Universidad de Murcia | Departamento Matemáticas

Topology and functional analysis

B. Cascales

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Meeting in Topology and Functional Analysis, Elx 27-28 Sep., 2013
(On the occasion of the 60th birthday of Jerzy Kakol)

Supported by GOBIERNO DE ESPAÑA MINISTERIO DE ECONOMÍA Y COMPETITIVIDAD Unión Europea Fondo Europeo de Desarrollo Regional

Contents:

2. Mathematical part:
   - old results;
   - newer results: a few indexes;
   - recent results: different indexes.

Contents:

1. My congrats and wishes: something to start with
2. Mathematical part:
   - old results;
   - newer results: a few indexes;
   - recent results: different indexes.

Contents:

1. My congrats and wishes: something to start with
2. Mathematical part:
   - old results;
   - newer results: a few indexes;
   - recent results: different indexes.
3. One last thing

Congrats and wishes

My congrats and wishes

1. Congrats on your 60th birthday.

My congrats and wishes

1. Congrats on your 60th birthday.
2. I wish you many more quality years.

My congrats and wishes

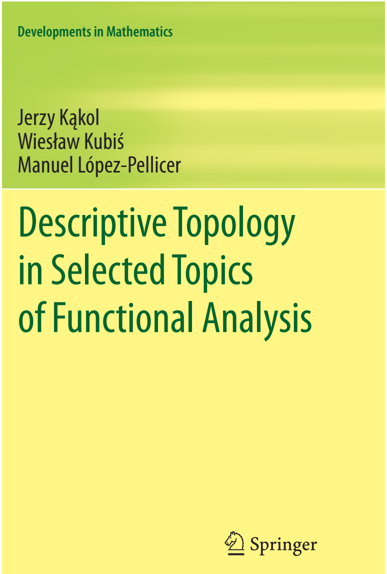
1. Congrats on your 60th birthday.
2. I wish you many more quality years.
3. I wish you mathematical recognition for your research.

My congrats and wishes

1. Congrats on your 60th birthday.
2. I wish you many more quality years.
3. I wish you mathematical recognition for your research.
4. I also wish you that people at your place express you recognition and respect for your teaching and the help that you gave them.

Something to start with

The book by Kąkol-Kubiś-López Pellicer



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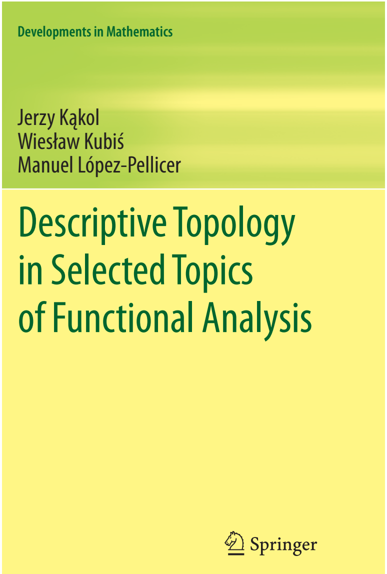
The book by Kąkol-Kubiś-López Pellicer



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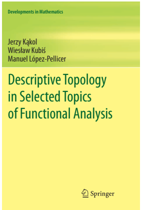
La Gaceta de la RSME, Vol. 15 (2012), Núm. 4, Págs. 9999–10002 (provisional) 9999

Reseña de libros

«Descriptive Topology in Selected Topics of Functional Analysis», de Jerzy Kąkol, Wiesław Kubiś y Manuel López-Pellicer



Título: Descriptive Topology in Selected Topics of Functional Analysis
Autores: Jerzy Kąkol, Wiesław Kubiś y Manuel López-Pellicer
Editorial: Springer, New York (Developments in Mathematics, 24)
Fecha de publicación: 2011
Páginas: xii+493
ISBN: 978-1-4614-0528-3

La *teoría descriptiva de conjuntos* es la parte de las Matemáticas que estudia determinadas clases de subconjuntos que originalmente fueron considerados en la recta real (borelianos, analíticos, etc.) y que desde hace ya bastante tiempo se ha extendido, por sus numerosas aplicaciones, al estudio de clases distinguidas de subconjuntos en espacios topológicos más generales, como los espacios polacos. La *teoría descriptiva de conjuntos* es un área de investigación muy activa que proporciona herramientas con aplicaciones notables, entre otros campos, al Análisis Funcional. El presente libro es un magnífico ejemplo de la potencia de los métodos de la teoría descriptiva de conjuntos y sus aplicaciones. Es de rigor comentar que el origen de esta «teoría descriptiva» se remonta a un famoso error de Lebesgue quien, al estudiar funciones definidas implícitamente por funciones continuas del plano, afirmó que las imágenes continuas de conjuntos de Borel son de Borel, cuando en realidad lo que ocurre es que las imágenes continuas de conjuntos de Borel son lo que se llaman conjuntos analíticos, y los borelianos se caracterizan como aquellos conjuntos que tanto ellos como sus complementarios

son analíticos —el error de Lebesgue fue subsanado posteriormente por Lusin y Souslin.

Informalmente hablando, lo que se hace una y otra vez en las aplicaciones de la teoría descriptiva de conjuntos es dar una *descripción* razonable de un espacio topológico a través de familias de subconjuntos notables del mismo, para a partir de ahí obtener conclusiones sorprendentes para este. Aquí va un ejemplo que nuestros estudiantes de cursos de Análisis Funcional de la Licenciatura en Matemáticas podrían entender. Tomemos $\Omega \subset \mathbb{R}^n$ un abierto y sea $\mathcal{D}'(\Omega)$ el espacio de distribuciones en Ω (funciones generalizadas, *i.e.*, el marco general para plantear determinadas ecuaciones en derivadas parciales y buscar soluciones débiles). Conocida la topología fuerte de $\mathcal{D}'(\Omega)$, tal y como se describe en cualquier manual clásico de Análisis Funcional, un ejercicio, debidamente guiado por el profesor, llevaría a nuestro hipotético estudiante a concluir que

$$\mathcal{D}'(\Omega) = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} A_\alpha \qquad (1)$$

