Universidad de Murcia | Departamento Matemáticas

Some new aspects of James' weak compactness theorem

B. Cascales

http://webs.um.es/beca

Function theory on infinite dimensional spaces XIII.
ICMAT, February 4th - 7th, 2014

Supported by GOBIERNO DE ESPAÑA MINISTERIO DE ECONOMÍA Y COMPETITIVIDAD Unión Europea Fondo Europeo de Desarrollo Regional

Contents:

2. Main part:
    - a few pictures explaining what we do;
    - a few contributions;

Contents:

1. Background
2. Main part:
   - a few pictures explaining what we do;
   - a few contributions;

Contents:

1. Background
2. Main part:
   - a few pictures explaining what we do;
   - a few contributions;
3. One last thing.

Background

James' theorem, 1964. Trans. Amer. Math. Soc.

WEAKLY COMPACT SETS

BY

ROBERT C. JAMES[1]

It has been conjectured that a closed convex subset C of a Banach space B is weakly compact if and only if each continuous linear functional on B attains a maximum on C [5]. This reduces easily to the case in which C is bounded, and will be answered in the affirmative [Theorem 4] after some preliminary results are established. Following suggestions by Namioka and Peck, the result is then generalized, first to weakly closed subsets of Banach spaces and then to weakly closed subsets of complete locally convex linear spaces.

THEOREM 4. *Let C be a bounded, closed, non-weakly-compact, convex subset of a Banach space B. Then there is a continuous linear functional defined on B that does not attain its sup on C.*

James'theorem

References

1. W. F. Eberlein, *Weak compactness in Banach spaces*. I, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 51–53.
2. R. C. James, *Reflexivity and supremum of linear functionals*, Ann. of Math. (2) **66** (1957), 159–169.
3. ———, *Characterizations of reflexivity*, Studia Math. **23** (1964), 205–216.
4. J. L. Kelley and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, N. J., 1963.
5. V. L. Klee, Jr., *A conjecture on weak compactness*, Trans. Amer. Math. Soc. **104** (1962), 398–402.
6. V. Šmulian, *On the principle of inclusion in the space of type* (B), Rec. Math. (Mat. Sb. (N. S.)) **5** (1939), 317–328.
7. J.W. Tukey, *Some notes on the separation of convex sets*, Portugal. Math. **3** (1942), 95–102.

Simons' inequality, 1972, Pac. J.

PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

A CONVERGENCE THEOREM WITH BOUNDARY

S. SIMONS

This paper contains a bounded-convergence type theorem that depends on the fact that certain functions attain their suprema. Among the applications discussed are Rainwater's theorem and two technical results, one used in the proof of the Choquet-Bishop-deLeeuw theorem and the other in the proof of Krein's Theorem.

3. THEOREM. *If the notation is as in Lemma 2 and μ is a linear functional on $l_\infty(X)$ dominated by S* (i.e., a positive linear functional of norm 1) *then*

$$\sup_{y\in Y} \limsup_{n\to\infty} f_n(y) \geqq \limsup_{n\to\infty} \mu(f_n)\ .$$

In particular, for all $x \in X$,

$$\sup_{y\in Y} \limsup_{n\to\infty} f_n(y) \geqq \limsup_{n\to\infty} f_n(x)\ .$$

Simons' inequality, 1972, Pac. J.

Theorem (Simons's theorem)

If E is a Banach space, $B \subset C$ are nonempty bounded subsets of E^* and (x_n) is a bounded sequence in E such that for every

$$x \in \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \text{ for all } n \geq 1,\ \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1 \right\}$$

there exists $b^* \in B$ with $\langle x, b^* \rangle = \sup\{\langle x, c^* \rangle : c^* \in C\}$, then

$$\sup\left\{ \limsup_n \langle x_n, b^* \rangle : b^* \in B \right\} = \sup\left\{ \limsup_n \langle x_n, c^* \rangle : c^* \in C \right\}.$$

Godefroy's theorem, 1987, Math. Ann.

Math. Ann. 277, 173–184 (1987)

Mathematische Annalen

Boundaries of a Convex Set and Interpolation Sets

Gilles Godefroy

Equipe d'Analyse, Université Paris 6 Tour 46-0, 4, Place Jussieu, F-75230 Paris Cedex 05, France
University of Missouri-Columbia, Department of Mathematics, Columbia, MO 65211, USA

0. Introduction. Notations

The structure of the "boundary" of a convex set C is a field of intensive research in functional analysis. The classical "boundary" is the set $\mathrm{Ext}(C)$ of the extreme points of C – at least when some compactness is assumed – and C is recovered from $\mathrm{Ext}(C)$ by means of the integral representation theory.

