Universidad de Murcia | Departamento Matemáticas

Brondsted-Rockafellar variational principle, Asplundness and operators attaining their norm

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Stay focused

- Kind of problems studied. Credit to co-authors.
- Bishop-Phelps property.
- Bishop-Phelps-Bollobás property.
- Our results: ingredients, proofs and applications.

What kind of problem are we going to talk about?

A BISHOP-PHELPS-BOLLOBAS TYPE THEOREM FOR UNIFORM ALGEBRAS

B. CASCALES, A. J. GUIRAO AND V. KADETS

1. INTRODUCTION

Mathematical optimization is associated to maximizing or minimizing real functions. James's compactness theorem [17] and Bishop-Pehlps's theorem [5] are two landmark results along this line in functional analysis. The former characterizes reflexive Banach spaces X as those for which continuous linear functionals $x^* \in X^*$ attain their norm in the unit sphere S_X. The latter establishes that for any Banach space X every continuous linear functional $x^* \in X^*$ can be approximated (in norm) by linear functionals that attain the norm in S_X. This paper is concerned with the study of a strengthening of Bishop-Phelps's theorem that mixes ideas of Bollobás [6] –see Theorem 3.1 here– and Lindenstrauss [21] –who initiated the study of the Bishop-Phelps property for bounded operators between Banach spaces.

The problem for $x^* : X \to \mathbb{R}$ form and $T : X \to Y$ operator

$$\|x^*\| = \sup\{|x^*(x)| : \|x\| = 1\} \overset{\text{not always}}{=} \max\{|x^*(x)| : \|x\| = 1\}$$

$$\|T\| = \sup\{\|T(x)\| : \|x\| = 1\} \overset{\text{not always}}{=} \max\{\|T(x)\| : \|x\| = 1\}$$

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A first glance to our result

Our paper is devoted to showing that Asplund operators with range in a uniform Banach algebra have the Bishop-Phelps-Bollobás property, *i.e.*, they are approximated by norm attaining Asplund operators at the same time that a point where the approximated operator almost attains its norm is approximated by a point at which the approximating operator attains it. To prove this result we establish a Uryshon type lemma producing peak complex-valued functions in uniform algebras that are small outside a given open set and whose image is inside a symmetric rhombus with main diagonal $[0, 1]$ and small height.

Credit to co-authors



R. M. Aron, B. Cascales and O. Kozhushkina, *The Bishop-Phelps-Bollobas theorem and Asplund operators*, Proc. Amer. Math. Soc. 139 (2011), no. 10, 3553–3560.

B. Cascales, A. J. Guirao and V. Kadets, *A Bishop-Phelps-Bollobás type theorem for uniform algebras*, Enviado para publicación 19/Abril/2012

Bishop-Phelps theorem

Theorem (Bishop-Phelps, 1961)

If X is a Banach, then $\overline{\mathbf{NA}X^*} = X^*$.

A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere S of E are norm-dense in E^*, i.e., if for each f in E^* and each $\epsilon>0$ there exist g in E^* and x in S such that $|g(x)| = \|g\|$ and $\|f-g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1][1] as well as incomplete spaces which *are* subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is sub-

The Bishop-Phelps property for operators

Definition

An operator $T : X \to Y$ is **norm attaining** if there exists $x_0 \in X$, $\|x_0\| = 1$, such that $\|T(x_0)\| = \|T\|$.

Definition (Lindenstrauss)

(X, Y) has the Bishop-Phelps Property (BPp) if every operator $T : X \to Y$ can be uniformly approximated by **norm attaining** operators.

1. $(X, \mathbb{K})$ has BPp for every X Bishop-Phelps (1961);
2. $\overline{\{T \in L(X;Y) : T^{**} \in NA(X^{**};Y^{**})\}} = L(X;Y)$ for every pair of Banach spaces X and Y, Lindenstrauss (1963);
3. X with RNP, then (X, Y) has BPp for every Y, Bourgain (1977);
4. there are spaces X, Y and Z such that $(X, C([0,1]))$, (Y, ℓ^p) $(1 < p < \infty)$ and $(Z, L^1([0,1]))$ fail BPp, Schachermayer (1983), Gowers (1990) and Acosta (1999);
5. $(C(K), C(S))$ has BPp for all compact spaces K, S, Johnson and Wolfe, (1979).
6. $(L^1([0,1]), L^\infty([0,1]))$ has BPp, Finet-Payá (1998),

Bollobás observation

AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

Bishop and Phelps proved in **[1]** that every real or complex Banach space is *subreflexive*, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by S and S' t

respectively.

