Universidad de Murcia | Departamento Matemáticas

Bishop-Phelps-Bollobás theorem and Asplund operators

B. Cascales

Websites: http://webs.um.es/beca (Personal)
http://www.um.es/beca (Group)

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Stay focused

1. Introduction: Bishop Phelps theorem
   - Credit to co-authors and a few papers by others
   - Bishop-Phelps theorem
   - The Bishop-Phelps property for operators
2. Bishop-Phelps-Bollobás theorem and Asplund operators
   - Bollobás observation and BPBp for operators
   - Our main result: applications
   - Remarks and further development
3. Final comments: other applications of fragmentability
   - Fragmentability, topology and boundaries
   - Fragmentability and measure theory

Notation

- X, Y, E, B Banach spaces;
- B_X **closed** unit ball; S_X unit sphere;
- $L(X, Y)$ bounded linear operators from X to Y;
- $C_0(L)$ space of continuous functions, vanishing at ∞.

$$\|f\| = \sup_{s \in L} |f(s)|,$$

  where L is a **locally compact** Hausdorff space.
- (Ω, Σ, μ) complete probability space.

. . . Bishop-Phelps property

Credit to co-authors and previous work

- María D. Acosta, Richard M. Aron, Domingo García, and Manuel Maestre, *The Bishop-Phelps-Bollobás theorem for operators*, J. Funct. Anal. **254** (2008), no. 11, 2780–2799.
- R. M. Aron, B. Cascales and O. Kozhushkina, *The Bishop-Phelps-Bollobas theorem and Asplund operators*, to appear PAMS 2011.
- Jerry Johnson and John Wolfe, *Norm attaining operators*, Studia Math. **65** (1979), no. 1, 7–19.
- C. Stegall, *The Radon-Nikodým property in conjugate Banach spaces. II*, Trans. Amer. Math. Soc. **264** (1981), no. 2, 507–519.

Bishop-Phelps theorem

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If X is a Banach, then $\overline{\mathbf{NA}X^*} = X^*$.

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A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere S of E are norm-dense in E^*, i.e., if for each f in E^* and each $\epsilon > 0$ there exist g in E^* and x in S such that $|g(x)| = \|g\|$ and $\|f - g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1][1] as well as incomplete spaces which *are* subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is sub-

A few words about reflexivity

WEAKLY COMPACT SETS

BY

ROBERT C. JAMES(1)

It has been conjectured that a closed convex subset C of a Banach space B is weakly compact if and only if each continuous linear functional on B attains a maximum on C [5]. This reduces easily to the case in which C is bounded, and will be answered in the affirmative [Theorem 4] after some preliminary results are established. Following suggestions by Namioka and Peck, the result is then generalized, first to weakly closed subsets

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A QUANTITATIVE VERSION OF JAMES' COMPACTNESS THEOREM

BERNARDO CASCALES, ONDŘEJ F.K. KALENDA AND JIŘÍ SPURNÝ

Abstract. We introduce two measures of weak non-compactness Ja_E and Ja that quantify, via distances, the idea of boundary behind James' compactness theorem. These measures tell us, for a bounded subset C of a Banach space E and for given $x^* \in E^*$, how far from E or C one needs to go to find $x^{**} \in \overline{C}^{w^*} \subset E^{**}$ with $x^{**}(x^*) = \sup x^*(C)$. A quantitative version of James'

Bishop-Phelps property

Question (Bishop-Phelps)

A possible generalization of this theorem remains open: Suppose E and F are Banach spaces, and let $\mathfrak{L}(E, F)$ be the Banach space of all continuous linear transformations from E into F, with the usual norm. For which E and F are those T such that $\|T\| = \|Tx\|$ (for some x in E, $\|x\| = 1$) dense in $\mathfrak{L}(E, F)$? This is true for arbitrary E if F is an ideal in $m(A)$ (the space of bounded functions on the set A, with the supremum norm).