In this paper, a more general notion of boundary (Definition 1.1) is considered. Such a boundary needs not contain, or even meet, the extreme points, and thus the classical tools are not available. However, through R. C. James's technique and a remarquable result of S. Simons, a convex set can often be "recovered", in a strong sense, from its boundary (Sect. 1). Tight connections are established between this

Godefroy's theorem, 1987, Math. Ann.

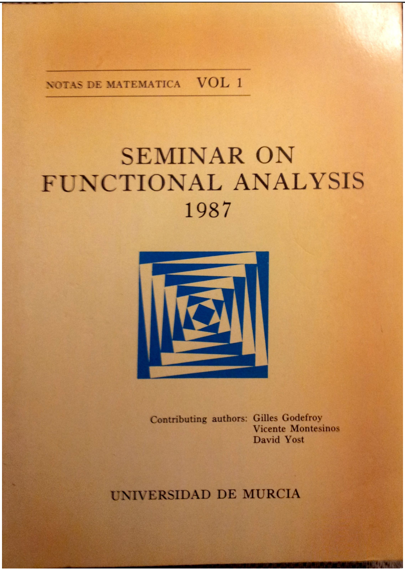
Definition I.1. *Let C be a closed bounded convex set in the dual E^* of a Banach space E, and let B be a subset of C. The set B is a boundary of C if:*

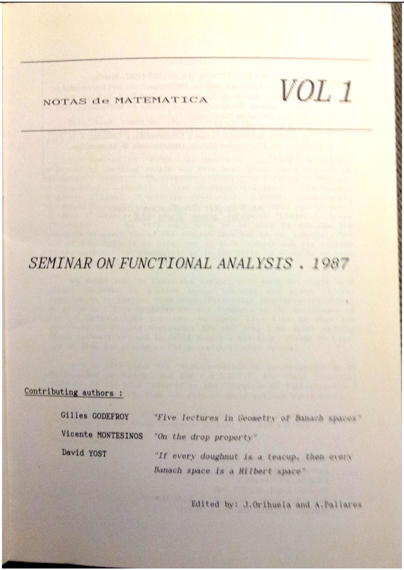
$$\forall x \in E, \quad \exists b \in B \quad \text{such that} \quad x(b) = \sup_C (x).$$

Our first result is a consequence of a deep result of Simons [26, Lemma 2]. According to [26], the roots of this result are lying in the works of James [13] and Pryce [21].

Theorem I.2. *Let K be a closed bounded convex set in E^*, and B a boundary of K. We assume that if C is a convex bounded set in E and $\varphi \in E^{**}$ is in the closure of C for the topology σ_B of pointwise convergence on B, there exists a sequence $(x_n)_{n \geqq 1}$ in C such that $\varphi = \lim_{n \to \infty} x_n$ for σ_B. Then K is w^*-compact and $K = \overline{\mathrm{cv}}''(B)$.*

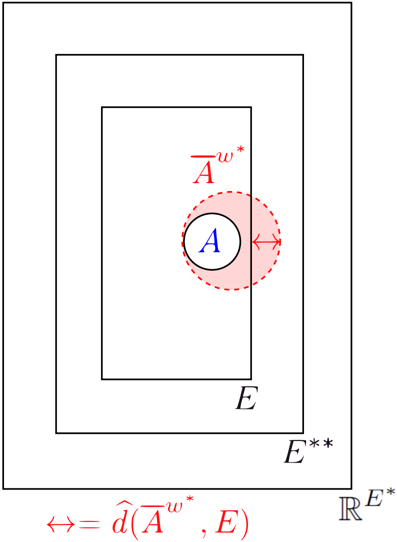
Theorem III.3. *Let K be a closed bounded convex set in the dual of a Banach space E. We assume that K has a boundary B, and that B is contained in a weakly $\mathscr{K}$-analytic subset A of E^*. Then K is w^*-compact and $K = \overline{\mathrm{cv}}''(B)$.*


