Measurable selectors, proximinality and integration of multi-functions

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The interplay between functional analysis, topology and measure theory.

What can one expect when mixing up analysis, topology and measure theory?

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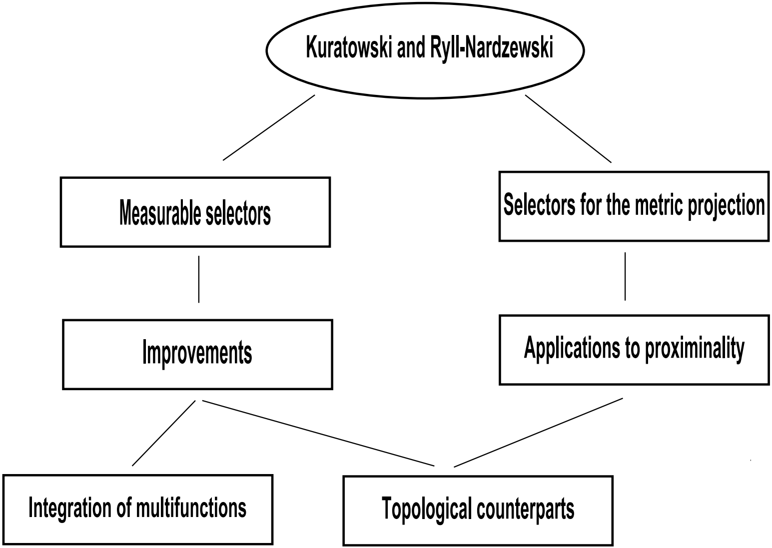
It's a common theme of mathematics that when one mixes different mathematical endeavors, like topology (geometry), algebra and analysis the end product is oftentimes much greater than a simple sum of the individual parts...

Respectfully yours Joe Diestel
Kent State University.


Kuratowski and Ryll-Nardzewski
Measurable selectors
Selectors for the metric projection
Improvements
Applications to proximinality
Integration of multifunctions
Topological counterparts

Notation

- X topological space; E Banach;
- 2^E subsets; $wk(E)$ weakly compact sets; $cwk(E)$ convex weakly compact sets;

Notation

- X topological space; E Banach;
- 2^E subsets; $wk(E)$ weakly compact sets; $cwk(E)$ convex weakly compact sets;
- (Ω, Σ, μ) complete probability space;
- Σ^+ measurable sets of positive measure; for $A \in \Sigma$, Σ_A^+ measurable subsets of A of positive measure;
- measurability and scalar measurability for $f : \Omega \to E$ standard; measurability for $F : \Omega \to 2^E$ will be defined;

Stay focused: kind of problems studied

Block 1 if $F : \Omega \longrightarrow 2^E$ is *nice* to find *nice* selectors $f : \Omega \longrightarrow E$ of F.
Application: integration of multi-functions.

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$$E \ni x \mapsto P_Y(x) := \{y \in Y : \|x - y\| = d(x, Y)\} \neq \emptyset$$

Application: proximinality of $L^1(\mu, Y) \subset L^1(\mu, E)$.

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Block 3 — to use ideas as above but in *topology* to measure distances to spaces of Baire one functions.
Application: quantitative versions of compactness results in spaces of Baire one functions.

Block 4 — to use ideas as above but in spaces of continuous functions.
Application: weak compactness in Banach spaces can be rewritten using inequalities the true compactness result is Tijonov theorem.

The co-authors

- B. Cascales and **M. Raja**, *Measurable selectors for the metric projection*, Math. Nachr. **254/255** (2003), 27–34.
- B. Cascales and **J. Rodríguez,** *The Birkhoff integral and the property of Bourgain*, Math. Ann. **331** (2005), no. 2, 259–279.
- B. Cascales, **V. Kadets, and J. Rodríguez,** *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*, J. Funct. Anal. **256** (2009).
- B. Cascales, **V. Kadets, and J. Rodríguez,** *Measurability and selections of multi-functions in Banach spaces*, J. Convex Analysis (2009 or 2010).
- **C. Angosto** y B. Cascales *The quantitative difference between countable compactness and compactness.* J. Math. Anal. Appl. (2008).
- **C. Angosto** y B. Cascales *Measures of weak noncompactness in Banach spaces.* Topology Appl. (2009)
- **C. Angosto, I. Namioka** and B. Cascales, *Distances to spaces of Baire one functions*, Math. Z. (2009 or 2010).