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Theorem (Lindenstrauss, 1963)

Let Y be a strictly convex Banach space, isomorphic to c_0, and let $X = Y \bigoplus c_0$ where c_0 has the usual norm and consider the supremum norm on the direct sum. Then $NA\mathscr{L}(X;X)$ is NOT dense in $\mathscr{L}(X;X)$.

The Bishop-Phelps property for operators

Definition

An operator $T : X \to Y$ is **norm attaining** if there exists $x_0 \in X$, $\|x_0\| = 1$, such that $\|T(x_0)\| = \|T\|$.

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(X, Y) has the Bishop-Phelps Property (BPp) if every operator $T : X \to Y$ can be uniformly approximated by **norm attaining** operators.

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4. there are spaces X, Y and Z such that $(X, C([0,1]))$, (Y, ℓ^p) $(1 < p < \infty)$ and $(Z, L^1([0,1]))$ fail BPp, Schachermayer (1983), Gowers (1990) and Acosta (1999);

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6. $(L^1([0,1]), L^\infty([0,1]))$ has BPp, Finet-Payá (1998),

. . . Bishop-Phelps-Bollobás property

Bollobás observation

AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

Bishop and Phelps proved in **[1]** that every real or complex Banach space is *subreflexive*, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by S and S' the unit spheres in a Banach space B and its dual space B', respectively.

THEOREM 1. *Suppose* $x \in S$, $f \in S'$ *and* $|f(x)-1| \leqslant \varepsilon^2/2$ $(0 < \varepsilon < \frac{1}{2})$. *Then there exist* $y \in S$ *and* $g \in S'$ *such that* $g(y) = 1$, $\|f-g\| \leqslant \varepsilon$ *and* $\|x-y\| < \varepsilon+\varepsilon^2$.

A different way of writing BPB

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Corollary. . . the way is oftentimes presented

Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

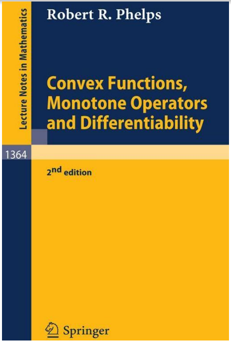
$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$

A variational principle implying BPB

Theorem 3.17 (Brøndsted-Rockafellar). *Suppose that f is a convex proper lower semicontinuous function on the Banach space E. Then given any point $x_0 \in \mathrm{dom}(f)$, $\epsilon > 0$, $\lambda > 0$ and any $x_0^* \in \partial_\epsilon f(x_0)$, there exist $x \in \mathrm{dom}(f)$ and $x^* \in E^*$ such that*

$$x^* \in \partial f(x), \quad \|x - x_0\| \leq \epsilon/\lambda \text{ and } \|x^* - x_0^*\| \leq \lambda.$$

In particular, the domain of ∂f is dense in $\mathrm{dom}(f)$.



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1. Take $f : E \to [0, +\infty]$ 0 at C and $+\infty$ at $E \setminus C$;
2. $\varepsilon^2/2$ instead of ε, $\lambda = \varepsilon/2$;
3. replace $x^* \in E^*$ in the corollary above by $x^*/\|x^*\|$

Corollary...the constants are better

Given $1 > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\varepsilon^2}{2},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

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Bishop-Phelps-Bollobás Property for operators

Definition: Acosta, Aron, García and Maestre, 2008

(X, Y) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any $\varepsilon > 0$ there are $\eta(\varepsilon) > 0$ such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

then there are $u_0 \in S_X$, $S \in S_{L(X,Y)}$ with

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3. thee is pair (ℓ^1, X) failing BPBp, but having BPp;
4. (ℓ_n^∞, Y) has BPBp Y uniformly convex no hope for c_0: $\eta(\varepsilon) = \eta(n, \varepsilon) \to 1$ with $n \to \infty$.

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PROBLEM?

No Y infinite dimensional is known s.t. (c_0, Y) has BPBP.

