Universidad de Murcia | Departamento Matemáticas

A complex Urysohn type lemma with applications

B. Cascales

http://webs.um.es/beca

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Stay focused

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    - Our result(s) in Functional Analysis needs the lemma.

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6. What should you take home with you?.

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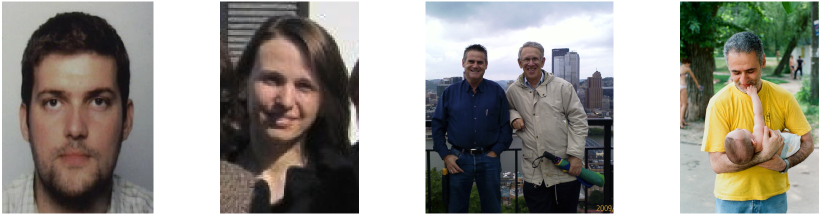

R. M. Aron, B. Cascales and O. Kozhushkina,
The Bishop-Phelps-Bollobás theorem and Asplund operators,
Proc. Amer. Math. Soc. 139 (2011), no. 10, 3553–3560.

B. Cascales, A. J. Guirao and V. Kadets,
A Bishop-Phelps-Bollobás type theorem for uniform algebras,
Preprint. April/2012

Topology behind the scenes

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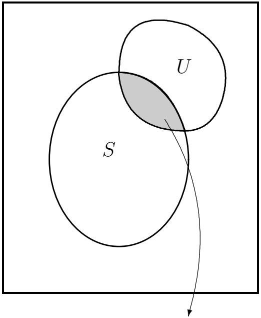


$\|\cdot\| - \mathrm{diam}(U \cap S) \leq \varepsilon$

Asplund spaces: Namioka, Phelps and Stegall

Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space, *i.e.*, whenever f is a convex continuous function defined on an open convex subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U.

(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

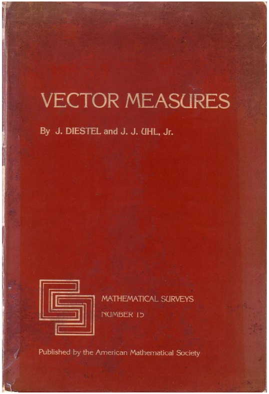
(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

To learn about

RNP in Banach spaces

read



VECTOR MEASURES

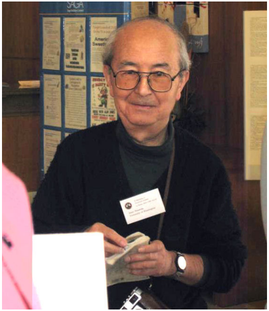
By J. DIESTEL and J. J. UHL, Jr.

MATHEMATICAL SURVEYS
NUMBER 15

Published by the American Mathematical Society

Fragmentability

Google **NAMIOKA**



Our result(s) in functional analysis: the need of the lemma

Our result(s)

Theorem (A. J. Guirao, V. Kadets and B. C. 2012)

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T\colon X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \le \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

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Just for the students

Uniform algebra

1. Closed vector subspace $A \subset C(K)$, with the properties:
    - the products of functions in A remains in A;
    - $1 \in A$.
    - A separates the points of K.

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2. In the real case, if A uniform algebra $\Rightarrow A = C(K)$.
3. The complex case is different: the disk algebra $A(\mathbb{D})$. Why the complex case can be different from the real case?

Asplund operators

Stegall, 1975

An **operator** $T \in L(X, Y)$ is **Asplund**, if it factors through an Asplund space:

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "T_1"'] & & Y \\
& Z \arrow[ur, "T_2"'] &
\end{tikzcd}$$

Z is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

T Asplund operator $\Leftrightarrow$ $T^*(B_{Y^*})$ is fragmented by the norm of X^*.

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and

$$\|T - \widetilde{T}\| < 2\varepsilon.$$

Corollary

Let $T \in L(X,\mathfrak{A})$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X,\mathfrak{A})$ **weakly compact** with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 2\varepsilon.$$

Corollary

$(X,\mathfrak{A})$ has the BPBP for any Asplund space X and any locally compact Hausdorff topological space L ($X = c_0(\Gamma)$**, for instance**).