MEASURABLE SELECTORS

Naive approach to find measurable selectors

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How good is this approach?

As good as the real applications you can get!!!

Starting point... an elementary result

Exercise

$f : \Omega \to \mathbb{R}$. TFAE:

1. f is (μ-)measurable;
2. For every $\varepsilon > 0$ $A \in \Sigma^+$ there is $B \in \Sigma_A^+$ such that

$$|\cdot| - \operatorname{diam} f(B) < \varepsilon.$$

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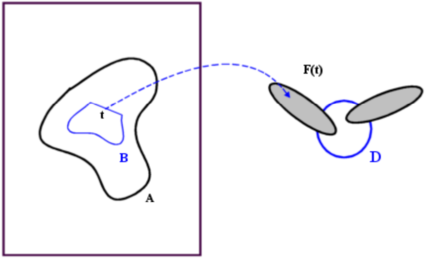

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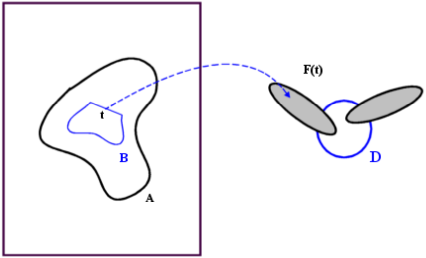

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(P) is the measure theory counterpart of σ-fragmentable multi-functions introduced by Jayne-Pallarés-Orihuela and Vera

Multi-functions

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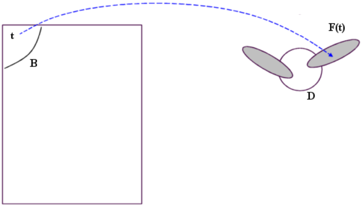

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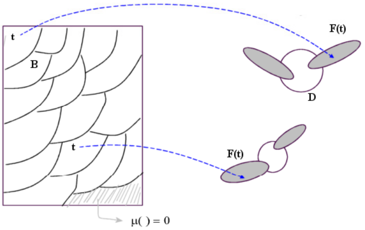

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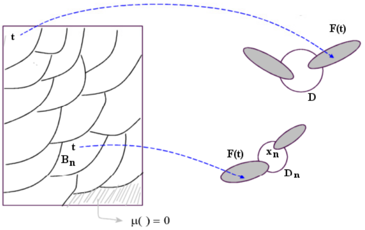

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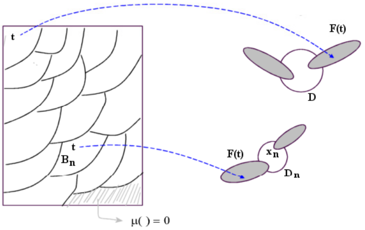

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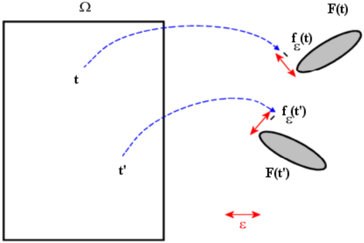

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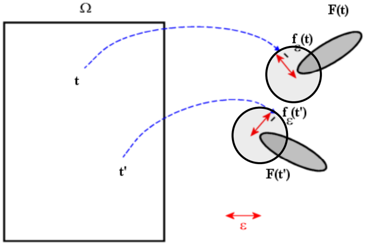

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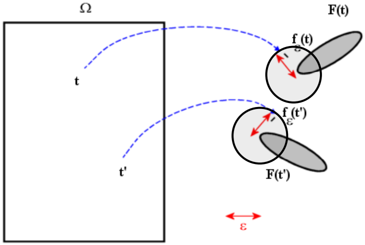

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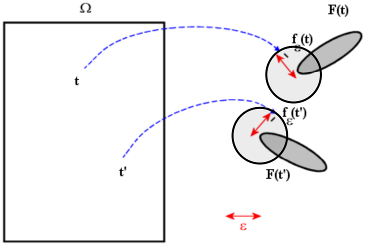

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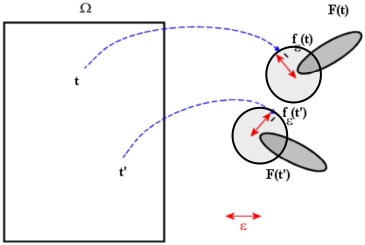

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Conclusion

We produce a sequence $(f_n):\Omega\to E$ of μ-measurable functions such that $(f_n(t))$ is Cauchy μ-a.e., hence it is convergent.