donde cada A_α es compacto y $A_\alpha \subset A_\beta$ si $\alpha \leq \beta$ (el orden coordenada a coordenada). Llegados aquí, se ha descrito $\mathcal{D}'(\Omega)$ de una manera muy especial, y tan buena, que con un poco más de esfuerzo se puede obtener que $\mathcal{D}'(\Omega)$ es lo que se llama un espacio *analítico*, y a partir de ahí concluir que toda aplicación lineal $T : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ cuya gráfica sea un conjunto de Borel de $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)$ es automáticamente continua (resultado debido a Schwartz dando solución a un problema de Grothendieck). Lo sorprendente del caso es que las representaciones como las de (1) están por doquier en Análisis Funcional, y saber que muchos espacios pueden describirse así tiene implicaciones no triviales. Además del resultado antes comentado de Schwartz, las descomposiciones de este tipo tienen consecuencias sobre la metrizabilidad de compactos o sobre el comportamiento sucesional de los mismos, que cuando se obtuvieron por primera vez sorprendieron tanto a topólogos como a analistas debido a sus aplicaciones y a lo inesperado de los resultados. Buena parte de este libro explota una y mil veces la posibilidad de describir un espacio topológico a través de igualdades como (1) para desde ahí obtener numerosas consecuencias de esta *descripción*.

Hasta hoy, muchos de los resultados expuestos en el volumen que estamos analizando sólo podían encontrarse dispersos en artículos de investigación destinados a especialistas. El libro contiene un buen número de resultados clásicos y nuevos, y muestra el *estado actual* de esta parte del Análisis Funcional que utiliza la teoría descriptiva de conjuntos como fuente principal de herramientas e inspiración. Presenta a veces pruebas originales y siempre rigurosas. Llegará a ser un buen texto de referencia para estudiosos del tema. Debe hacerse énfasis en que es autocontenido en gran medida y está escrito de forma que será particularmente útil para alumnos de máster, doctorado e investigadores en general. El potencial lector encontrará muchas veces la referencia adecuada al resultado clásico al que alguna vez quiso poner nombre y nunca encontró debidamente referenciado.

Los autores son consumados especialistas en la temática de la que trata el libro. No sólo la conocen bien, sino que han publicado numerosos trabajos

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en este campo. Técnicamente hablando, la experiencia de los autores garantiza el alto nivel de los temas seleccionados y la forma en la que los mismos son tratados. Es una obra escrita por analistas con un profundo conocimiento de los métodos de topología y teoría descriptiva de conjuntos. El punto de vista que la experiencia de los autores da a los asuntos tratados contribuye, sin duda, a la riqueza de intercambios entre las áreas de análisis y topología.

El libro está bien organizado, claramente escrito y la secuencia de los capítulos responde a la lógica dependencia entre ellos. Tiene una larga lista de referencias que ayudan a fijar autoría de resultados así como a posibles ampliaciones de estudio. Está dividido en 20 capítulos de los cuales los 16 primeros constituyen el bloque dedicado a la topología y aplicaciones a espacios vectoriales topológicos (o localmente convexos) y espacios de funciones, mientras que los capítulos del 17 al 20 se centran más en el estudio de algunos aspectos de los espacios de Banach.

Después de un capítulo primero donde se da la panorámica general del libro, el capítulo 2 se dedica al estudio de espacios *tipo* Baire, con aplicaciones a espacios de funciones continuas $C(X)$. Los capítulos 3, 4 y 5 están dedicados al estudio de las nociones de espacios K-analíticos, quasi-Suslin, *web*-compactos y fuertemente *web*-compactos. Se hace énfasis en la noción de espacio quasi-LB, debida a Valdivia, que responde a una estructura como la dada en (1) pero donde en este caso los A_α son conjuntos que generan espacios de Banach. Como aplicación se obtienen resultados de buen comportamiento sucesional de compactos y teoremas de gráfica cerrada. El capítulo 6 se dedica al estudio de espacios débilmente analíticos y el capítulo 7 a espacios K-analíticos de Baire. La propiedad de los tres espacios para la clase de espacios analíticos es analizada en el capítulo 8 y las estructuras analíticas y K-analíticas en espacios de funciones continuas $C(X)$ dotados de su topología de convergencia puntual se analiza en el capítulo 9.

El capítulo 10 sirve de preámbulo para el estudio que se hace en el capítulo 11 de la clase $\mathfrak{G}$ que fue introducida por Orihuela y el autor de esta recensión: la clase $\mathfrak{G}$ está formada por espacios cuyos duales satisfacen la igualdad (1), donde la propiedad de compacidad de los A_α es sustituida por una propiedad de equicontinuidad. En los capítulos 12 y 13 se presenta la relación de la propiedad (C) de Corson, real-compacidad, propiedad de Lindelöf, *estrechez* numerable, etc. en algunas de las clases de espacios considerados anteriormente (K-analíticos, espacios de la clase $\mathfrak{G}$, etc.); se presta atención a los espacios de Fréchet débilmente compactamente generados y algunas de las propiedades anteriores se aplican para espacios $C(X)$. El capítulo 14 se dedica al estudio de la propiedad de Fréchet-Urysohn (accesibilidad de puntos de la clausura de subconjuntos arbitrarios por sucesiones) y grupos topológicos. Los capítulos 15 y 16 vuelven a estar inspirados por la clase $\mathfrak{G}$ y en ellos se estudia, entre otras cosas, la relación entre metrizabilidad y la propiedad de Fréchet-Urysohn en la susodicha clase.

El bloque de capítulos del 17 al 20, se inicia con el estudio de espacios de Banach con abundancia de familias de proyecciones: resoluciones proyectivas de la identidad, *projectional skeletons*

y propiedad de complementación separable. Aquí se introducen técnicas de submodelos que vienen de la lógica y que simplifican algunas pruebas. El capítulo 18 se dedica al estudio de una clase particular de espacios $C(K)$ donde K es un compacto linealmente ordenado. El capítulo 19 analiza espacios compactos no metrizables que se corresponden con clases de espacios de Banach con abundancia de proyecciones: compactos de Corson, de Eberlein, de Valdivia, etc. El capítulo 20 trata sobre la complementabilidad universal en los espacios de Banach y en él se presenta la construcción de uno de estos espacios universales, bajo la hipótesis del continuo, para la clase de espacios con resoluciones proyectivas de la identidad.

