal84, 9-5-4].

# Fréchet interpretation of Lebesgue integral

Given $f : \Omega \longrightarrow \mathbb{R}$, for each partition $\Gamma$ of $\Omega$ into countably many sets $(A_n)$ of $\Sigma$ consider the relative *upper* and *lower* sums:

$$J^*(f,\Gamma) = \sum_n \sup_{A_n} f \ \mu(A_n) \quad \text{and} \quad J_*(f,\Gamma) = \sum_n \inf_{A_n} f \ \mu(A_n),$$

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- The intersection of the "relative integral ranges" $J_*(f,\Gamma) \leq x \leq J^*(f,\Gamma)$, for variable $\Gamma$ is not empty.
- This intersection is a single point $x$ if, and only if, $f$ is Lebesgue integrable and $x = \int_\Omega f \ d\mu$.

# Fréchet interpretation of Lebesgue integral

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grand avantage de la définition de M. J. Radon, avantage que celui-ci ne paraît pas avoir remarqué. M. J. Radon avait pour but de réaliser un progrès dans la Théorie des fonctions en unifiant les définitions de Stieltjes et de M. Lebesgue. Mais, en fait, on remarque que, moyennant quelques légères modifications, la définition et les propriétés de l'intégrale de M. Radon s'étendent bien au delà du Calcul intégral classique, *elles sont presque immédiatement applicable au domaine infiniment plus vaste du calcul fonctionnel.*

En d'autres termes, on peut conserver la majeure partie des définitions et des raisonnements de M. J. Radon en négligeant l'hypothèse faite sur la nature de l'argument P à savoir que P est un point de l'espace à $n$ dimensions.

Nous pourrions nous contenter de cette indication à laquelle nous nous sommes bornés dans une Note présentée à l'Académie des Sciences, le 28 juin 1915.

Mais il nous a paru utile de présenter brièvement ce que devient la définition de l'intégrale ainsi étendue et de préciser la forme sous laquelle se présentent ses propriétés. Il est vraiment remarquable de pouvoir constater combien peu d'entre elles doivent être abandonnées en opposition avec ce qui a lieu presque toujours quand on veut obtenir un haut degré de généralisation; et combien celles qui se conservent changent peu de physionomie.

J'ai profité de l'occasion pour utiliser un mode de présentation de l'intégrale de M. Lebesgue qui a l'avantage de se rapprocher beaucoup plus que celui de M. Lebesgue de celui de Riemann-Darboux avec lequel un grand nombre d'étudiants sont plus familiers ([1]).

Je tiens à insister à nouveau, avant de commencer cet exposé, sur ce que la nouvelle définition va se trouver applicable non plus seulement à un espace à $n$ dimensions mais à un ensemble abstrait quelconque. C'est-à-dire qu'il n'est même pas nécessaire, par exemple, de supposer qu'on sache ce que c'est que la limite d'une suite d'éléments de cet ensemble (comme cela est au contraire indispensable pour la généralisation de la théorie des ensembles

([1]) Un essai dans ce sens a été fait par J. Pierpont dans son Ouvrage *Theory of Functions of real variables*, t. II, dans le cas particulier où P est un point de l'espace à $n$ dimensions. Mais son exposé prête à des objections très sérieuses.

**Fréchet 1915**

This way of presenting the theory of integration due to M. Lebesgue has the advantage, over the way M. Lebesgue presented his theory himself, that is very much close to the views of Riemann-Darboux to which many students are familiar with.

# Birkhoff views

Let $f : \Omega \longrightarrow X$ be a function. If $\Gamma$ is a partition of $\Omega$ into countably many sets $(A_n)$ of $\Sigma$, the function $f$ is called **summable** with respect to $\Gamma$ if the restriction $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set of sums

$$J(f,\Gamma) = \left\{ \sum_n f(t_n)\mu(A_n) : t_n \in A_n \right\}$$

is made up of unconditionally convergent series.

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The function $f$ is said to be **Birkhoff integrable** if for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ for which $f$ is summable and

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In this case, the **Birkhoff integral** $(B)\int_\Omega f \, d\mu$ of $f$ is the only point in the intersection

$$\bigcap \{ \overline{\mathrm{co}(J(f,\Gamma))} : f \text{ is summable with respect to } \Gamma \}.$$

## Theorem

*Let $f : \Omega \to X$ be a bounded function. The following statements are equivalent:*

(i) *$f$ is Birkhoff integrable;*

(ii) *for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ such that for each $t_k, t'_k \in A_k$, $k \in \mathbb{N}$, we have*

$$\left| \sum_{k=1}^{m} \langle x^*, f \rangle (t_k) \mu(A_k) - \sum_{k=1}^{m} \langle x^*, f \rangle (t'_k) \mu(A_k) \right| < \varepsilon$$

*for every $m \in \mathbb{N}$ and every $x^* \in B_{X^*}$;*

(iii) *$Z_f = \{\langle x^*, f \rangle : x^* \in B_{X^*}\}$ has Bourgain property.*

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where $J(f,\Gamma) = \left\{ \sum_n f(t_n)\mu(A_n) : t_n \in A_n \right\}$.

*Let* ... *nt:*

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*Proof.-*

- (i) $\Leftrightarrow$ (ii) pretty easy since (ii) is a reformulation of (i).
- (ii) $\Rightarrow$ (iii) straightforward.

