Universidad de Murcia | Departamento Matemáticas

The Radon-Nikodým property, multifunctions and norm attaining operators

B. Cascales

http://webs.um.es/beca

Integration, vector measures and related topics V
Palermo, August 28th - September 1st, 2012

Stay focused

3. **First part:** Bishop-Phelps-Bollobás property.

4. **Second part:** Multifunctions in non separable Banach spaces.

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   - Our result(s).
   - The problem: a bit of history and people involved.
   - Why I like the results: the proofs.
4. **Second part:** Multifunctions in non separable Banach spaces.
   - A result about derivatives of multifunctions is presented.

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5. What you should **take home** with you.

Co-authors

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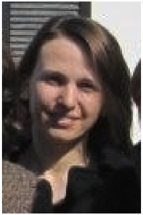


R. M. Aron, B. Cascales and O. Kozhushkina,
The Bishop-Phelps-Bollobás theorem and Asplund operators,
Proc. Amer. Math. Soc. 139 (2011), no. 10, 3553–3560.

B. Cascales, A. J. Guirao and V. Kadets,
A Bishop-Phelps-Bollobás type theorem for uniform algebras,
Preprint. April/2012

B. Cascales, V. Kadets and J. Rodríguez,
Radon-Nikodym theorems for multimeasures in non-separable spaces,
Preprint February/2012

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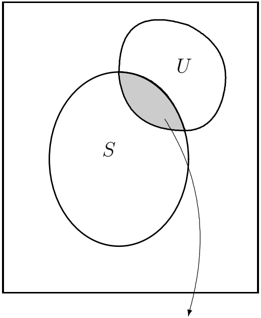

$\|\cdot\| - \text{diam}(U \cap S) \leq \varepsilon$

Asplund spaces: Namioka, Phelps and Stegall

Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space, *i.e.*, whenever f is a convex continuous function defined on an open convex subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U.

(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

Bishop-Phelps-Bollobás property

Our result(s)

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T : X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

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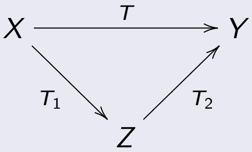
Asplund operators

Stegall, 1975

An **operator** $T \in L(X, Y)$ is **Asplund**, if it factors through an Asplund space:



Z is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

T Asplund operator $\Leftrightarrow$ $T^*(B_{Y^*})$ is fragmented by the norm of X^*.

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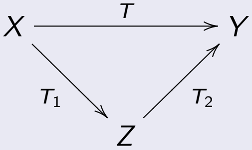

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$$\|T - \widetilde{T}\| < 2\varepsilon.$$

Corollary

Let $T \in L(X, \mathfrak{A})$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, \mathfrak{A})$ **weakly compact** with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 2\varepsilon.$$

Corollary

$(X, \mathfrak{A})$ has the BPBP for any Asplund space X and any locally compact Hausdorff topological space L ($X = c_0(\Gamma)$**, for instance**).

Corollary

$(X, C_0(L))$ has the BPBP for any X and any scattered locally compact Hausdorff topological space L.

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

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Corollary

Let $T \in L(X, A(\overline{\mathbb{D}}))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

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Remark

The theorem applies in particular to the ideals of finite rank operators $\mathscr{F}$, compact operators $\mathscr{K}$, p-summing operators Π_p and of course to the weakly compact operators $\mathscr{W}$ themselves. To the best of our knowledge even in the case $\mathscr{W}(X, \mathfrak{A})$ the Bishop-Phelps property that follows is a brand new result.

Bishop-Phelps theorem

Theorem (Bishop-Phelps, 1961)

If X is a Banach, then $\overline{\mathbf{NA}X^*} = X^*$.

A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere S of E are norm-dense in E^*, i.e., if for each f in E^* and each $\epsilon > 0$ there exist g in E^* and x in S such that $|g(x)| = \|g\|$ and $\|f-g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1][1] as well as incomplete spaces which *are* subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is sub-

The Bishop-Phelps property for operators

Definition

An operator $T : X \to Y$ is **norm attaining** if there exists $x_0 \in X$, $\|x_0\| = 1$, such that $\|T(x_0)\| = \|T\|$.

Definition (Lindenstrauss)

(X, Y) has the Bishop-Phelps Property (BPp) if every operator $T : X \to Y$ can be uniformly approximated by **norm attaining** operators.