Theorem (R. Aron, B. Cascales, O. Kozhushkina, 2011)

Let $T : X \to C_0(L)$ be an **Asplund operator** with $\|T\| = 1$. Suppose that $\frac{1}{2} > \varepsilon > 0$ and $x_0 \in S_X$ are such that

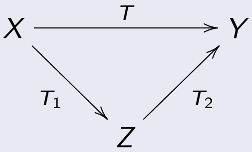
$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and an **Asplund operator** $S \in S_{L(X,C_0(L))}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

Stegall, 1975

An **operator** $T \in L(X, Y)$ is **Asplund**, if it factors through an Asplund space:



Z is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

Stegall, 1975

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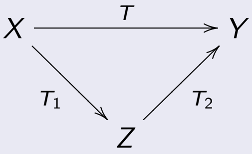

Z is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

T Asplund operator $\Leftrightarrow$ $T^*(B_{Y^*})$ is fragmented by the norm of X^*.

Asplund spaces: Namioka, Phelps and Stegall

Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space, *i.e.*, whenever f is a convex continuous function defined on an open convex subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U.

(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

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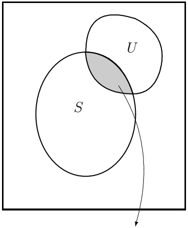

$\|\cdot\| - \mathrm{diam}(U \cap S) \leq \varepsilon$

Definition

B_{X^*} is fragmented if for every $\varepsilon > 0$ and every non empty subset $S \subset B_{X^*}$ there exists a w^*-open subset $U \subset X$ such that $U \cap S \neq \emptyset$ and

$$\|\cdot\| - \mathrm{diam}(U \cap S) \leq \varepsilon.$$

Theorem (R. Aron, B. Cascales, O. Kozhushkina, P.A.M.S. 2011)

Let $T : X \to C_0(L)$ be an **Asplund operator** with $\|T\| = 1$. Suppose that $\frac{1}{2} > \varepsilon > 0$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and an **Asplund operator** $S \in S_{L(X, C_0(L))}$ satisfying

$$\|S(u_0)\| = 1$$

and

$$\|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

Corollary

Let $T \in L(X, C_0(L))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, C_0(L))$ **weakly compact** with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

Corollary

$(X, C_0(L))$ has the BPBP for any Asplund space X and any locally compact Hausdorff topological space L ($X = c_0(\Gamma)$**, for instance**).

Corollary

$(X, C_0(L))$ has the BPBP for any X and any scattered locally compact Hausdorff topological space L.

An idea of the proof

Theorem

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An idea of the proof

1. let $\phi : L \to X^*$ given by $\phi(s) = \delta_s \circ T$;

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Then there are $u_0 \in S_X$ and an **Asplund operator** $S \in S_{L(X,C_0(L))}$ satisfying

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Lemma

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2. take $s_0 \in L$ such that $|\phi(s_0)(x_0)| = |T(x_0)(s_0)| > 1 - \frac{\varepsilon^2}{4}$;
3. $U_1 = \{x^* \in X^* : |x^*(x_0)| > 1 - \frac{\varepsilon^2}{4}\}$,
4. $\phi(s_0) \in U_1 \cap \phi(L)$;
5. $\phi(L) \subset B_{X^*}$ is fragmented;
6. $U_2 \subset X^*$ such that $(U_1 \cap \phi(L)) \cap U_2 \neq \emptyset$ and
$$\|\cdot\|\text{-diam}\left((U_1 \cap \phi(L)) \cap U_2\right) \leq \varepsilon;$$
7. Let $U := U_1 \cap U_2$;
8. Pick a point, $x_0^* \in U \cap \phi(L)$ normalize it $\frac{x_0^*}{\|x_0^*\|}$ and use. . .

BPB in the scalar case

Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that $|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4}$, then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$

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Theorem (R. Aron, B. Cascales, O. Kozhushkina, 2011)

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Operator Ideals

Approximating operator $S : X \to C_0(L)$,:

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Observe:

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Consequence:
If $\mathscr{I} \subset \mathscr{A} = \mathscr{A}(X, C_0(L))$ is a sub-ideal of Asplund operators then

$$T \in \mathscr{I} \Rightarrow S \in \mathscr{I}.$$

The above applies to:

- Finite rank operators $\mathscr{F}$;
- Compact operators $\mathscr{K}$;
- p-summing operators Π_p;
- Weakly compact operators $\mathscr{W}$.