Corollary

$(X, C_0(L))$ has the BPBP for any X and any scattered locally compact Hausdorff topological space L.

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Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T : X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

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Corollary

Let $T \in L(X, A(\mathbb{D}))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

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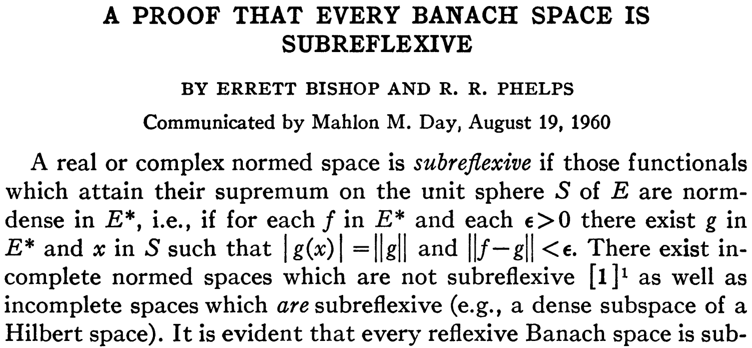
Remark

The theorem applies in particular to the ideals of finite rank operators $\mathscr{F}$, compact operators $\mathscr{K}$, p-summing operators Π_p and of course to the weakly compact operators $\mathscr{W}$ themselves. To the best of our knowledge even in the case $\mathscr{W}(X, \mathfrak{A})$ the Bishop-Phelps property that follows is a brand new result.

Bishop-Phelps theorem

Theorem (Bishop-Phelps, 1961)

If X is a Banach, then $\overline{\mathbf{NA}X^*} = X^*$.



A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere S of E are norm-dense in E^*, i.e., if for each f in E^* and each $\epsilon > 0$ there exist g in E^* and x in S such that $|g(x)| = \|g\|$ and $\|f - g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1][1] as well as incomplete spaces which *are* subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is sub-

The Bishop-Phelps property for operators

Definition

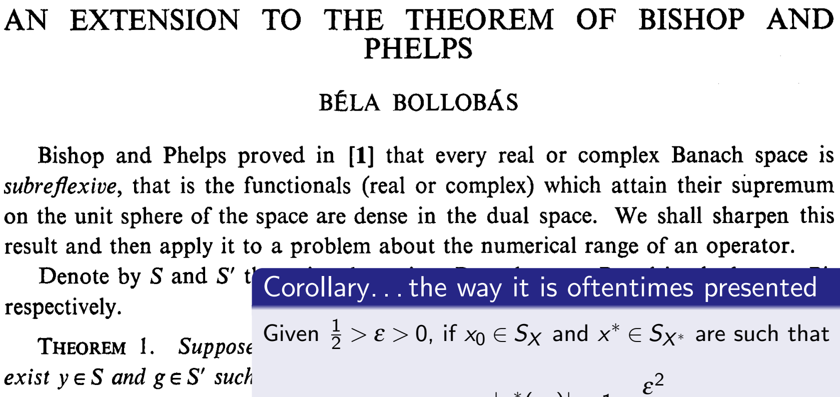
An operator $T : X \to Y$ is **norm attaining** if there exists $x_0 \in X$, $\|x_0\| = 1$, such that $\|T(x_0)\| = \|T\|$.

Definition (Lindenstrauss)

(X, Y) has the Bishop-Phelps Property (BPp) if every operator $T : X \to Y$ can be uniformly approximated by **norm attaining** operators.