En resumidas cuentas, considero que este libro es de lectura recomendada para aquellos analistas que quieren conocer técnicas de topología aplicada al Análisis Funcional y, viceversa, para aquellos topólogos que necesitan encontrar dónde pueden ser utilizadas sus herramientas. Hay una comunidad internacional amplia que actualmente estudia y trabaja en los temas sobre los que este texto trata.

Señalemos, finalmente, que los autores escriben literalmente en la introducción: *«Our material, much of it in book form for the first time, carries forward the rich legacy of Köthe, [3], Jarchow, [2], Valdivia, [4] and Bonet and Pérez Carreras, [1]».* Si yo como lector tuviera que elegir uno de estos cuatro para considerarlo inspirador del libro de Kąkol, Kubiś y López-Pellicer, elegiría sin duda el de Valdivia.

Referencias

[1] P. Pérez Carreras y J. Bonet, *Barrelled locally convex spaces*, North-Holland Mathematics Studies, vol. **131**, North-Holland Publishing Co., Amsterdam, 1987.

[2] H. Jarchow, *Locally convex spaces*, B. G. Teubner, Stuttgart, 1981.

[3] G. Köthe, *Topological vector spaces. I*, Die Grundlehren der mathematischen Wissenschaften, Band **159**, Springer-Verlag, New York, 1969.

[4] M. Valdivia, North-Holland Mathematics Studies, vol. **67**, North-Holland Publishing Co., Amsterdam, 1982.

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From the review

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[2] H. JARCHOW, *Locally convex spaces*, B. G. Teubner, Stuttgart, 1981.

[3] G. KÖTHE, *Topological vector spaces. I*, Die Grundlehren der mathematischen Wissenschaften, Band **159**, Springer-Verlag, New York, 1969.

[4] M. VALDIVIA, North-Holland Mathematics Studies, vol. **67**, North-Holland Publishing Co., Amsterdam, 1982.

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Collaborators

Total Author/Related Publications: **124**
Total Citations: **253**

Co-Authors
Collaboration
Mathematics (
Citations

⊞ Published as: Kąkol, J. ...

Co-authors (by number of collaborations)

Abul-Ez, M. A. Alexiewicz, Andrzej Angosto, Carlos Cascales, Bernardo De Grande-De Kimpe, Nicole Dierolf, Susanne Ferrando, Juan Carlos Ferrer-Llopis, Jesús Gilsdorf, Thomas E. Jurkiewicz, Lidia Kubiś, Wiesław Kubzdela, Albert López Pellicer, Manuel Martín Peinador, Elena Moll, Santiago Muñoz Guillermo, María Pérez-García, Cristina Roelcke, Walter Sánchez Ruiz, Luis M. Saxon, Stephen A. Schikhof, Wilhelmus H. Śliwa, Wiesław Sorjonen, Pekka Tarieladze, Vazha I. Todd, Aaron R. Tweddle, Ian Wójtowicz, Marek

Publications (by number in area)

Functional analysis Functions of a complex variable General topology Operator theory Topological groups, Lie groups

Old results

What I believe that got Kąkol interested about our results

Cascales-Orihuela, 1987

Corollary 22 *For a compact space K the following statements are equivalent:*

(i) K *is metrizable;*
(ii) Δ *is* G_δ *in* $K \times K$*;*
(iii) $\Delta = \bigcap_{n=1}^{\infty} G_n$ *where* $\{G_n : n \in \mathbb{N}\}$ *is basis of open neighborhoods of* Δ*;*
(iv) $(K \times K)\setminus\Delta = \bigcup_{n=1}^{\infty} F_n$*, with* $\{F_n : n \in \mathbb{N}\}$ *increasing family of compact sets that swallows all the compact subsets in* $(K \times K)\setminus\Delta$*;*
(v) $(K \times K)\setminus\Delta = \bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} F_\alpha$*, with* $\{F_\alpha : \alpha \in \mathbb{N}\}$ *increasing family of compact sets* (i.e. $F_\alpha \subset F_\beta$ *whenever* $\alpha \leq \beta$ *in the coordinatewise order of* $\mathbb{N}^{\mathbb{N}}$) *that swallows all the compact subsets in* $(K \times K)\setminus\Delta$*;*
(vi) $(K \times K)\setminus\Delta$ *is strongly dominated by a Polish space;*
(vii) $(K \times K)\setminus\Delta$ *is strongly dominated by a separable metric space;*
(viii) $(K \times K)\setminus\Delta$ *is Lindelöf.*

A way of presenting the results that you might recognize

Math. Z. 195, 365–381 (1987)

Mathematische Zeitschrift

On Compactness in Locally Convex Spaces

B. Cascales and J. Orihuela

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1. Introduction and Terminology

The purpose of this paper is to show that the behaviour of compact subsets in many of the locally convex spaces that usually appear in Functional Analysis is good as the corresponding behaviour of compact subsets in Banach spaces. Our results can be intuitively formulated in the following terms: *Dealing with metrizable spaces or their strong duals, and carrying out any of the usual operations of countable type with them, we ever obtain spaces with their precompact subsets metrizable, and they even give good performance for the weak topology, indeed they are weakly angelic,* [14], *and their weakly compact subsets are metrizable if and only if they are separable.*

How the metrizability result was presented, consequences:

Cascales-Orihuela, 1987

Theorem 1. *Let $(X, \mathscr{U})$ be a uniform space and let us suppose that the uniformity $\mathscr{U}$ has a basis $\mathscr{B} = \{N_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ verifying the following condition:*

(a) *For any α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha \leq \beta$ we have that $N_\beta \subset N_\alpha$.*

Then the precompact subsets of $(X, \mathscr{U})$ are metrizable in the induced uniformity.

The natural consequence

Theorem 2. *Let $E[\mathfrak{T}]$ be a LCS with a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of E' verifying the following conditions:*

(a) $\bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = E'$.