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Corollary

Let $T \in L(X, C_0(L))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, C_0(L))$ **weakly compact** with $\|S\| = 1$ satisfying

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Corollary

$(X, C_0(L))$ has the BPBP for any Asplund space X and any locally compact Hausdorff topological space L ($X = c_0(\Gamma)$**, for instance**).

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Remarks and further development

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2. The technicality that leads to our results is really better:

Lemma: Aron, Cascales and Kozhushkina, 2011

Let $T : X \to Y$ be an Asplund operator with $\|T\| = 1$, let $\frac{1}{2} > \varepsilon > 0$ and choose $x_0 \in S_X$ such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

For any given 1-norming set $B \subset B_{Y^*}$ if we write $M := T^*(B)$ then there are:

(a) a w^*-open set $U \subset X^*$ with $U \cap M \neq \emptyset$ and

(b) points $y^* \in S_{X^*}$ and $u_0 \in S_X$ with $|y^*(u_0)| = 1$ such that

$$\|x_0 - u_0\| < \varepsilon \text{ and } \|z^* - y^*\| < 3\varepsilon \quad \text{for every} \quad z^* \in U \cap M.$$

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3. The previous lemma has been used already as it is to establish the BPBp for Asplund operators $T : X \to C(K, Y)$, for some Y's (Acosta, Maestre and Garcia; to be published);

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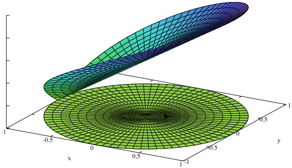

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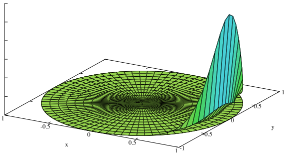
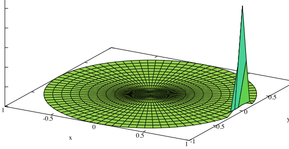


$z \to \left| \left(\frac{z+1}{2} \right)^{50} \right|$



$z \to \left| \left(\frac{z+1}{2} \right)^{1000} \right|$

Very hot refrences

- R.M. Aron, Y.S. Choi, D. García and M. Maestre, The Bishop-Phelps-Bollobás theorem for $\mathscr{L}(L_1(\mu), L_\infty[0,1])$, *Advances of Math.*, **228** (2011), 617–628.
- Y. S. Choi and S. K. Kim, The Bishop-Phelps-Bollobás theorem for operators From $L_1(\mu)$ to Banach spaces with the Radon Nikodým property, preprint 2011.

THANK YOU!

... other applications of fragmentability

Fragmentability $\Rightarrow$ topology and boundaries

- B. Cascales. and **I. Namioka**, *The Lindelöf property and σ-fragmentability*, **Fund. Math.** 180 (2003), no. 2, 161–183.
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Boundaries and selectors

Let $J : X \to 2^{B_{X^*}}$ be the duality mapping: defined at each $x \in X$ by

$$J(x) := \{x^* \in B_{X^*} : x^*(x) = \|x\|\}.$$

There is a *reasonable* selector $f : X \to X^*$ for J iff X is Asplund (in this case $\overline{f(X)}^{\|\cdot\|} = B_{X^*}$).

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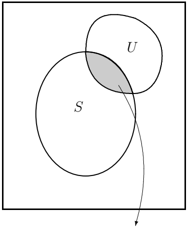

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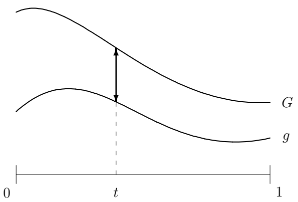
$F : \Omega \longrightarrow cwk(X)$ –convex w-compact



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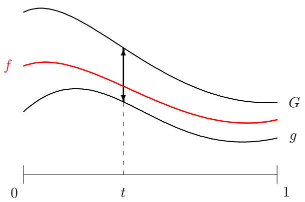

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2. (Aumann Nobel prize in 2005) to take all *integrable* selectors f of F and consider

$$\int F\, d\mu = \left\{ \int f\, d\mu : f \text{ integra. sel.} F \right\}.$$

Fragmentability and measure theory

$f : \Omega \to E$

For every $\varepsilon > 0$ $A \in \Sigma^+$ there is $B \in \Sigma_A^+$ such that

$$\|\cdot\| - \operatorname{diam} f(B) < \varepsilon.$$

Is there a reasonable extension of the above for multi-functions?