NOTAS DE MATEMATICA VOL 1
SEMINAR ON FUNCTIONAL ANALYSIS 1987
Contributing authors: Gilles Godefroy
Vicente Montesinos
David Yost
UNIVERSIDAD DE MURCIA

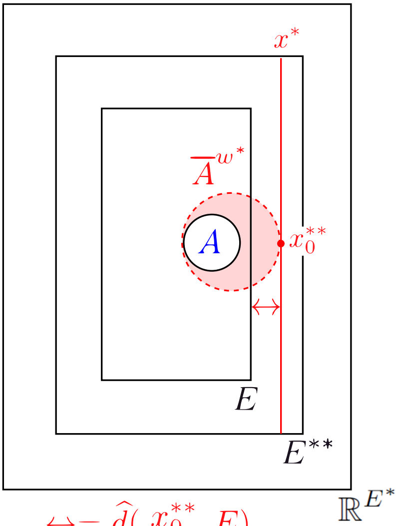

NOTAS de MATEMATICA VOL 1
SEMINAR ON FUNCTIONAL ANALYSIS . 1987
Contributing authors :
Gilles GODEFROY "Five lectures in Geometry of Banach spaces"
Vicente MONTESINOS "On the drop property"
David YOST "If every doughnut is a teacup, then every Banach space is a Hilbert space"
Edited by: J.Orihuela and A.Pallares

A few pictures explaining what we do

Tychonoff's theorem

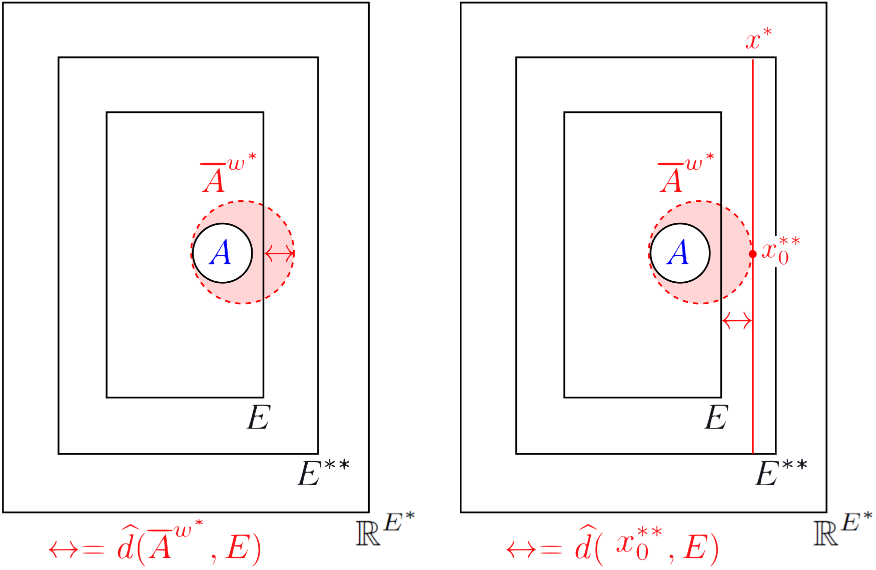


James' compactness theorem

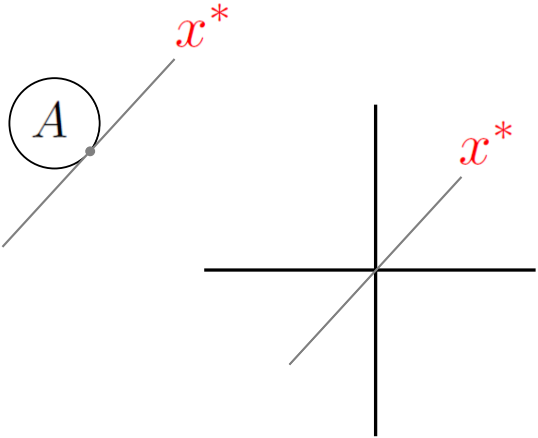


$$\leftrightarrow = \widehat{d}(\,x_0^{**}, E)$$

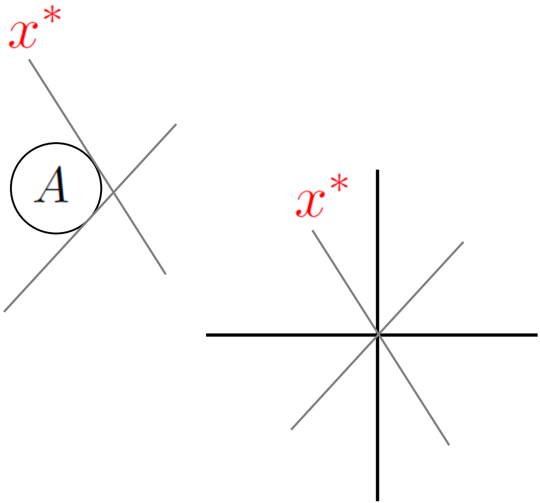
Tychonoff's theorem vs. James' compactness theorem