1. $(X, \mathbb{K})$ has BPp for every X Bishop-Phelps (1961);
2. $\overline{\{T \in L(X; Y) : T^{**} \in NA(X^{**}; Y^{**})\}} = L(X; Y)$ for every pair of Banach spaces X and Y, Lindenstrauss (1963);
3. X with RNP, then (X, Y) has BPp for every Y, Bourgain (1977);
4. there are spaces X, Y and Z such that $(X, C([0,1]))$, (Y, ℓ^p) $(1 < p < \infty)$ and $(Z, L^1([0,1]))$ fail BPp, Schachermayer (1983), Gowers (1990) and Acosta (1999);
5. $(C(K), C(S))$ has BPp for all compact spaces K, S, Johnson and Wolfe, (1979).
6. $(L^1([0,1]), L^\infty([0,1]))$ has BPp, Finet-Payá (1998),

Bollobás observation, 1970

AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

Bishop and Phelps proved in **[1]** that every real or complex Banach space is *subreflexive*, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by S and S' t

respectively.

THEOREM 1. *Suppose*

exist $y \in S$ *and* $g \in S'$ *such*

Corollary... the way it is oftentimes presented

Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$

Bishop-Phelps-Bollobás Property for operators

Definition: Acosta, Aron, García and Maestre, 2008

(X, Y) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any $\varepsilon > 0$ there are $\eta(\varepsilon) > 0$ such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

then there are $u_0 \in S_X$, $S \in S_{L(X,Y)}$ with

$$\|S(u_0)\| = 1$$

and

$$\|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| < \varepsilon.$$

1. Y has *certain* almost-biorthogonal system (X, Y) has BPBp any X;
2. (ℓ^1, Y) BPBp is characterized through a condition called AHSP: it holds for Y finite dimensional, uniformly convex, $Y = L^1(\mu)$ for a σ-finite measure or $Y = C(K)$;
3. there is pair (ℓ^1, X) failing BPBp, but having BPp;
4. (ℓ^∞_n, Y) has BPBp Y uniformly convex no hope for c_0: $\eta(\varepsilon) = \eta(n, \varepsilon) \to 1$ with $n \to \infty$.

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PROBLEM?

No Y infinite dimensional was known s.t. (c_0, Y) has BPBP.

An idea of the proof for $\mathfrak{A} = C(K)$

Theorem (R. M. Aron, O. Kozhushkina and B. C. 2011)

Let $T: X \to C(K)$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,C(K))}$ satisfying that

$$\|\widetilde{T} u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

1. **Black box** provides a suitable open set $U \subset K$, $y^* \in S_{X^*}$ and $\rho < 2\varepsilon$ with
$$1 = |y^*(u_0)| = \|u_0\| \text{ and } \|x_0 - u_0\| < \varepsilon \;\&\; \|T^*(\delta_t) - y^*\| < \rho \;\forall t \in U$$
2. Uryshon's lemma that applied to an arbitrary $t_0 \in U$ produces a function $f \in C(K)$ satisfying
$$f(t_0) = \|f\|_\infty = 1, f(K) \subset [0,1] \text{ and } \operatorname{supp}(f) \subset U.$$
3. $\widetilde{T}$ is explicitly defined by
$$\widetilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - f(t)) \cdot T(x)(t), x \in X, t \in K,$$
4. The suitability of U is used to prove that $\|T - \widetilde{T}\| < 2\varepsilon$.

An idea of the proof for $\mathfrak{A} = A(\overline{\mathbb{D}})$

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

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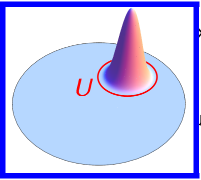

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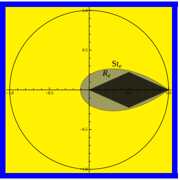

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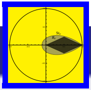
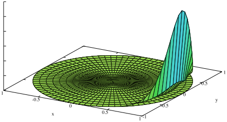
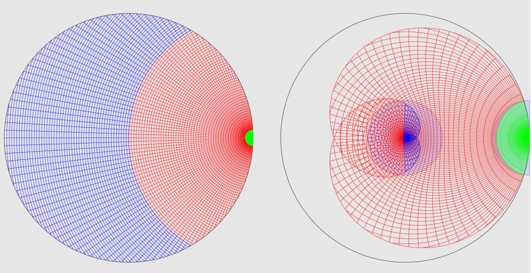


Uryshon type lemma for $A(\overline{\mathbb{D}})$

$U \overset{open}{\subset} \overline{\mathbb{D}}, 1 \in U, \exists f(1) = \|f\|_\infty = 1,\ f(\overline{\mathbb{D}}) \subset R_{\varepsilon'}$ and f small in $\overline{\mathbb{D}} \setminus U$,

$$f_n(z) := \left(\frac{z+1}{2}\right)^n$$





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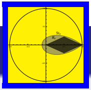