(b) *For any α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha \leq \beta$ we have that $A_\alpha \subset A_\beta$.*

(c) *For any α in $\mathbb{N}^{\mathbb{N}}$ the countable subsets of A_α are equicontinuous.*

Then the precompact subsets of $E[\mathfrak{T}]$ are metrizable.

How the class $\mathfrak{G}$ was introduced:

From the natural consequence. . .

Theorem 2. *Let $E[\mathfrak{T}]$ be a LCS with a family $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ of subsets of E' verifying the following conditions:*

(a) $\bigcup\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}=E'$.

(b) *For any α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha\leqq\beta$ we have that $A_\alpha\subset A_\beta$.*

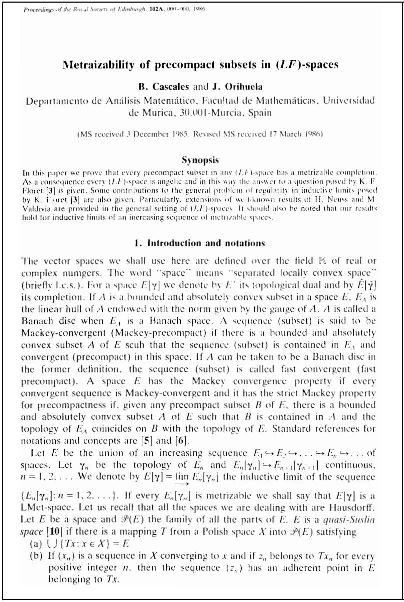
(c) *For any α in $\mathbb{N}^{\mathbb{N}}$ the countable subsets of A_α are equicontinuous.*

Then the precompact subsets of $E[\mathfrak{T}]$ are metrizable.

. . . to the class $\mathfrak{G}$

Definition 3. *Let $\mathfrak{G}$ be the class of LCS E that fulfill the conditions of Theorem* 2. *A family $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ in E' verifying the conditions* (a), (b) *and* (c) *of Theorem* 2 *shall be called a $\mathfrak{G}$-representation of E in E'.*

There were previous results:



Proceedings of the Royal Society of Edinburgh, 102A, 000–000, 1986

Metraizability of precompact subsets in (LF)-spaces

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(MS received 3 December 1985. Revised MS received 17 March 1986)

Synopsis

In this paper we prove that every precompact subset in any (LF)-space has a metrizable completion. As a consequence every (LF)-space is angelic and in this way the answer to a question posed by K. F. Floret [3] is given. Some contributions to the general problem of regularity in inductive limits posed by K. Floret [3] are also given. Particularly, extensions of well-known results of H. Neuss and M. Valdivia are provided in the general setting of (LF)-spaces. It should also be noted that our results hold for inductive limits of an increasing sequence of metrizable spaces.

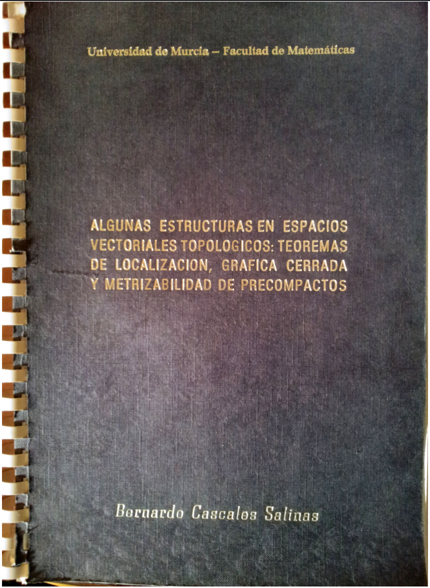
1. Introduction and notations

The vector spaces we shall use here are defined over the field $\mathbb{K}$ of real or complex numgers. The word "space" means "separated locally convex space" (briefly l.c.s.). For a space $E[\gamma]$ we denote by E' its topological dual and by $\hat{E}[\hat{\gamma}]$ its completion. If A is a bounded and absolutely convex subset in a space E, E_A is the linear hull of A endowed with the norm given by the gauge of A. A is called a Banach disc when E_A is a Banach space. A sequence (subset) is said to be Mackey-convergent (Mackey-precompact) if there is a bounded and absolutely convex subset A of E scuh that the sequence (subset) is contained in E_A and convergent (precompact) in this space. If A can be taken to be a Banach disc in the former definition, the sequence (subset) is called fast convergent (fast precompact). A space E has the Mackey convergence property if every convergent sequence is Mackey-convergent and it has the strict Mackey property for precompactness if, given any precompact subset B of E, there is a bounded and absolutely convex subset A of E such that B is contained in A and the topology of E_A coincides on B with the topology of E. Standard references for notations and concepts are [5] and [6].

Let E be the union of an increasing sequence $E_1 \hookrightarrow E_2 \hookrightarrow \dots \hookrightarrow E_n \hookrightarrow \dots$ of spaces. Let γ_n be the topology of E_n and $E_n[\gamma_n] \hookrightarrow E_{n+1}[\gamma_{n+1}]$ continuous, $n = 1, 2, \dots$ We denote by $E[\gamma] = \varinjlim E_n[\gamma_n]$ the inductive limit of the sequence $\{E_n[\gamma_n]: n = 1, 2, \dots\}$. If every $E_n[\gamma_n]$ is metrizable we shall say that $E[\gamma]$ is a LMet-space. Let us recall that all the spaces we are dealing with are Hausdorff. Let E be a space and $\mathscr{P}(E)$ the family of all the parts of E. E is a *quasi-Suslin space* [10] if there is a mapping T from a Polish space X into $\mathscr{P}(E)$ satisfying

(a) $\bigcup \{Tx : x \in X\} = E$

(b) If (x_n) is a sequence in X converging to x and if z_n belongs to Tx_n for every positive integer n, then the sequence (z_n) has an adherent point in E belonging to Tx.