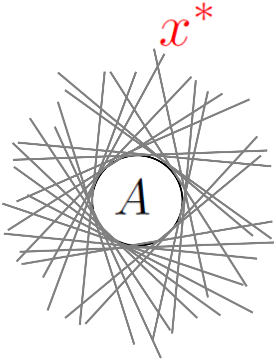
One side James theorem

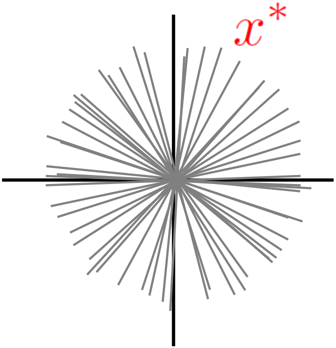


One side James theorem

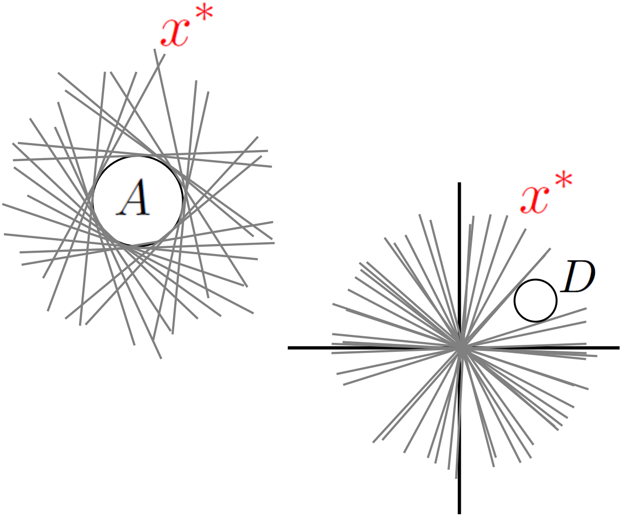


One side James theorem





One side James theorem



A few contributions

Co-authors

B. Cascales, O. F. K. Kalenda and J. Spurný. *A quantitative version of James's compactness theorem.* Proc. Edinb. Math. Soc. (2) 55 (2012), no. 2, 369–386.

B. Cascales and J. Orihuela. *One side James' compactness theorem.* Work in progress. Available at

Measures of weak non-compactness

Theorem 3.7. *For any bounded subset A of a Banach space E the following inequalities hold true*

$$\begin{array}{ccccccccc} & & & & & & \sigma(A) & \leq & 2\omega(A) \\ & & & & & \rotatebox{45}{$\leq$} & & & \rotatebox{90}{$\geq$} \\ \frac{1}{2}\gamma(A) & \leq & \mathrm{Ja}_{\mathrm{E}}(A) & \leq & \mathrm{ck}_E(A) & \leq & k(A) & \leq & \gamma(A). \end{array} \tag{3.6}$$

Moreover for any $x^{**} \in \overline{A}^{\mathrm{w}^*}$, there is a sequence $\{x_n\}_{n\geq 1}$ in A such that

$$\|x^{**} - y^{**}\| \leq \gamma(A) \tag{3.7}$$

for any w^*-cluster point y^{**} of $\{x_n\}_{n\geq 1}$ in E^{**}.

Furthermore, A is weakly relatively compact in E if, and only if, one (equivalently all) of the numbers $\gamma(A), \mathrm{Ja}_{\mathrm{E}}(A), \mathrm{ck}_E(A), \mathrm{k}(A), \sigma(A)$ and $\omega(A)$ is zero.

