Universidad de Murcia
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On fragmentability and its applications

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http://webs.um.es/beca

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On the occasion of the 60th birthday of Andreas Defant

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Contents:

2. Mathematical part:
    - Old results:
    - Newer results: fragmentability and boundaries.
    - Recent results: fragmentability and norm attaning operators.
    - Unpublished results: fragmentability and representable operators.

Contents:

1. A picture to start with
2. Mathematical part:
    - Old results: . . . related to the picture
    - Newer results: fragmentability and boundaries.
    - Recent results: fragmentability and norm attaning operators.
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Contents:

A picture to start with

Old results. . . related to the picture

1987, Orihuela-Cascales

For a compact space K, T.F.A.E.:

1. K is metrizable;
2. Δ is a G_δ;
3. $\Delta = \bigcap_n G_n$ with G_n open and $\{G_n\}_n$ a basis of neighb. of Δ;
4. $(K \times K) \setminus \Delta = \cup_n F_n$, with $\{F_n\}$ an increasing fundamental family of compact sets in $(K \times K) \setminus \Delta$;
5. $(K \times K) \setminus \Delta = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ with each $\{A_\alpha\}$ a fundamental family of compact sets such that $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;
6. $(K \times K) \setminus \Delta$ is Lindelöf.

A way of presenting the results that you might recognize

Math. Z. 195, 365–381 (1987)

Mathematische Zeitschrift

On Compactness in Locally Convex Spaces

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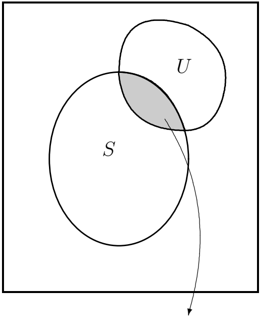
1. Introduction and Terminology

The purpose of this paper is to show that the behaviour of compact subsets in many of the locally convex spaces that usually appear in Functional Analysis is as good as the corresponding behaviour of compact subsets in Banach spaces. Our results can be intuitively formulated in the following terms: *Dealing with metrizable spaces or their strong duals, and carrying out any of the usual operations of countable type with them, we ever obtain spaces with their precompact subsets metrizable, and they even give good performance for the weak topology, indeed they are weakly angelic,* [14], *and their weakly compact subsets are metrizable if and only if they are separable.*

Topology behind the scenes

Topology behind the scenes.



$\|\cdot\| - \text{diam}(U \cap S) \leq \varepsilon$

Asplund spaces: Namioka, Phelps and Stegall

Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space, *i.e.*, whenever f is a convex continuous function defined on an open convex subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U.

(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

Fragmentability and boundaries

Definition

Given a Banach space X and a w^*-compact subset $K \subset X^*$, a James boundary for K is a subset B of K such that for every $x \in X$ there exists some $b \in B$ such that $b(x) = \sup\{k(x) : k \in K\}$. If K is convex then $K = \overline{\operatorname{conv} B}^{w^*}$.

The question?

K conv. w^*-comp. $B \subset K$ boundary, conditions (X, B or K?) $\Rightarrow$

$$K = \overline{\operatorname{conv} B}^{\|\cdot\|}.$$

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What are the techniques that have been used?

1. 1976, Haydon: if $\ell^1 \not\subset X$ then $K = \overline{\operatorname{conv} \operatorname{Ext} K}^{\|\cdot\|}$ using independent sequences and Bishop-Phelps theorem.
2. 1987, Namioka: $K \subset X^*$ is norm fragmented, then $\overline{\operatorname{conv} K}^{w^*} = \overline{\operatorname{conv} K}^{\|\cdot\|}$ using the existence of barycenters.
3. 1987, Godefroy: if $B \subset K$ is norm separable then $K = \overline{\operatorname{conv} B}^{\|\cdot\|}$ using Simons' inequality.
4. 1987, Godefroy using Simons' inequality proves that if X is separable and $\ell^1 \not\subset X$ then $K = \overline{\operatorname{conv} B}^{\|\cdot\|}$.

Orihuela-Cascales 2008

1. We prove that when B is "w^*-descriptive" then $K = \overline{\operatorname{conv} B}^{\|\cdot\|}$.
2. Let $J : X \to 2^{B_{X^*}}$ be the duality mapping: defined at each $x \in X$ by
$$J(x) := \{x^* \in B_{X^*} : x^*(x) = \|x\|\}.$$
If $B \subset B_{X^*}$ is boundary there is a selector $f : X \to X^*$ for J such that $f(X) \subset B$: if f is reasonable (for instance Borel), B is strong.