Multi-functions: measurable selections

Corollary, Kuratowski-Ryll Nardzewski, 1965

Let $F:\Omega\to 2^E$ be a multi-function with closed non empty values of E. If E is separable and F satisfies that

$$\{t\in\Omega : F(t)\cap O\neq\emptyset\}\in\Sigma \text{ for each open set } O\subset X. \tag{E}$$

Then F admits a μ-measurable selector f.

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Theorem

For a multi-function $F:\Omega\to wk(E)$ TFAE:

(i) *F admits a strongly measurable selector.*

(iii) *F satisfies property (P).*

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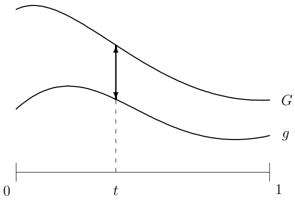
(i) *F admits a strongly measurable selector.*

(ii) *There exist a set of measure zero $\Omega_0 \in \Sigma$, a separable subspace $Y \subset X$ and a multi-function $G : \Omega \setminus \Omega_0 \to wk(Y)$ that is Effros measurable and such that $G(t) \subset F(t)$ for every $t \in \Omega \setminus \Omega_0$;*

(iii) *F satisfies property (P).*

Our interest in selections: the integral of a multifunction

$F : \Omega \longrightarrow cwk(E)$ –convex w-compact



There are several possibilities to define the integral of F:

1. to take a reasonable embedding j from $cwk(E)$ into the Banach space $Y(= \ell_\infty(B_{E^*}))$ and then study the integrability of $j \circ F$;

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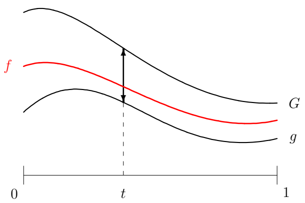

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$$\int F\,d\mu = \left\{ \int f\,d\mu : f \text{ integra. sel.} F \right\}.$$

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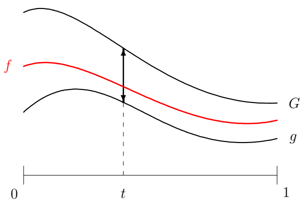

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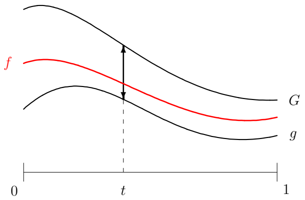

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2. Aumann, [Aum65], used the selectors technique;

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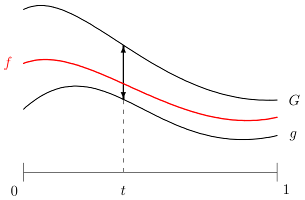

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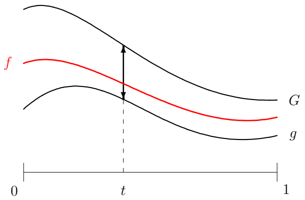
Our interest in selections: the integral of a multifunction

$F : \Omega \longrightarrow cwk(E)$ –convex w-compact



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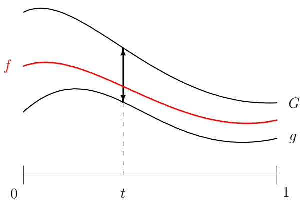

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The non-separable case

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SCALARLY MEASURABLE SELECTORS

$cwk(E)$ embeds into $Y(= \ell_\infty(B_{E^*}))$

Definition

For $C \subset E$ bounded and $x^* \in E^*$, we write

$$\delta^*(x^*, C) := \sup\{x^*(x) : \ x \in C\}.$$

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Theorem, Rådström embedding [Råd52]

The map $j : cwk(E) \longrightarrow \ell_\infty(B_{E^*})$ given by por $j(C)(x^*) = \delta^*(x^*, C)$ satisfies the following properties:

(i) $j(C+D) = j(C) + j(D)$ for each $C, D \in cwk(E)$;

(ii) $j(\lambda C) = \lambda j(C)$ for each $\lambda \geq 0$ and $C \in cwk(E)$;

(iii) $h(C, D) = \|j(C) - j(D)\|_\infty$ for each $C, D \in cwk(E)$;

(iv) $j(cwk(E))$ is closed in $\ell_\infty(B_{E^*})$.