Angosto, Cascales, Ruiz Galán, Fabian, Kalenda, Hajek, Montesinos, Suarez Granero, Orihuela, Spurný, Zizler, 2005-2013

Definition

$$\omega(A) := \inf\{\varepsilon > 0 : A \subset K_\varepsilon + \varepsilon B_E \textit{ and } K_\varepsilon \subset E \textit{ is } \text{w}\textit{-compact}\},$$

$$\gamma(A) := \sup\{|\lim_n \lim_m x_m^*(x_n) - \lim_m \lim_n x_m^*(x_n)| : \{x_m^*\}_{m\geq 1} \subset B_{E^*}, \{x_n\}_{n\geq 1} \subset A\},$$

assuming the involved limits exist,

$$\mathrm{ck_E}(A) := \sup_{\{x_n\}_{n\geq 1} \subset A} d(L_{E^{**}}\{x_n\}, E),$$

$$\mathrm{k}(A) := \widehat{\mathrm{d}}(\overline{A}^{\mathrm{w}^*}, E) = \sup_{x^{**} \in \overline{A}^{\mathrm{w}^*}} d(x^{**}, E)$$

and

$$\begin{aligned} \mathrm{Ja_E}(A) = \inf\{\varepsilon > 0 :\ & \textit{for every } x^* \in E^*, \textit{ there is } x^{**} \in \overline{A}^{\mathrm{w}^*} \\ & \textit{such that } x^{**}(x^*) = S_A(x^*) \textit{ and } d(x^{**}, E) \leq \varepsilon\}. \end{aligned}$$

$$\sigma(A) := \sup_{\{x_n^*\}_{n\geq 1} \subset B_{E^*}} \mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}).$$

We followed the proof of

WEAK COMPACTNESS IN LOCALLY CONVEX SPACES

J. D. PRYCE

1. **Introduction.** A recently published paper of R. C. James [1] proves the following Theorem: A weakly closed set C in a Banach space B is weakly compact if and only if every bounded linear functional on B attains its supremum on C at some point of C. The proof given by James is rather long and involved: the following, while not employing any basically different ideas, is a simpler version and extends the theorem with no extra effort to deal with a locally convex linear topological space rather than a Banach space, using the Eberlein criterion for weak compactness (see e.g. [2, p. 159]).

Kalenda, Spurný and Cascales, 2012

Theorem 3.8 ([30, Theorem 6.1]). *Let E be a Banach space such that (B_{E^*}, w^*) is angelic. Then for any bounded set $A \subset E$ we have*

$$\frac{1}{2}\gamma(A) \leq \gamma_0(A) = \mathrm{Ja_E}(A) = \mathrm{ck_E}(A) = \mathrm{k}(A) \leq \gamma(A),$$

where

$$\gamma_0(A) = \sup\{|\lim_i \lim_j x_i^*(x_j)| : \{x_j\}_{j\geq 1} \subset A, \{x_i^*\}_{i\geq 1} \subset B_{E^*}, x_i^* \xrightarrow{w^*} 0\}.$$

One side James' compactness theorem

Orihuela-Cascales

Given $D \subset E$ we write

$$L_D(E^*) := \{x^* \in E^* : \langle x, x^* \rangle < 0, \text{ for every } x \in D\}.$$

Theorem 3 (One-side James compactness theorem). *Let E be a weakly countably K-determined Banach space and let A be a bounded, convex and closed subset of E. The following statements are equivalent:*

(i) *A is weakly compact;*

(ii) *there is a weakly compact set $D \subset E$ with $0 \notin D$ and with the property that every element of $L_D(E^*)$ attains its supremum on A.*

Theorem 2 (Unbounded Godefroy's Theorem). *Let E be a Banach space, B a nonempty subset of E^* and $D \subset E^*$ weakly compact. Let us assume that,*

(i) *for every $x \in E$ with $\langle x, d^* \rangle < 0$ for all $d^* \in D$ we have that*

$$\sup\{\langle x, c^* \rangle : c^* \in B\} = \langle x, b^* \rangle,$$

for some $b^* \in B$;

(ii) *for every convex bounded subset $L \subset E$ and every $x^{**} \in \overline{L}^{w^*} \subset E^{**}$ there is a sequence (y_n) in L such that $\langle x^{**}, z^* \rangle = \lim_n \langle y_n, z^* \rangle$ for every $z^* \in B \cup D$.*[2]