THEOREM 1. *Suppose*

exist $y \in S$ *and* $g \in S'$ *such*

Corollary. . . the way it is oftentimes presented

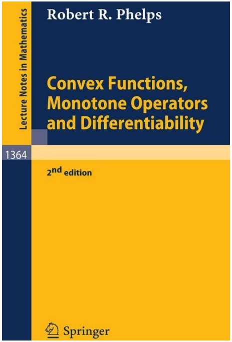
Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$

A variational principle implying BPB



Theorem 3.17 (Brøndsted-Rockafellar). *Suppose that f is a convex proper lower semicontinuous function on the Banach space E. Then given any point $x_0 \in \operatorname{dom}(f)$, $\epsilon > 0$, $\lambda > 0$ and any $x_0^* \in \partial_\epsilon f(x_0)$, there exist $x \in \operatorname{dom}(f)$ and $x^* \in E^*$ such that*

$$x^* \in \partial f(x), \quad \|x - x_0\| \leq \epsilon/\lambda \text{ and } \|x^* - x_0^*\| \leq \lambda.$$

In particular, the domain of ∂f is dense in $\operatorname{dom}(f)$.

Corollary. . . the constants are better

Given $1 > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\varepsilon^2}{2},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

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Bishop-Phelps-Bollobás Property for operators

Definition: Acosta, Aron, García and Maestre, 2008

(X,Y) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any $\varepsilon > 0$ there are $\eta(\varepsilon) > 0$ such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

then there are $u_0 \in S_X$, $S \in S_{L(X,Y)}$ with

$$\|S(u_0)\| = 1$$

and

$$\|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| < \varepsilon.$$

1. Y has *certain* almost-biorthogonal system (X,Y) has BPBp any X;
2. (ℓ^1, Y) BPBp is characterized through a condition called AHSP: it holds for Y finite dimensional, uniformly convex, $Y = L^1(\mu)$ for a σ-finite measure or $Y = C(K)$;
3. there is pair (ℓ^1, X) failing BPBp, but having BPp;
4. (ℓ^∞_n, Y) has BPBp Y uniformly convex no hope for c_0: $\eta(\varepsilon) = \eta(n,\varepsilon) \to 1$ with $n \to \infty$.

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PROBLEM?

No Y infinite dimensional was known s.t. (c_0, Y) has BPBP.

Our main result

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T\colon X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

Asplund operators

Stegall, 1975

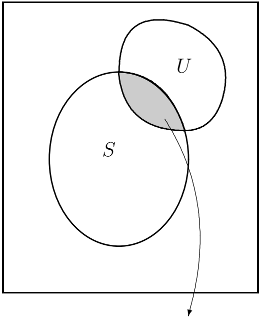
An **operator** $T \in L(X, Y)$ is **Asplund**, if it factors through an Asplund space:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "T_1"'] & & Y \\ & Z \arrow[ur, "T_2"'] & \end{tikzcd}$$

Z is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

T Asplund operator $\Leftrightarrow$ $T^*(B_{Y^*})$ is fragmented by the norm of X^*.

The black box... fragmentability



$\|\cdot\| - \text{diam}(U \cap S) \leq \varepsilon$

Asplund spaces: Namioka, Phelps and Stegall

Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space, *i.e.*, whenever f is a convex continuous function defined on an open convex subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U.

(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

An idea of the proof for $\mathfrak{A} = C(K)$

Theorem (R. M. Aron, O. Kozhushkina and B. C. 2011)

Let $T: X \to C(K)$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,C(K))}$ satisfying that

$$\|\widetilde{T} u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

1. **Black box** provides a suitable open set $U \subset K$, $y^* \in S_{X^*}$ and $\rho < 2\varepsilon$ with
$$1 = |y^*(u_0)| = \|u_0\| \text{ and } \|x_0 - u_0\| < \varepsilon \;\&\; \|T^*(\delta_t) - y^*\| < \rho \;\forall t \in U$$
2. Uryshon's lemma that applied to an arbitrary $t_0 \in U$ produces a function $f \in C(K)$ satisfying
$$f(t_0) = \|f\|_\infty = 1, f(K) \subset [0,1] \text{ and } \operatorname{supp}(f) \subset U.$$
3. $\widetilde{T}$ is explicitly defined by
$$\widetilde{T}(x)(t) = f(t) \cdot y^*(x) + (1 - f(t)) \cdot T(x)(t), x \in X, t \in K,$$
4. The suitability of U is used to prove that $\|T - \widetilde{T}\| < 2\varepsilon$.