Proofs based on techniques producing K-analytic structures

Arch. Math., Vol. 49, 232–244 (1987)

0003-889 X/87/4903-0232 $ 4.10/0

On K-analytic locally convex spaces

By

B. Cascales*)

1. Introduction and terminology. The earliest approach to K-analytic spaces is that of Choquet [4], who defined them as the image of a $K_{\sigma\delta}$-set in some compact Hausdorff space. For our purposes it will be more convenient to deal with the equivalent notion, [16], which allows us to look at K-analytic spaces as the image under an upper semicontinuous (usco) compact set-valued mapping of a Polish space (or even $\mathbb{N}^{\mathbb{N}}$). The images under usco compact set-valued mappings of Polish spaces are also known as K-Suslin spaces, [12], [20]. Throughout this paper we shall use the term K-Suslin space instead of K-analytic space because we are going to deal with another related notion called quasi-Suslin space, [20].

All the topological spaces, considered here, will be Hausdorff as well as all the topological vector spaces (TVS) and all the locally convex spaces (LCS) which will be defined over the field $\mathbb{K}$ of real or complex numbers. We shall denote by $\mathbb{N}$ the set of positive integers endowed with the discrete topology and by $\mathbb{N}^{\mathbb{N}}$ the set of sequences of positive integers, $\alpha = (a_n)$, endowed with the product topology. In $\mathbb{N}^{\mathbb{N}}$ we consider the following relation of order $\leqq$, for $\alpha = (a_n)$ and $\beta = (b_n) \in \mathbb{N}^{\mathbb{N}}$ we say that $\alpha \leqq \beta$ if and only if $a_n \leqq b_n$ for every positive integer n.

Let E be a topological space and let $\mathscr{K}(E)$ (resp. $\mathscr{P}(E)$) denote the family of all the compact subsets (resp. all the parts) of E. E is a K-Suslin, [20], (resp. quasi-Suslin, [20]) space, if there is a mapping T from $\mathbb{N}^{\mathbb{N}}$ into $\mathscr{K}(E)$ (resp. $\mathscr{P}(E)$) satisfying:

(a) $\bigcup \{T_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = E$.

(b) If (α_n) is a sequence in $\mathbb{N}^{\mathbb{N}}$ converging to α and x_n belongs to T_{α_n} for every positive integer n, then the sequence (x_n) has an adherent point in E belonging to T_α.

Such a mapping T will be called a K-Suslin (resp. quasi-Suslin) mapping in E. The class of quasi-Suslin spaces is strictly wider than the class of K-Suslin spaces, [20], and the difference between them is a wide as that of the countably compact and the compact subsets of a topological space: let us observe that if $T: \mathbb{N}^{\mathbb{N}} \to \mathscr{P}(E)$ is a quasi-Suslin mapping, then T_α is countably compact in E for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. A family of parts of E, $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, is said to be an ordered family if $A_\alpha \subset A_\beta$ whenever $\alpha \leqq \beta$ in $\mathbb{N}^{\mathbb{N}}$.

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*) This paper constitutes part of the doctoral thesis of the author which has been made under the direction of Prof. M. Valdivia.

Newer results: some indexes

Alternative proofs were given

Montel (DF)-Spaces, Sequential (LM)-Spaces and the Strongest Locally Convex Topology

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Abstract

Topologists say that a space is sequential if every sequentially closed set is closed. Directly from the definitions, metrizable ⇒ Fréchet-Urysohn ⇒ sequential ⇒ *k*-space. Kąkol showed that for an (LM)-space (the inductive limit of a sequence of locally convex metrizable spaces), metrizable , Fréchet-Urysohn. The Cascales and Orihuela result that every (LM)-space is angelic proved that for an (LM)-space, sequential ⇔ k-space. Within the class of (LM)-spaces, then, the four notions become only two distinct ones bearing the relation metrizable ⇒ sequential. Webb proved that every infinite-dimensional Montel (DF)-space is sequential but not Fréchet-Urysohn, and equivalently, not metrizable, since Montel (DF)-spaces are (LB)-spaces and, *a fortiori*, (LM)-spaces. Does the converse hold in the (LB)-space, (DF)-space or (LM)-space settings? Yes, in all cases. If a (DF)-space or (LM)-space is sequential, then it is either metrizable or it is a Montel (DF)-space. Pfister's result that every (DF)-space is angelic is needed, and the paper provides elementary proofs for this and the similar theorem by Cascales and Orihuela. The strongest locally convex topology plays a vital role throughout.

We collaborated

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Weight of precompact subsets and tightness ☆

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Submitted by R.M. Aron

Abstract

Pfister (1976) and Cascales and Orihuela (1986) proved that precompact sets in (DF)- and (LM)-spaces have countable weight, i.e., are metrizable. Improvements by Valdivia (1982), Cascales and Orihuela (1987), and Kąkol and Saxon (preprint) have varying methods of proof. For these and other improvements a refined method of upper semi-continuous compact-valued maps applied to uniform spaces will suffice. At the same time, this method allows us to dramatically improve Kaplansky's theorem, that the weak topology of metrizable spaces has countable tightness, extending it to include all (LM)-spaces and all quasi-barrelled (DF)-spaces, both in the weak and original topologies. One key is showing that for a large class $\mathfrak{G}$ including all (DF)- and (LM)-spaces, countable tightness of the weak topology of E in $\mathfrak{G}$ is equivalent to realcompactness of the weak* topology of the dual of E.

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METRIZABILITY VS. FRÉCHET-URYSOHN PRO

B. CASCALES, J. KĄKOL, AND S. A. SAXON

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. In metrizable spaces, points in the closure of a subset A are limits of sequences in A; i.e., metrizable spaces are Fréchet-Urysohn spaces. The aim of this paper is to prove that metrizability and the Fréchet-Urysohn property are actually equivalent for a large class of locally convex spaces that includes (LF)- and (DF)-spaces. We introduce and study countable bounded tightness of a topological space, a property which implies countable tightness and is strictly weaker than the Fréchet-Urysohn property. We provide applications of our results to, for instance, the space of distributions $\mathfrak{D}'(\Omega)$. The space $\mathfrak{D}'(\Omega)$ is not Fréchet-Urysohn, has countable tightness, but its bounded tightness is uncountable. The results properly extend previous work in this direction.