We have that,

(a) *if $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, then*

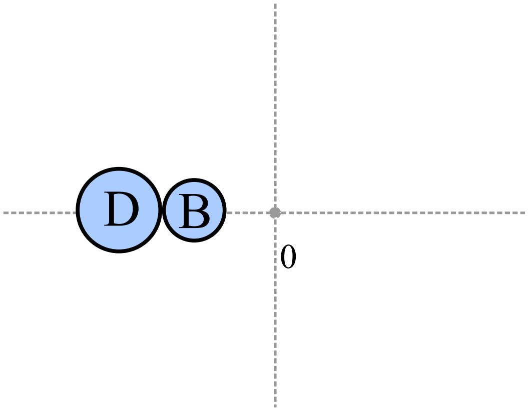
$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup \{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$

(b) *if B is bounded, weakly countably K-determined and $0 \notin D$,*[3] *then*[4]

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\mathrm{co}(B)}^{\|\cdot\|} + \bigcup \{\xi \overline{\mathrm{co}(D)}^{\|\cdot\|} : \xi \geq 0\}.$$

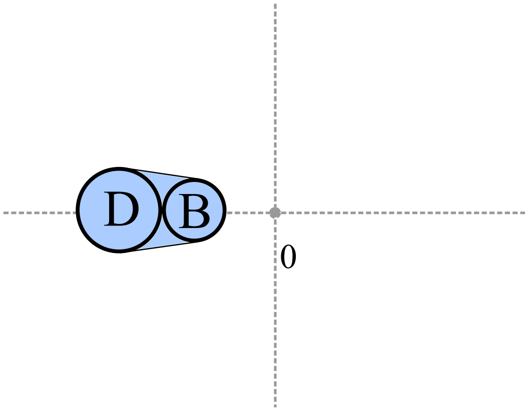
(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup\{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$



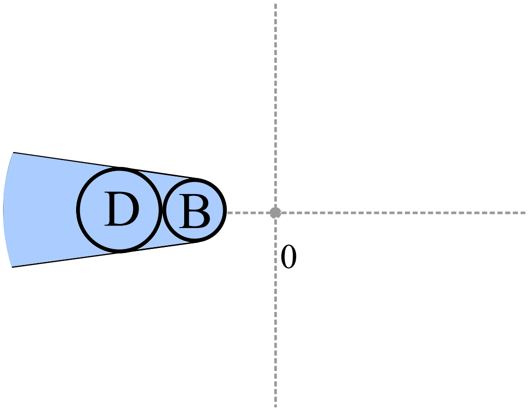
(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup \{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$



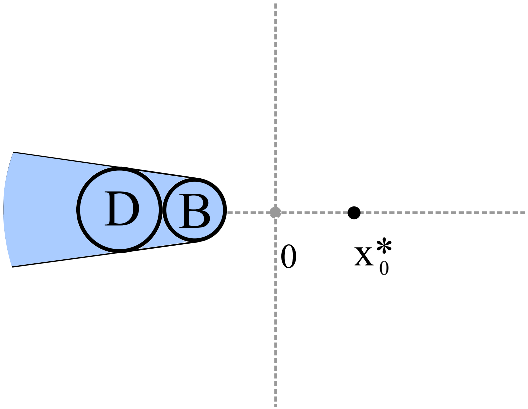
(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup \{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$



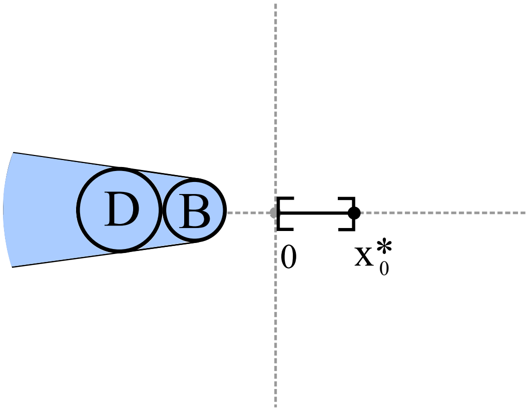
(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup\{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$



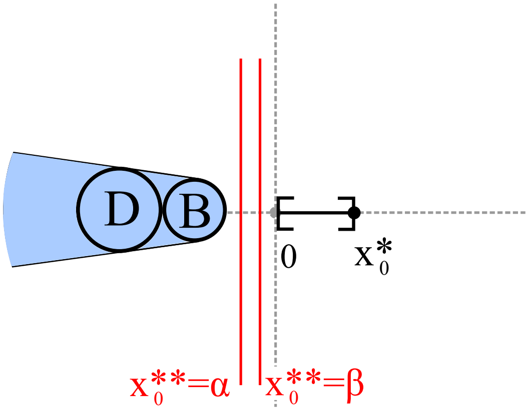
(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup\{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$