1. $(X, \mathbb{K})$ has BPp for every X Bishop-Phelps (1961);
2. $\overline{\{T \in L(X; Y) : T^{**} \in NA(X^{**}; Y^{**})\}} = L(X; Y)$ for every pair of Banach spaces X and Y, Lindenstrauss (1963);
3. X with RNP, then (X, Y) has BPp for every Y, Bourgain (1977);
4. there are spaces X, Y and Z such that $(X, C([0, 1]))$, (Y, ℓ^p) $(1 < p < \infty)$ and $(Z, L^1([0, 1]))$ fail BPp, Schachermayer (1983), Gowers (1990) and Acosta (1999);
5. $(C(K), C(S))$ has BPp for all compact spaces K, S, Johnson and Wolfe, (1979).
6. $(L^1([0, 1]), L^\infty([0, 1]))$ has BPp, Finet-Payá (1998),

Bollobás observation, 1970



AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

Bishop and Phelps proved in **[1]** that every real or complex Banach space is *subreflexive*, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by S and S' t

respectively.

THEOREM 1. *Suppose*

exist $y \in S$ *and* $g \in S'$ *such*

Corollary. . . the way it is oftentimes presented

Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$

Bishop-Phelps-Bollobás Property for operators

Definition: Acosta, Aron, García and Maestre, 2008

(X,Y) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any $\varepsilon > 0$ there are $\eta(\varepsilon) > 0$ such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

then there are $u_0 \in S_X$, $S \in S_{L(X,Y)}$ with

$$\|S(u_0)\| = 1$$

and

$$\|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| < \varepsilon.$$

1. Y has *certain* almost-biorthogonal system (X,Y) has BPBp any X;
2. (ℓ^1, Y) BPBp is characterized through a condition called AHSP: it holds for Y finite dimensional, uniformly convex, $Y = L^1(\mu)$ for a σ-finite measure or $Y = C(K)$;
3. there is pair (ℓ^1, X) failing BPBp, but having BPp;
4. (ℓ^∞_n, Y) has BPBp Y uniformly convex no hope for c_0: $\eta(\varepsilon) = \eta(n, \varepsilon) \to 1$ with $n \to \infty$.

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PROBLEM?

No Y infinite dimensional was known s.t. (c_0, Y) has BPBP.

An idea of the proof for $\mathfrak{A} = C(K)$

Theorem (R. M. Aron, O. Kozhushkina and B. C. 2011)

Let $T\colon X \to C(K)$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,C(K))}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \le \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

1. **Topological tools** provide a suitable open set $U \subset K$, $y^* \in S_{X^*}$ and $\rho < 2\varepsilon$ with

$$1 = |y^*(u_0)| = \|u_0\| \text{ and } \|x_0 - u_0\| < \varepsilon \;\&\; \|T^*(\delta_t) - y^*\| < \rho \;\forall t \in U$$

2. Uryshon's lemma that applied to an arbitrary $t_0 \in U$ produces a function $f \in C(K)$ satisfying

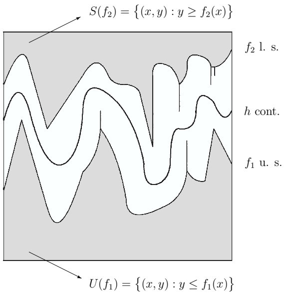
$$f(t_0) = \|f\|_\infty = 1, f(K) \subset [0,1] \text{ and } \operatorname{supp}(f) \subset U.$$

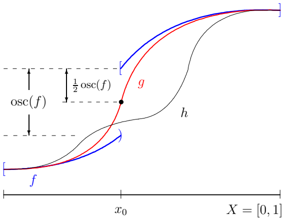
3. $\widetilde{T}$ is explicitly defined by

$$\widetilde{T}(x)(t) = f(t)\cdot y^*(x) + (1 - f(t))\cdot T(x)(t), x \in X, t \in K,$$

4. The suitability of U is used to prove that $\|T - \widetilde{T}\| < 2\varepsilon$.

Just for the students: A few words about Uryshon's lemma





Normal spaces - Exercise for students

For a topological space T the following statements are equivalent:

1. T is normal.
2. Urysohn's lemma holds for T.
3. Tietze's extension theorem holds for T.
4. Katetov's *"sandwich"* theorem holds for T.
5. For every function $f \in \mathbb{R}^T$ the distance to $C_b(T)$ is given by

$$d(f, C_b(T)) = \frac{1}{2}\,\mathrm{osc}(f).$$

An idea of the proof for $\mathfrak{A} = A(\mathbb{D})$

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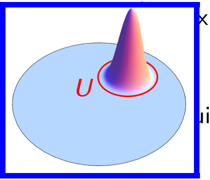