1. INTRODUCTION

The *tightness* $t(X)$ [resp., *bounded tightness* $t_b(X)$] of a topological space X is the smallest infinite cardinal number m such that for any set A of X and any point $x \in \overline{A}$ (the closure in X) there is a set [resp., bounding set] $B \subset A$ for which $|B| \le m$ and $x \in \overline{B}$. Recall that a subset B of X is *bounding* if every continuous real-valued function on X is bounded on B. The notion of countable tightness arises as a natural weakening of the Fréchet-Urysohn notion. Recall that X is *Fréchet-Urysohn* if for every set $A \subset X$ and every $x \in \overline{A}$ there is a sequence in A which converges to x. Clearly,

Fréchet-Urysohn ⇒ countable bounded tightness ⇒ countable tightness.

Franklin [9] recorded an example of a compact topological space with countable tightness, hence countable bounded tightness, which is not Fréchet-Urysohn.

In [5] Cascales and Orihuela introduced the class $\mathfrak{G}$ of those locally convex spaces (lcs) $E = (E, \mathfrak{T})$ for which there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets in the topological dual E' of E (called its $\mathfrak{G}$-representation) such that:

(1)
(a) $E' = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$;
(b) $A_\alpha \subset A_\beta$ when $\alpha \le \beta$ in $\mathbb{N}^{\mathbb{N}}$;
(c) in each A_α, sequences are $\mathfrak{T}$-equicontinuous.

In the set $\mathbb{N}^{\mathbb{N}}$ of sequences of positive integers the inequality $\alpha \le \beta$ for $\alpha = (a_n)$ and $\beta = (b_n)$ means that $a_n \le b_n$ for all $n \in \mathbb{N}$.

The original results still had a saying

More players into this game. . .

⋮

- **MR2541044** Reviewed Kąkol, J.; López Pellicer, M.; Todd, A. R. A topological vector space
(2009), no. 2, 313–317. (Reviewer: Juan Carlos Ferrando) 46A50 (46A30 54C35)
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(46A03 46A08)
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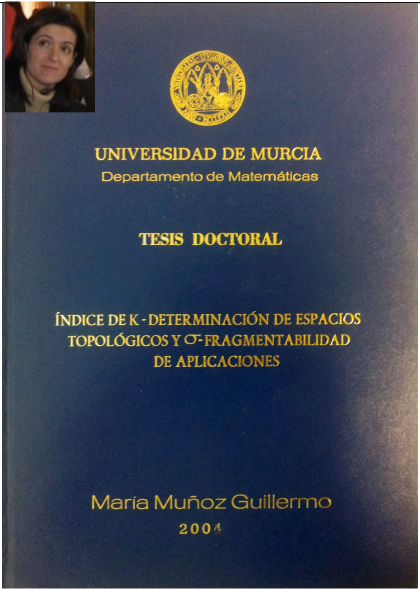
In Murcia we went in a different direction



In Murcia we went in a different direction

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Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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Compactoid filters and USCO maps

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Received 8 October 2002

Submitted by R.M. Aron

Abstract

The aim of this paper is to report in a short and self-contained way on the properties of compactoid and countably compactoid filters. We apply them to some questions in both topology and analysis such as the generation and extension of USCO maps, the study of some properties of K-analytic spaces and the study of bounds for the weight of compact sets in spaces obtained through inductive operations.

Keywords: Filters; Compact spaces; USCO map; K-analytic spaces

1. Introduction

All our topologies, hence all our topological spaces, are assumed to be Hausdorff. We use the concept of filter, filter base, ultrafilter, net and subnet as introduced in [10, pp. 76–77] and [19, p. 65]. A filter in a topological space is said to be *compactoid* if every finer ultrafilter converges—see Definition 1 below and [8,24] for historical references. Compactoid filters generalize both convergent filters and compact sets. Compactoid filters have been widely applied in optimization, generalized differentiation, existence of upper semi-continuous compact-valued maps—recall that a multi-valued map $\psi : X \to 2^Y$ is said to be *USCO* if it is compact valued and upper semicontinuous, i.e., for every $x \in X$ the set $\psi(x)$ is compact nonempty and for every open set V in Y with $\psi(x) \subset V$ there is an open neighborhood U of x in X such that $\psi(U) \subset V$, etc.—see, for instance, [5,8,9,21,24]

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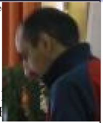

THE NUMBER OF WEAKLY COMPACT S WHICH GENERATE A BANACH SPAC

BY

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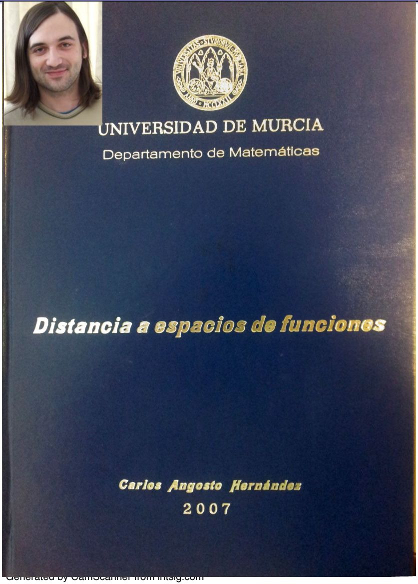
ABSTRACT

We consider the cardinal invariant $CG(X)$ of the minimal number of weakly compact subsets which generate a Banach space X. We study the behavior of this index when passing to subspaces, its relation with the Lindelöf number in the weak topology and other related questions.