## Theorem

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For every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ such that

$$\sum_{\mu(A_n)>0} |\cdot|\text{-diam } (\langle x^*, f\rangle(A_n))\mu(A_n) < \varepsilon$$

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(i)

(ii) *for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ such that for each $t_k, t_k' \in A_k$, $k \in \mathbb{N}$, we have*

$$\left|\sum_{k=1}^{m} \langle x^*, f\rangle(t_k)\mu(A_k) - \sum_{k=1}^{m} \langle x^*, f\rangle(t_k')\mu(A_k)\right| < \varepsilon$$

*for every $m \in \mathbb{N}$ and every $x^* \in B_{X^*}$;*

(iii) *$Z_f = \{\langle x^*, f\rangle : \ x^* \in B_{X^*}\}$ has Bourgain property.*

*Proof.-*

- (i) $\Leftrightarrow$ (ii) pretty easy since (ii) is a reformulation of (i).
- (ii) $\Rightarrow$ (iii) straightforward.
- (iii) $\Rightarrow$ (ii) mimic a result by Talagrand.

**Theorem**

*Let $f : \Omega \longrightarrow X$ be a function. TFAE:*

(i) *$f$ is Birkhoff integrable;*

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- (iii) $\Rightarrow$ (i) requires some extra work.

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$$\left\| \sum_k f(t_{n,k})\mu(A_{n,k}) - \sum_k f(t'_{n,k})\mu(A_{n,k}) \right\| < \frac{\varepsilon}{2^n}$$

for arbitrary choices $(t_{n,k})$ and $(t'_{n,k})$ in $\Gamma_n$.

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- For $\Gamma := (A_{n,k})_{n,k}$, the series $\sum_{n,k} f(t_{n,k})\mu(A_{n,k})$ c. u. for every choice $T = (t_{n,k})$ in $\Gamma$ because:

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  - for each finite set $Q\subset\mathbb{N}$

$$\left\|\sum_{k\in Q}\left(f(t_{n,k})\mu(A_{n,k})-\int_{A_{n,k}} f\,d\mu\right)\right\|\leq\frac{\varepsilon}{2^n}.$$

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$$\left\|\sum_{k\in Q}\left(f(t_{n,k})\mu(A_{n,k})-\int_{A_{n,k}} f\,d\mu\right)\right\|\le\frac{\varepsilon}{2^n}.$$

- $\|\ \|$-diam $\left(\left\{\sum_{n,k} f(t_{n,k})\mu(A_{n,k}) : t_{n,k}\in A_{n,k}\right\}\right)\le\varepsilon.$

## Theorem

*Let* $f:\Omega\longrightarrow X$ *be a function. TFAE:*

(i) $f$ *is Birkhoff integrable;*

(ii) *there is* $x\in X$ *satisfying: for every* $\varepsilon>0$ *there is a countable partition* $\Gamma$ *of* $\Omega$ *in* $\Sigma$ *for which* $f$ *is summable and*

$$\|S(f,\Gamma,T)-x\|<\varepsilon \text{ for every choice } T \text{ in } \Gamma;$$

(iii) *there is* $y\in X$ *satisfying: for every* $\varepsilon>0$ *there is a countable partition* $\Gamma$ *of* $\Omega$ *in* $\Sigma$ *such that* $f$ *is summable with respect to each countable partition* $\Gamma'$ *finer than* $\Gamma$ *and*

$$\|S(f,\Gamma',T')-y\|<\varepsilon \qquad \text{for every choice } T' \text{ in } \Gamma'.$$

*In this case,* $x=y=\int_\Omega f\,d\mu$.

## Theorem

*Let $X$ be a Banach space. The following statements are equivalent:*

1. *$X^*$ has the weak Radon-Nikodým property;*
2. *$X$ does not contain a copy of $\ell^1$;*
3. *for every complete probability space $(\Omega,\Sigma,\mu)$ and for every $\mu$-continuous countably additive vector measure $\nu:\Sigma\longrightarrow X^*$ of $\sigma$-finite variation there is a Birkhoff integrable function $f:\Omega\longrightarrow X^*$ such that*

$$\nu(E)=\int_E f\;d\mu$$

*for every $E\in\Sigma$.*

## The integral as limits of Riemann-Lebesgue sums

Let $F : \Omega \longrightarrow cwk(\mathbb{R}^n)$ be a multi-valued function. The following conditions are equivalent:

1. $F$ is Debreu integrable;
2. there is $B \in cwk(X)$ with the following property: for every $\varepsilon > 0$ there is a countable partition $\Gamma_0$ of $\Omega$ in $\Sigma$ such that for every countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ finer than $\Gamma_0$ and any choice $T = (t_n)$ in $\Gamma$, the series $\sum_n \mu(A_n)F(t_n)$ is unconditionally convergent and

$$h\Big(\sum_n \mu(A_n)F(t_n), B\Big) \leq \varepsilon.$$

In this case, $B = (B)\int_\Omega F \, d\mu$.

## Boundary problem

If $X$ is a Banach space not containing $\ell^1(\mathbb{R})$ and $B \subset B_{X^*}$ a boundary (i.e. for every $x \in X$ there exists $e^* \in B$ such that $e^*(x) = \| x \|$) then the norm bounded $\sigma(X, B)$- relatively compact subsets $H$ of $X$ are relatively weakly compact.

*Proof.-*

- It is enough to prove that $\overline{\mathrm{co}(H)}^{\sigma(X,B)}$ is $\sigma(X, B)$-compact.
- We prove that for each Radon probability measure $\mu$ on $H$ the identity $\mathrm{id} : H \to X$ is $\mu$- Pettis (Birkhoff) integrable.

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