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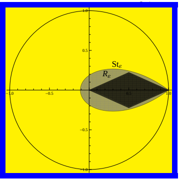

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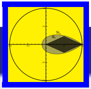
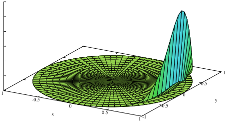
Complex Urysohn type lemma: a few pictures

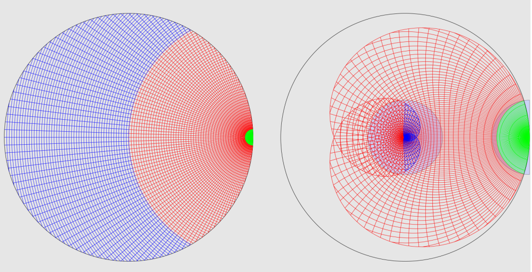
Uryshon type lemma for $A(\mathbb{D})$



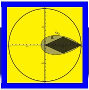
$U \overset{open}{\subset} \overline{\mathbb{D}}, 1 \in U, \exists f(1) = \|f\|_\infty = 1, f(\overline{\mathbb{D}}) \subset R_{\varepsilon'}$ and f small in $\overline{\mathbb{D}} \setminus U$,

$$f_n(z) := \left(\frac{z+1}{2}\right)^n$$





Uryshon type lemma for $A(\mathbb{D})$

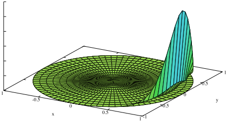


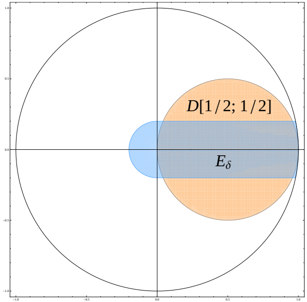
$$U \overset{open}{\subset} \overline{\mathbb{D}}, 1 \in U, \exists f(1) = \|f\|_\infty = 1,\ f(\overline{\mathbb{D}}) \subset R_{\varepsilon'} \text{ and } f \text{ small in } \overline{\mathbb{D}} \setminus U,$$

$$f_n(z) := \left(\frac{z+1}{2}\right)^n$$

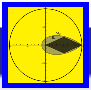
$$\exists\, n_1, n_2, \ldots, n_k$$

$$f := \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_2}}{k}$$



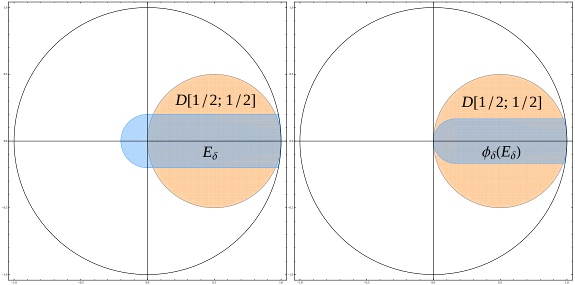


Uryshon type lemma for $A(\mathbb{D})$

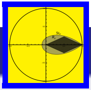


$U \overset{open}{\subset} \overline{\mathbb{D}}, 1 \in U, \exists f(1) = \|f\|_\infty = 1, f(\overline{\mathbb{D}}) \subset R_{\varepsilon'}$ and f small in $\overline{\mathbb{D}} \setminus U$,