Introduction

A Banach space is weakly compactly generated if there is a weakly compact subset which is linearly dense and weakly Lindelöf if it is a Lindelöf space in its weak topology. Corson [10] asked what the relation was between these two concepts. The answer was that every weakly compactly generated space is weakly Lindelöf but the converse is not true, and in order to clarify what was in the middle the class of weakly $\mathcal{K}$-analytic was introduced by Talagrand [18] who, together with Pol [15], was the first to solve this problem. Here we shall analyze the question of Corson from a more general point of view: What is the relation between the number of weak compacta which are necessary to generate a Banach space and the Lindelöf number of the space in the weak topology? Again, an intermediate class analogous to that introduced by Talagrand plays a clarifying role in the theory. Thus, our starting point is the following (cf. Sections 1 and 2 for notation):

In Murcia we went in a different direction



Math. Z. (2009) 263:103–124
DOI 10.1007/s00209-008-0412-8

Mathematisc

Distances to spaces of Baire one functions

C. Angosto · B. Cascales · I. Namioka

Received: 27 March 2007 / Accepted: 4 July 2008 / Published online: 26 August 2008

Abstract Given a metric space X and a Banach space $(E, \|\cdot\|)$ we use an index of σ-fragmentability for maps $f \in E^X$ to estimate the distance of f to the space $B_1(X, E)$ of Baire one functions from X into $(E, \|\cdot\|)$. When X is Polish we use our estimations for these distances to give a quantitative version of the well known Rosenthal's result stating that in $B_1(X, \mathbb{R})$ the pointwise relatively countably compact sets are pointwise relatively compact. We also obtain a quantitative version of a Srivatsa's result that states that whenever X is metric any weakly continuous function $f \in E^X$ belongs to $B_1(X, E)$: our result here says that for an arbitrary $f \in E^X$ we have

$$d(f, B_1(X, E)) \leq 2 \sup_{x^* \in B_{E^*}} \operatorname{osc}(x^* \circ f),$$

where $\operatorname{osc}(x^* \circ f)$ stands for the supremum of the oscillations of $x^* \circ f$ at all points $x \in X$. As a consequence of the above we prove that for functions in two variables $f : X \times K \to \mathbb{R}$, X complete metric and K compact, there exists a G_δ-dense set $D \subset X$ such that the oscillation of f at each $(x, k) \in D \times K$ is bounded by the oscillations of the *partial* functions f_x and f^k. A representative result in this direction, that we prove using games, is the following: if X is a σ–β-unfavorable space and K is a compact space, then there exists a dense G_δ-subset

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A representative result

Cascales-Muñoz-Orihuela

Theorem 21 *Let K be a compact space and $\mathfrak{m}$ a cardinal number. The following statements are equivalent:*

(i) $w(K) \leq \mathfrak{m}$;

(ii) *There exists a metric space M with $w(M) \leq \mathfrak{m}$ and a family $\mathcal{O} = \{O_L : L \in \mathcal{K}(M)\}$ of open sets in $K \times K$ that is basis of the neighborhoods of Δ such that $O_{L_1} \subset O_{L_2}$ whenever $L_2 \subset L_1$ in $\mathcal{K}(M)$;*

(iii) $(K \times K)\setminus\Delta$ *is strongly dominated by a metric M with $w(M) \leq \mathfrak{m}$.*

To finish we prove that (ii)⇒(i). Let us assume that (ii) holds and given $m \in \mathbb{N}$ and a sequence $(L_1, L_2, \ldots)$ in $\mathcal{K}(M)$ we define

$$\varphi(m, L_1, L_2, \ldots) := \bigcap_{n\in\mathbb{N}} \{f \in mB_{C(K)} : |f(x) - f(y)| \leq \frac{1}{n}, \text{ for all } (x, y) \in O_{L_n}\}. \tag{7}$$

Note that each $\varphi(m, L_1, L_2, \ldots)$ is $\|\cdot\|_\infty$-bounded, closed and equicontinuous as a family of functions defined on K. Therefore, Ascoli's theorem, see [19, p. 234], implies that $\varphi(m, L_1, L_2, \ldots)$ is compact in $(C(K), \|\cdot\|_\infty)$. If $(\mathcal{K}(M), h)$ is the lattice of compact subsets of M with the Hausdorff distance, then $w(\mathcal{K}(M), h) = w(M)$ [28, Proposition 2.4.14]. Therefore the product $M' := \mathbb{N} \times \prod_{n=1}^{\infty}(\mathcal{K}(M), h)$ of countably many copies of $(\mathcal{K}(M), h)$ and $\mathbb{N}$ is still a metric space with $w(M') = w(M)$. Note that the formula (7) defines a multi-map $\varphi : M' \to \mathcal{K}(C(K), \|\cdot\|_\infty)$. Being $\mathcal{O}$ a basis of neighborhoods of Δ implies that $C(K) = \bigcup\{\varphi(x) : x \in M'\}$.

Another approach

Topology and its Applications 158 (2011) 204–214

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Topology and its Applications

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Domination by second countable spaces and Lindelöf Σ-property

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Orderings by irrationals
Orderings by a second countable space
Compact cover
Function spaces
Cosmic spaces

A B S T R A C T

Given a space M, a family of sets $\mathcal{A}$ of a space X is *ordered by* M if $\mathcal{A} = \{A_K\colon K$ is a compact subset of $M\}$ and $K \subset L$ implies $A_K \subset A_L$. We study the class $\mathcal{M}$ of spaces which have compact covers ordered by a second countable space. We prove that a space $C_p(X)$ belongs to $\mathcal{M}$ if and only if it is a Lindelöf Σ-space. Under MA(ω_1), if X is compact and $(X \times X)\setminus\Delta$ has a compact cover ordered by a Polish space then X is metrizable; here $\Delta = \{(x,x)\colon x \in X\}$ is the diagonal of the space X. Besides, if X is a compact space of countable tightness and $X^2\setminus\Delta$ belongs to $\mathcal{M}$ then X is metrizable in ZFC.
We also consider the class $\mathcal{M}^*$ of spaces X which have a compact cover $\mathcal{F}$ ordered by a second countable space with the additional property that, for every compact set $P \subset X$ there exists $F \in \mathcal{F}$ with $P \subset F$. It is a ZFC result that if X is a compact space and $(X \times X)\setminus\Delta$ belongs to $\mathcal{M}^*$ then X is metrizable. We also establish that, under CH, if X is compact and $C_p(X)$ belongs to $\mathcal{M}^*$ then X is countable.

Metrizability of compact sets again

K compact space & $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ subsets of $(K \times K) \setminus \Delta$. We write:

(A) each A_α is compact;

(B) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$;

(C) $(K \times K) \setminus \Delta = \bigcup \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.