An idea of the proof for $\mathfrak{A} = A(\overline{\mathbb{D}})$

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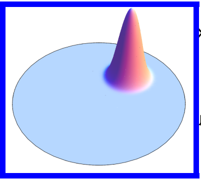

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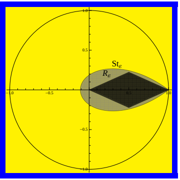

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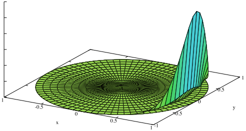
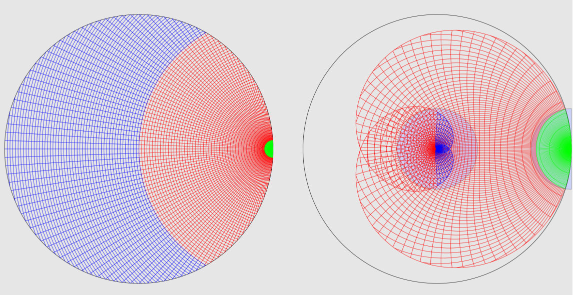
Our key Uryshon type lemma for $A(\overline{\mathbb{D}})$

Lemma 2.8. *Let $\mathfrak{A} \subset C(K)$ be a unital uniform algebra and Γ_0 its Choquet boundary. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0 < \varepsilon < 1$, there exist $f \in \mathfrak{A}$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and $f(K) \subset R_\varepsilon$. In particular,*

$$|f(t)| + (1-\varepsilon)|1 - f(t)| \leq 1, \text{ for all } t \in K. \tag{2.8}$$





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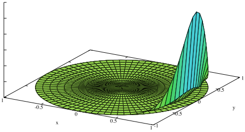

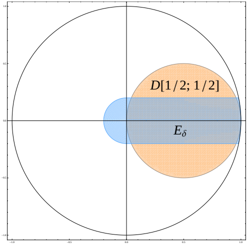

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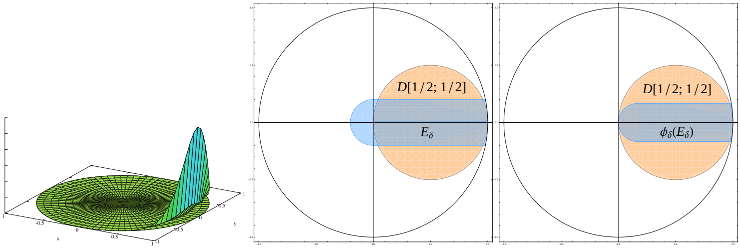

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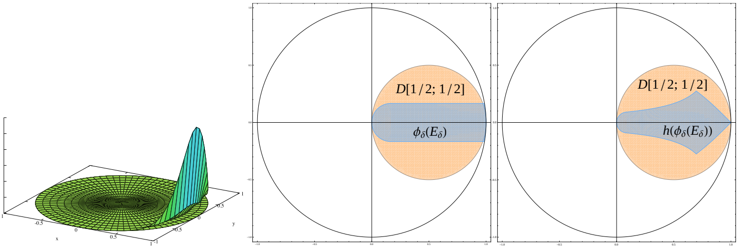

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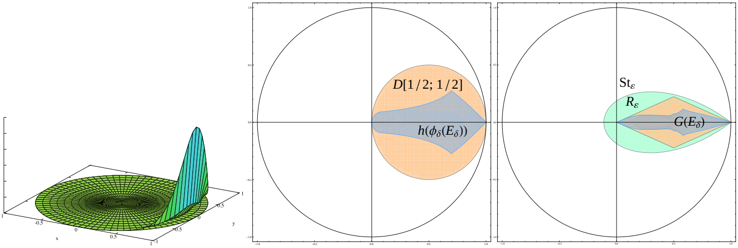

Our Uryshon type lemma is suited for calculations with a computer.

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon$$

and

$$\|T - \widetilde{T}\| < 2\varepsilon.$$

Corollary

Let $T \in L(X, C_0(L))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, C_0(L))$ **weakly compact** with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

Corollary

$(X, C_0(L))$ has the BPBP for any Asplund space X and any locally compact Hausdorff topological space L ($X = c_0(\Gamma)$, **for instance**).

Corollary

$(X, C_0(L))$ has the BPBP for any X and any scattered locally compact Hausdorff topological space L.

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \le \varepsilon$$

and

$$\|T - \widetilde{T}\| < 2\varepsilon.$$

Corollary

Let $T \in L(X, A(\overline{\mathbb{D}}))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, A(\overline{\mathbb{D}}))$ **weakly compact** with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \le 3\varepsilon.$$

Remark

The theorem applies in particular to the ideals of finite rank operators $\mathscr{F}$, compact operators $\mathscr{K}$, p-summing operators Π_p and of course to the weakly compact operators $\mathscr{W}$ themselves. To the best of our knowledge even in the case $\mathscr{W}(X, \mathfrak{A})$ the Bishop-Phelps property that follows is a brand new result.

GRACIAS!