Scalar measurability and Pettis integrability

Definition

$F : \Omega \longrightarrow cwk(E)$ is said to be scalarly measurable if

$$\delta^*(x^*, F) : t \mapsto \delta^*(x^*, F(t)).$$

is measurable for each $x^* \in E^*$.

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Definition (Amri, Hess, Ziat)

Let E be a separable Banach space. A multi-function $F : \Omega \to cwk(E)$ is said to be *Pettis integrable* if

- $\delta^*(x^*, F)$ is integrable for each $x^* \in E^*$;
- for each $A \in \Sigma$, there is $\int_A F \, d\mu \in cwk(E)$ such that

$$\delta^*\left(x^*, \int_A F \, d\mu\right) = \int_A \delta^*(x^*, F) \, d\mu \quad \text{for every } x^* \in E^*.$$

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Definition (Amri, Hess, Ziat)

Let E be an arbitrary Banach space. A multi-function $F : \Omega \to cwk(E)$ is said to be *Pettis integrable* if

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Multi-functions: scalarly measurable selections

Theorem

Let $F : \Omega \to wk(E)$ be a scalarly measurable multi-function. Then F admits a scalarly measurable selector.

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1. Note that if $\Delta F_0 := \sup_{x^*\in S_{E^*}} \int_\Omega \big(\delta^*(x^*,F_0)-\delta_*(x^*,F_0)\big)\,d\mu = 0$ implies any selector f of F_0 is scalarly measurable because for every $x^*\in E^*$

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4. $F_0:\Omega\to wk(E)$ given by $F_0(t):=\bigcap_{n\in\mathbb{N}}F_n(t)$ is scalarly measurable and $\Delta F_0=0$. Then (1) applies.

Two consequences

Theorem

$F:\Omega\to cwk(E)$ scalarly measurable. Then there is a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(E^*,w^*)}$ of scalarly meas. selectors of F such that

$$F(t)=\overline{\{f_\alpha(t):\ \alpha<\mathrm{dens}(E^*,w^*)\}}\quad \textit{for every } t\in\Omega.$$

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Theorem

If $F:\Omega\to cwk(E)$ a Pettis integrable multi-function, then:

- every scalarly measurable selector is Pettis integrable;
- F admits a scalarly measurable selector.

Furthermore, F admits a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(E^*,w^*)}$ of Pettis integrable selectors such that

$$F(t)=\overline{\{f_\alpha(t):\ \alpha<\mathrm{dens}(E^*,w^*)\}}\quad \text{for every } t\in\Omega.$$

Moreover, $\int_A F\,d\mu=\overline{IS_F(A)}$ for every $A\in\Sigma$.

PROXIMINALITY, TOPOLOGY

The problem

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4. take $g : Z \to Y_0$ measurable selector for P_Y;
5. then $g \circ f \in L^1(\mu, Y)$ is best approximation of f.

A topological version of the proximal result

Theorem

Let Y be a topological space, Z Polish and $F : Y \times Z \to \mathbb{R}$ a map satisfying:

H1. F^z is upper semi-continuous for every $z \in Z$;

H2. F_y is lower semi-continuous for every $y \in Y$;

H3. For every $y \in Y$ there is $z \in Z$ such that $F(y,z) = \inf_{w \in Z} F(y,w)$.

Then there is a Čech-analytic measurable map $h : Y \to Z$ such that

$$F(y,h(y)) = \inf_{z \in Z} F(y,z)$$

for every $y \in Y$.