Theorem (Orihuela, Tkachuk, B.C. 2011)

(A) + **(B)** + **(C)** + *$MA(\omega_1)$* $\Rightarrow$ *K is metrizable.*

Open question

(A) + **(B)** + **(C)** $\overset{?}{\Rightarrow}$ K is metrizable.

More comments and problems can be found in:

A Biased View of Topology as a Tool in Functional Analysis

Bernardo Cascales and José Orihuela

1 Introduction

The interaction between functional analysis and topology goes back to their origins and has deepened and widened over the years. Going back to history we have to highlight Banach's 1932 monograph [20] that made the theory of Banach spaces ("espaces du type (B)" in the book) an indispensable tool of modern analysis. The novel idea of Banach is to combine point-set topological ideas with the linear theory in order to obtain such powerful theorems as Banach–Steinhaus theorem, open-mapping theorem and closed graph theorem. For almost a century already general topology and functional analysis continue to benefit from each other.

The aim of this survey is to give *"Our biased views of topology as a tool in functional analysis"*, with particular stress in *"recent"* results. It would be, of course, too pretentious if we even tried to write about the general role of topology as a tool for functional analysis. Without any doubt, there are results much more important than those collected here and, of course, other authors might have different views.

Let us describe very briefly without any further ado the contents of this survey. Here are the different sections of the paper:

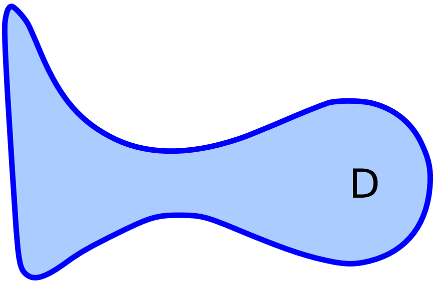
1. Introduction.
2. Metrizability of compact spaces with applications to functional analysis.
3. Topological networks meet renorming theory in Banach spaces.
4. Recent views about pointwise and weak compactness.
5. Concluding references and remarks.

K. P. Hart et al. (eds.), *Recent Progress in General Topology III*,
DOI: 10.2991/978-94-6239-024-9_3, © Atlantis Press and the authors 2013

Recent results: indexes with different flavour

Definition (Rieffel, 1967)

$D \subset E$ is dentable if for each $\varepsilon > 0$ there is a point $x \in D$ such that $x \notin \overline{\text{co}}(D \setminus U_\varepsilon(x))$



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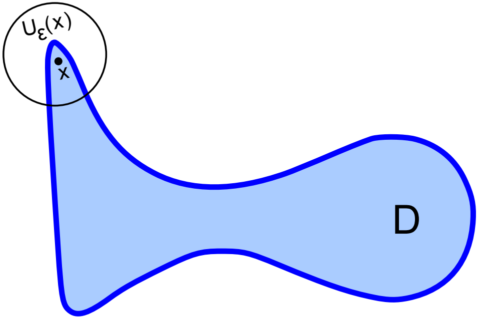

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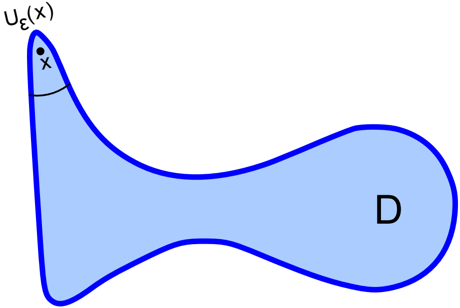

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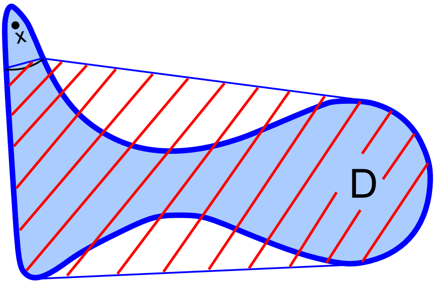
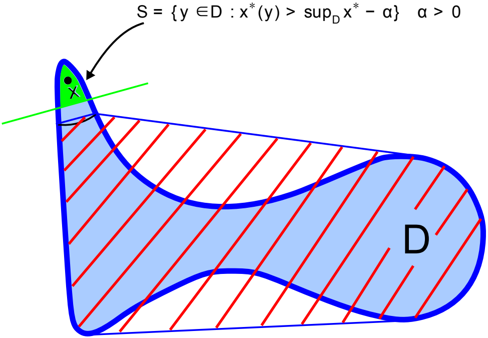


Proposition (Small slices)

$D \subset E$ is dentable if, and only if, D has slices of arbitrarily small diameter.



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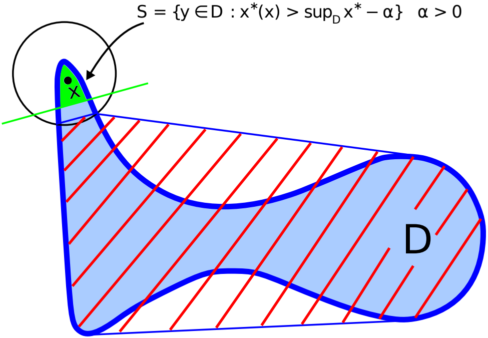


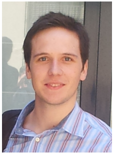
Define:

- $\text{Dent}(C) = \inf\{\varepsilon > 0 : \exists S \text{ slice of } C \text{ with } \text{rad}(S) < \varepsilon\}$
- $\text{dent}(D) = \sup\{\text{Dent}(C) : C \subseteq D\}$

Theorem

Let (Ω, Σ, μ) be a finite measure space and $T : L^1(\mu) \to E$ a continuous linear operator. Then

$$d(T, \mathscr{L}_{rep}(L^1(\mu), E)) \leq 2\gamma(T(B_{L^1(\mu)})).$$



B. Cascales, A. Pérez and M. Raja,
Radon-Nikodým indexes and measures of non weak compactness.
Preprint, 2013

One last thing

Thanks for helping people in Murcia

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- J.C. Ferrando, Jerzy Kąkol, Manuel López-Pellicer and M. Muñoz, *Some topological cardinal inequalities for spaces* $C_p(X)$, Topology and its Applications (2013), 1102–1107.

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