Techniques related to

Theorem (Namioka-Orihuela-Cascales, 2003)

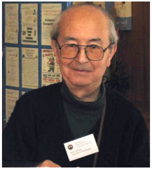
K compact subset of M^D, (M,ρ) metric space. T.F.A.E:

(a) *The space (K,τ_p) is fragmented by d.*

(b) *For each $A \in \mathscr{C}$, the pseudo-metric space (X, d_A) is separable.*

(c) *$(X, \gamma(D))$ is Lindelöf.*

(d) *$(X, \gamma(D))^{\mathbb{N}}$ is Lindelöf.*



Theorem (Namioka-Orihuela-Cascales, 2003)

If X is a Banach space and H is a weak*-compact subset of X^* which is weak-Lindelöf, then $\overline{co(H)}^{w^*} = \overline{co(H)}^{\|\cdot\|}$ and this closed convex hull is weakly Lindelöf again.

Fragmentability and norm attaning operators

Theorem (Guirao-Kadets-Cascales, 2013)

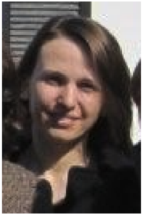
Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \le \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

This gives

- for $C(K)$ an example of the BPBp for c_0 as domain and an infinite dimensional Banach space as range (answer a question by Acosta-Aron-García-Maestre, 2008);
- new cases, in particular, disk algebra as range.

Co-authors

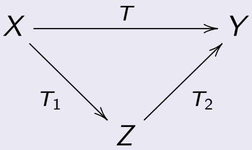

R. M. Aron, B. Cascales and O. Kozhushkina,
The Bishop-Phelps-Bollobás theorem and Asplund operators,
Proc. Amer. Math. Soc. 139 (2011), no. 10, 3553–3560.

B. Cascales, A. J. Guirao and V. Kadets,
A Bishop-Phelps-Bollobás type theorem for uniform algebras,
Advances in Mathematics 240 (2013) 370?382

Asplund operators

Stegall, 1975

An **operator** $T \in L(X, Y)$ is **Asplund**, if it factors through an Asplund space:



Z is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

T Asplund operator $\Leftrightarrow$ $T^*(B_{Y^*})$ is fragmented by the norm of X^*.

Theorem (Guirao-Kadets-Cascales 2013)

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \le \varepsilon$$

and

$$\|T - \widetilde{T}\| < 2\varepsilon.$$

Corollary

Let $T \in L(X, \mathfrak{A})$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, \mathfrak{A})$ **weakly compact** with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \le 2\varepsilon.$$

Corollary

$(X, \mathfrak{A})$ has the BPBP for any Asplund space X and any locally compact Hausdorff topological space L ($X = c_0(\Gamma)$, **for instance**).

Corollary

$(X, C_0(L))$ has the BPBP for any X and any scattered locally compact Hausdorff topological space L.

An idea of the proof for $\mathfrak{A}$...in particular for $A(\mathbb{D})$

Theorem (Guirao-Kadets-Cascales, 2013)

Let $T\colon X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

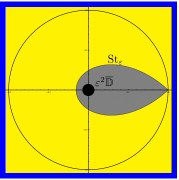
$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \le \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

1. **Fragmentability** gives an open set $U \cap \Gamma_0 \neq \emptyset$, $y^* \in S_{X^*}$ & $\rho < 2\varepsilon$ with

   $$1 = |y^*(u_0)| = \|u_0\| \text{ and } \|x_0 - u_0\| < \varepsilon \;\&\; \|T^*(\delta_t) - y^*\| < \rho \;\forall t \in U.$$

2. Uryshon's lemma that applied to an arbitrary $t_0 \in U \cap \Gamma_0$ produces a function $f \in \mathfrak{A}$ satisfying

   $$f(t_0) = \|f\|_\infty = 1, f(K) \subset St\varepsilon' \text{ and } f \text{ small in } K \setminus U.$$



explicitly defined by

$$\widetilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - \varepsilon')(1 - f(t)) \cdot T(x)(t)$$

suitability of U is used to prove that $\|T - \widetilde{T}\| < 2\varepsilon$.

A Urysohn type lemma for uniform algebras

Proposition 2.8. *Let $A \subset C(K)$ be a unital uniform algebra, $\Omega \subset \mathbb{C}$ a bounded simply connected region such that all points in its boundary $\partial\Omega$ are simple. Let us fix two different points a and b with $b \in \partial\Omega$, $a \in \overline{\Omega}$ and a neighborhood $V_a \subset \overline{\Omega}$ of a. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and for every $t_0 \in U \cap \Gamma_0$, there exists $f \in A$ such that*

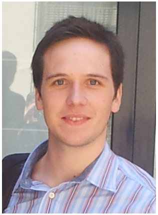
(i) $f(K) \subset \overline{\Omega}$;
(ii) $f(t_0) = b$;
(iii) $f(K \setminus U) \subset V_a$.

Fragmentability and representable operators

Pérez-Raja-Cascales, 2013

Let (Ω,Σ,μ) be a finite measure space and $T : L^1(\mu) \to X$ a continuous linear operator. Then

$$d(T,\mathscr{L}_{rep}(L^1(\mu),E)) \leq 2\,\gamma(T(B_{L^1(\mu)})).$$



B. Cascales, A. Pérez and M. Raja,
Radon-Nikodým indexes and measures of non weak compactness.
Preprint, 2013

A closing picture

Get him educated!