$$z \to \frac{z+\delta}{1+\delta}$$

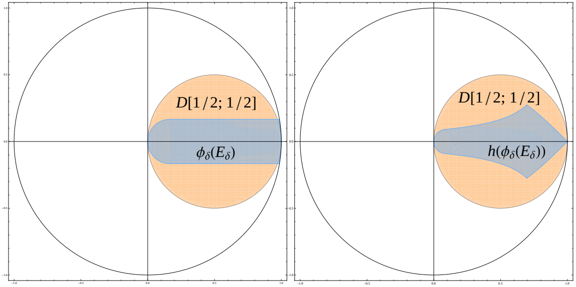


Uryshon type lemma for $A(\mathbb{D})$

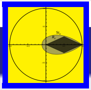


$U \overset{open}{\subset} \overline{\mathbb{D}}, 1 \in U, \exists f(1) = \|f\|_\infty = 1, f(\overline{\mathbb{D}}) \subset R_{\varepsilon'}$ and f small in $\overline{\mathbb{D}} \setminus U$,

$$z \to 1 - \sqrt{1-z}$$

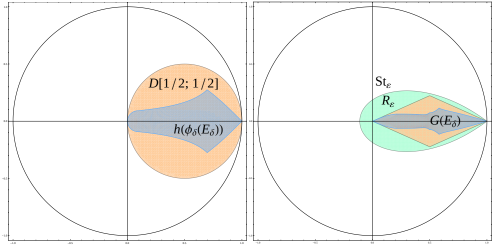


Uryshon type lemma for $A(\mathbb{D})$



$U \overset{open}{\subset} \overline{\mathbb{D}}, 1 \in U, \exists f(1) = \|f\|_\infty = 1, f(\overline{\mathbb{D}}) \subset R_{\varepsilon'}$ and f small in $\overline{\mathbb{D}} \setminus U$,

Riemann conformal mapping



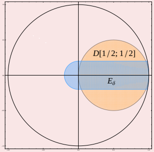
Our Uryshon type lemma is suited for calculations with a computer.

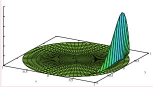
Why can we use the computer? Because of the provided proofs.

$$f_n(z) := \left(\frac{z+1}{2}\right)^n$$

$$\exists\, n_1, n_2, \ldots, n_k$$

$$f := \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_2}}{k}$$





Proof. Let us fix $0 < \eta < \min\{\delta/6, 1/2\}$ and $n \in \mathbb{N}$, $n > \frac{2}{\eta}$. Let us define $U_1 := U$. We shall construct inductively a collection of points $\{t_j\}_{j=1}^{n}$, a decreasing finite sequence $\{U_j\}_{j=1}^{n+1}$ of open subsets of U, $t_j \in U_{j+1} \cap \Gamma_0$, and functions $\{f_j\}_{j=1}^{n} \subset A$, satisfying for any $j \in \{1, \ldots, n\}$ the following conditions:

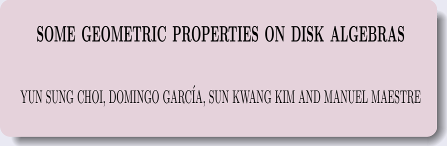
(i) $f_j(t_j) = 1$.
(ii) $|f_j(t)| < \frac{\eta}{2}$ for $t \in K \setminus U_j$.
(iii) $|f_j(t) - 1| < \frac{\eta}{2}$ for $t \in U_{j+1}$.

Indeed, Lemma 2.1 allows us to find a norm one function $f_1 \in A$ and a $t_1 \in U_1 \cap \Gamma_0$ such that $f_1(t_1) = 1$ and $|f_1(t)| < \frac{\eta}{2}$ for $t \in K \setminus U_1 \ldots$ THE PROOF GOES ON

$\ldots$

1+4+1 applications

- (1) The one already presented for the BPBp by Guirao-Kadets-Cascales;
- In the unfinished paper



SOME GEOMETRIC PROPERTIES ON DISK ALGEBRAS

YUN SUNG CHOI, DOMINGO GARCÍA, SUN KWANG KIM AND MANUEL MAESTRE

- (2) Daugavet prop.;
- (3) Lusheness;
- (4) Numerical radius;
- (5) AHSP.

- (6) in the Ph. dissertation by O. Kozhushkina for BPBp for $\mathfrak{A}(K, Y)$

What should you take home with you?

What should you take home with you?

The grad students should take home the exercises, at least!!!

Thank you