(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup \{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$



(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup\{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$

1. $\langle x_0^{**}, x^* \rangle < \alpha < \beta < \langle x_0^{**}, x_0^* \rangle$ for every $x^* \in H$ and $\beta < 0$.
2. Define $L := \{y \in B_E : \langle y, x_0^* \rangle > \beta\}$.
3. By Goldstine theorem, we have that $x_0^{**} \in \overline{L}^{w^*}$. Our assumptions imply that there is a sequence

   $$(x_n) \text{ in } L \text{ that converges to } x_0^{**} \text{ pointwise on } B \cup D.$$

4. Assume D is finite: then we can a assume that $\langle x_n, x^* \rangle < \alpha < 0$ for every $n \in \mathbb{N}$ and $x^* \in D$;
5. all convex series of (x_n) attains its maximum at B;

(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup\{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$

1. $\langle x_0^{**}, x^* \rangle < \alpha < \beta < \langle x_0^{**}, x_0^* \rangle$ for every $x^* \in H$ and $\beta < 0$.
2. Define $L := \{y \in B_E : \langle y, x_0^* \rangle > \beta\}$.
3. By Goldstine theorem, we have that $x_0^{**} \in \overline{L}^{w^*}$. Our assumptions imply that there is a sequence

Theorem (Simons's theorem)

If E is a Banach space, $B \subset C$ are nonempty bounded subsets of E^* and (x_n) is a bounded sequence in E such that for every

$$x \in \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \text{ for all } n \geq 1,\ \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1 \right\}$$

there exists $b^* \in B$ with $\langle x, b^* \rangle = \sup\{\langle x, c^* \rangle : c^* \in C\}$, then

$$\sup\left\{ \limsup_n \langle x_n, b^* \rangle : b^* \in B \right\} = \sup\left\{ \limsup_n \langle x_n, c^* \rangle : c^* \in C \right\}.$$

(a) *if* $0 \notin \overline{\mathrm{co}(B \cup D)}^{\|\cdot\|}$, *then*

$$\overline{\mathrm{co}(B)}^{w^*} \subset \overline{\bigcup\{\lambda \mathrm{co}(B \cup D) : \lambda \in [1, +\infty)\}}^{\|\cdot\|}.$$

1. $\langle x_0^{**}, x^* \rangle < \alpha < \beta < \langle x_0^{**}, x_0^* \rangle$ for every $x^* \in H$ and $\beta < 0$.
2. Define $L := \{y \in B_E : \langle y, x_0^* \rangle > \beta\}$.
3. By Goldstine theorem, we have that $x_0^{**} \in \overline{L}^{w^*}$. Our assumptions imply that there is a sequence

   (x_n) in L that converges to x_0^{**} pointwise on $B \cup D$.

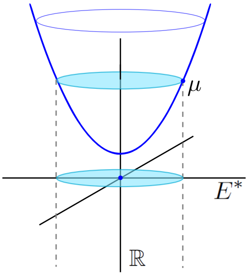
4. Assume D is finite: then we can a assume that $\langle x_n, x^* \rangle < \alpha < 0$ for every $n \in \mathbb{N}$ and $x^* \in D$;
5. all convex series of (x_n) attains its maximum at B;
6. Simons inequality apply to obtain that

$$\sup\{\limsup_n \langle x_n, b^* \rangle : b^* \in B\} = \sup\{\limsup_n \langle x_n, c^* \rangle : c^* \in \overline{\mathrm{co}(B)}^{w^*}\}.$$

7. The conditions imply that the previous equality cannot hold.

One last thing.

1. This study has been done as a consequence of a question by Delbaen for $L^1(\mu)$.
2. There are non-bounded versions of the results: one needs for that of an unbounded type Simons theorem proved by Ruiz Galán and Orihuela.
3. There are applications of the unbounded cases to convex functions.



B. Cascales, J. Orihuela and M. Ruiz Galán. *Compactness, optimality and Risk* Computational and Analytical Mathematics. Edited by D. Bailey, H.H. Bauschke, P. Borwein, F. Garvan, M. Therá, J.D. Vanderwerff and H.Wolkovicz. Springer, Chapter 10, 153–207, (2013).

THANKS