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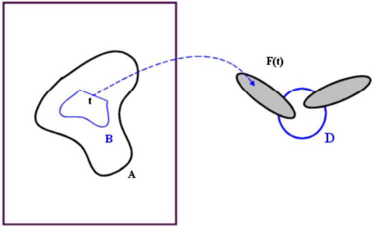


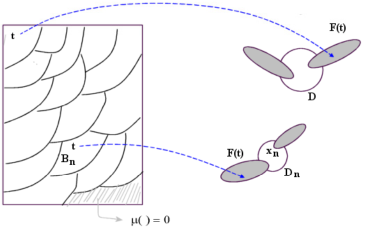
Definition

$F:\Omega \to 2^E$ satisfies property (P) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma_A^+$ and $D \subset E$ with $\operatorname{diam}(D) < \varepsilon$ such that

$$F(t) \cap D \neq \emptyset \text{ for every } t \in B.$$

Property (P)

$F:\Omega\to 2^E$ satisfies property (P) if for each $\varepsilon>0$ and each $A\in\Sigma^+$ there exist $B\in\Sigma_A^+$ and $D\subset E$ with $\operatorname{diam}(D)<\varepsilon$ such that $F(t)\cap D\neq\emptyset$ for every $t\in B$.



1. Fix $n=0$;
2. take $\varepsilon:=(1/2)^n$;
3. apply (P) for $A=\Omega$, ε and F;
4. a maximality argument produces a partition of $B's$;
5. enumerate $B's$ as $\{B_n\}$ and choose any $x_n\in D_n$;
6. define $f_\varepsilon:=\sum_n \chi_{B_n}x_n$;
7. f_ε is μ-measurable and $d(f_\varepsilon(t),F(t))<\varepsilon$ μ-a.e.;
8. define $F_\varepsilon(t):=F(t)\cap B(f_\varepsilon(t),\varepsilon)$;
9. IF F_ε satisfies (P) GOTO 11;
10. STOP;
11. $n:=n+1$;
12. GOTO 2.

Conclusion

We produce a sequence $(f_n):\Omega\to E$ of μ-measurable functions such that $(f_n(t))$ is Cauchy μ-a.e., hence it is convergent.

Fragmentability and measure theory: measurable selections

Corollary (Kuratowski-Ryll Nardzewski, 1965)

Let $F : \Omega \to 2^E$ be a multi-function with closed non empty values of E. If E is separable and F satisfies that

$$\{t \in \Omega : F(t) \cap O \neq \emptyset\} \in \Sigma \text{ for each open set } O \subset E. \tag{E}$$

Then F admits a μ-measurable selector f.

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Very little is known in the non separable case

Theorem (Kadets, Rodríguez and B. C. -2009)

For a multi-function $F:\Omega\to wk(E)$ TFAE:

(i) *F admits a strongly measurable selector.*

(ii) *There exist a set of measure zero $\Omega_0\in\Sigma$, a separable subspace $Y\subset E$ and a multi-function $G:\Omega\setminus\Omega_0\to wk(Y)$ that is Effros measurable and such that $G(t)\subset F(t)$ for every $t\in\Omega\setminus\Omega_0$;*

(iii) *F satisfies property (P).*

Consequences

NEW THINGS: the theory was stuck in the separable case

1. Characterization of multi-functions admitting strong selectors;
2. scalarly measurable selectors for scalarly measurable multi-functions;
3. Pettis integration; the theory was stuck in the separable case;
4. existence of w^*-scalarly measurable selectors;
5. Gelfand integration; relationship with the previous notions.
6. RNP for multi-functions;
7. set selectors.

GRACIAS!