Baire one functions

$f : \Omega \to E$

For every $\varepsilon > 0$ $A \in \Sigma^+$ there is $B \in \Sigma_A^+$ such that

$$\| \; \| - \operatorname{diam} f(B) < \varepsilon.$$

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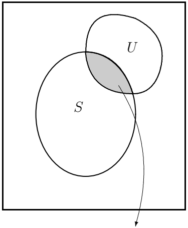

$$\| \ \| - \operatorname{diam}(f(U \cap S)) \leq \varepsilon$$

Definition

$f : X \to E$ is ε-fragmented if for every non empty subset $S \subset X$ there exist an open subset $U \subset X$ such that $U \cap S \neq \emptyset$ and

$$\| \ \| - \operatorname{diam}(f(U \cap S)) \leq \varepsilon.$$

Distances to Baire one functions

Definition

For $f \in E^X$ we define:

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Theorem

If X is a complete metric space, E a Banach space and $f \in E^X$ then

$$\frac{1}{2}\text{frag}(f) \leq d(f, B_1(X,E)) \leq \text{frag}(f).$$

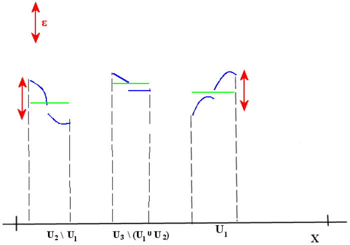
In the particular case $E = \mathbb{R}$ we precisely have

$$d(f, B_1(X)) = \frac{1}{2}\text{frag}(f).$$

Distances to Baire one functions

Theorem

If X is a complete metric space, $d(f, B_1(X)) = \frac{1}{2}\,\mathrm{frag}(f)$



1. $\frac{1}{2}\,\mathrm{frag}(f) \le d(f, B_1(X))$ easy;
2. we prove $d(f, B_1(X)) \le \frac{1}{2}\,\mathrm{frag}(f)$ for X Polish;
3. assume the quantities above are finite; fix $\varepsilon > \mathrm{frag}(f)$
4. use ε-fragmentability;
5. REPEAT 4 produce a sequence of open sets $\{U_n\}_n$ with:
   - $X = \cup_n U_n$;
   - $B_n := U_n \setminus \cup_{k=1}^{n} U_k \neq \emptyset$;
   - $\mathrm{diam}(f(B_n)) < \varepsilon$.
6. pick x_n the middle point of $\mathrm{co} f(B_n)$ and define $f_\varepsilon := \sum_n \chi_{B_n} x_n$;
7. $|f_\varepsilon(t) - f(t)| < \frac{1}{2}\varepsilon$ for every $t \in X$;
8. use that B_n are disjoint F_σ sets and Tietze theorem to conclude $f_\varepsilon \in B_1(X)$;
9. 3, 7 and 8 imply that $d(f, B_1(X) \le \frac{1}{2}\,\mathrm{frag}(f)$.

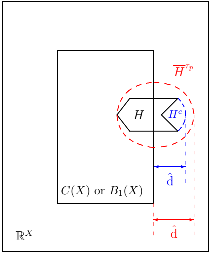
For complete metric space X is much more involved: there is no countability helping; in fact our results are far more general.

Application: Quantitative Rosenthal's result

Let X be a Polish space, $H \subset \mathbb{R}^X$ pointwise bounded and

$$\hat{\mathrm{d}} := \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{\mathbb{R}^X}, B_1(X)\Big).$$

$$\hat{\mathrm{d}} := \hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X))$$



$\hat{\mathrm{d}} = \hat{\mathrm{d}}$

Quantitative Rosenthal's result

$$\hat{\mathrm{d}} = \hat{\mathrm{d}}.$$

Application: quantitative angelicity

Let X be a Lindelöf Σ-space, $H \subset \mathbb{R}^X$ pointwise bounded and

$$\hat{\mathrm{d}} := \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{\mathbb{R}^X}, C(X)\Big).$$

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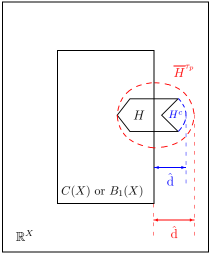

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Quantitative angelicity

$$\hat{\mathrm{d}} \le \hat{\mathrm{d}} \le 5\hat{\mathrm{d}}$$

And...?

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THANK YOU!

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