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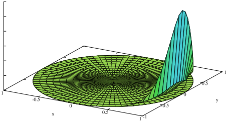
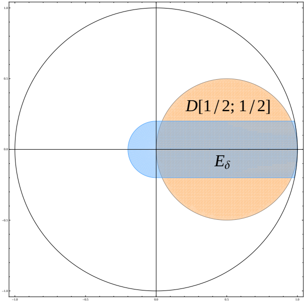
$$f_n(z) := \left(\frac{z+1}{2}\right)^n$$

$$\exists\, n_1, n_2, \ldots, n_k$$

$$f := \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_2}}{k}$$





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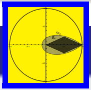

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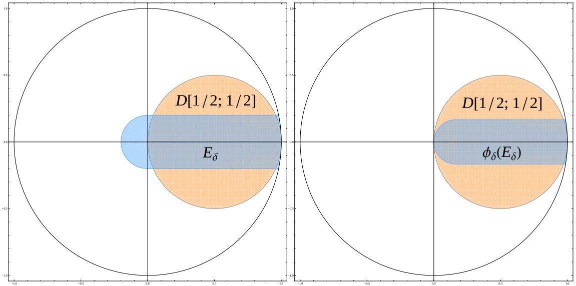
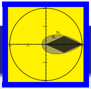
$$z \to \frac{z+\delta}{1+\delta}$$





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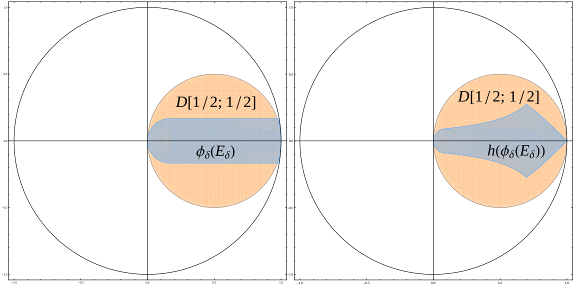
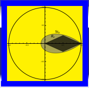
$U \overset{open}{\subset} \overline{\mathbb{D}}, 1 \in U, \exists f(1) = \|f\|_\infty = 1,\ f(\overline{\mathbb{D}}) \subset R_{\varepsilon'}$ and f small in $\overline{\mathbb{D}} \setminus U$,

$$z \to 1 - \sqrt{1-z}$$

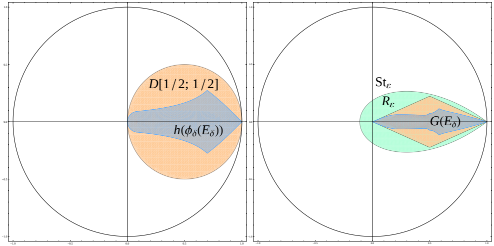




Uryshon type lemma for $A(\overline{\mathbb{D}})$

$U \overset{open}{\subset} \overline{\mathbb{D}}, 1 \in U, \exists f(1) = \|f\|_\infty = 1, f(\overline{\mathbb{D}}) \subset R_{\varepsilon'}$ and f small in $\overline{\mathbb{D}} \setminus U$,

Riemann conformal mapping



Our Uryshon type lemma is suited for calculations with a computer.

Multifunctions in non separable Banach spaces.

A result about derivatives of multifunctions is presented.

V. Kadets, J. Rodríguez and B. C., 2012

Theorem 4.1 *Suppose X has the RNP. Let $M : \Sigma \to \mathrm{bcc}(X)$ be a strong multimeasure of bounded variation with $M \ll \mu$. Then there is a Pettis integrable multifunction $F : \Omega \to \mathrm{bcc}(X)$ such that:*

(i) For every countably additive selector m of M there is a Bochner integrable selector of F such that $m(A) = \int_A f\, d\mu$ for all $A \in \Sigma$.

(ii) For every $A \in \Sigma$ the following equalities hold:

$$M(A) = \int_A F\, d\mu = \overline{\left\{\int_A f\, d\mu : f \text{ is a Bochner integrable selector of } F\right\}}.$$

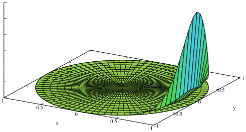
What you should take home with you.

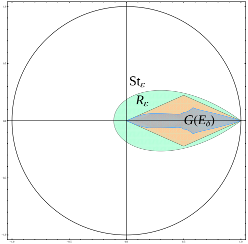
What you should take home with you

Our key Uryshon type lemma for $A(\overline{\mathbb{D}})$

Lemma 2.8. *Let $\mathfrak{A} \subset C(K)$ be a unital uniform algebra and Γ_0 its Choquet boundary. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0 < \varepsilon < 1$, there exist $f \in \mathfrak{A}$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and $f(K) \subset R_\varepsilon$. In particular,*

$$|f(t)| + (1-\varepsilon)|1 - f(t)| \leq 1, \text{ for all } t \in K. \tag{2.8}$$





Gracias