Universidad de Murcia | Departamento Matemáticas

# Aproximación de operadores por operadores que alcanzan la norma

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http://webs.um.es/beca

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## Para estar centrados

- Tipo de roblemas a estudiar... algo que todavía se entiende.
- Crédito a coautores, y trabajos previos.
- Un poco de historia.
- El problema que estudiamos ¿Qué resuelve?
- El alcance de los resultados.
- El *gusto* de alguna demostración: técnicas involucradas de AF + topología + AC.
- Un problema abierto.

## What kind of problem are we going to talk about?

# A BISHOP-PHELPS-BOLLOBAS TYPE THEOREM FOR UNIFORM ALGEBRAS

B. CASCALES, A. J. GUIRAO AND V. KADETS

## 1. Introduction

Mathematical optimization is associated to maximizing or minimizing real functions. James's compactness theorem [17] and Bishop-Pehlps's theorem [5] are two landmark results along this line in functional analysis. The former characterizes reflexive Banach spaces $X$ as those for which continuous linear functionals $x^* \in X^*$ attain their norm in the unit sphere $S_X$. The latter establishes that for any Banach space $X$ every continuous linear functional $x^* \in X^*$ can be approximated (in norm) by linear functionals that attain the norm in $S_X$. This paper is concerned with the study of a strengthening of Bishop-Phelps's theorem that mixes ideas of Bollobás [6] –see Theorem 3.1 here– and Lindenstrauss [21] –who initiated the study of the Bishop-Phelps property for bounded operators between Banach spaces.

## The problem for $x^* : X \to \mathbb{R}$ form and $T : X \to Y$ operator

$$\|x^*\| = \sup\{|x^*(x)| : \|x\| = 1\} \overset{\text{not always}}{=} \max\{|x^*(x)| : \|x\| = 1\}$$

$$\|T\| = \sup\{\|T(x)\| : \|x\| = 1\} \overset{\text{not always}}{=} \max\{\|T(x)\| : \|x\| = 1\}$$

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## What can do it about it?

Our paper is devoted to showing that Asplund operators with range in a uniform Banach algebra have the Bishop-Phelps-Bollobás property, *i.e.*, they are approximated by norm attaining Asplund operators at the same time that a point where the approximated operator almost attains its norm is approximated by a point at which the approximating operator attains it. To prove this result we establish a Uryshon type lemma producing peak complex-valued functions in uniform algebras that are small outside a given open set and whose image is inside a symmetric rhombus with main diagonal $[0, 1]$ and small height.

# Credit to co-authors and previous work

María D. Acosta, Richard M. Aron, Domingo García, and Manuel Maestre, *The Bishop-Phelps-Bollobás theorem for operators*, J. Funct. Anal. **254** (2008), no. 11, 2780–2799.

R. M. Aron, B. Cascales and O. Kozhushkina, *The Bishop-Phelps-Bollobas theorem and Asplund operators*, Proc. Amer. Math. Soc. 139 (2011), no. 10, 3553–3560.

B. Cascales, A. J. Guirao and V. Kadets, *A Bishop-Phelps-Bollobás type theorem for uniform algebras*, Enviado para publicación 19/Abril/2012

Jerry Johnson and John Wolfe, *Norm attaining operators*, Studia Math. **65** (1979), no. 1, 7–19.

C. Stegall, *The Radon-Nikodým property in conjugate Banach spaces. II*, Trans. Amer. Math. Soc. **264** (1981), no. 2, 507–519.

# Bishop-Phelps theorem

**Theorem (Bishop-Phelps, 1961)**

If $X$ is a Banach, then $\overline{\mathbf{NA}X^*} = X^*$.

**A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE**

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere $S$ of $E$ are norm-dense in $E^*$, i.e., if for each $f$ in $E^*$ and each $\epsilon > 0$ there exist $g$ in $E^*$ and $x$ in $S$ such that $|g(x)| = \|g\|$ and $\|f - g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1][1] as well as incomplete spaces which *are* subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is sub-

# The Bishop-Phelps property for operators

**Definition**

An operator $T : X \to Y$ is **norm attaining** if there exists $x_0 \in X$, $\|x_0\| = 1$, such that $\|T(x_0)\| = \|T\|$.

**Definition (Lindenstrauss)**

$(X, Y)$ has the Bishop-Phelps Property (BPp) if every operator $T : X \to Y$ can be uniformly approximated by **norm attaining** operators.

1. $(X, \mathbb{K})$ has BPp for every $X$ Bishop-Phelps (1961);
2. $\overline{\{T \in L(X;Y) : T^{**} \in NA(X^{**};Y^{**})\}} = L(X;Y)$ for every pair of Banach spaces $X$ and $Y$, Lindenstrauss (1963);
3. $X$ with RNP, then $(X, Y)$ has BPp for every $Y$, Bourgain (1977);
4. there are spaces $X$, $Y$ and $Z$ such that $(X, C([0,1]))$, $(Y, \ell^p)$ $(1 < p < \infty)$ and $(Z, L^1([0,1]))$ fail BPp, Schachermayer (1983), Gowers (1990) and Acosta (1999);
5. $(C(K), C(S))$ has BPp for all compact spaces $K, S$, Johnson and Wolfe, (1979).
6. $(L^1([0,1]), L^\infty([0,1]))$ has BPp, Finet-Payá (1998),

# . . . Bishop-Phelps-Bollobás property

# Bollobás observation

## AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

Bishop and Phelps proved in **[1]** that every real or complex Banach space is *subreflexive*, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by $S$ and $S'$ the unit spheres in a Banach space $B$ and its dual space $B'$, respectively.

THEOREM 1. *Suppose* $x \in S$, $f \in S'$ *and* $|f(x)-1| \leqslant \varepsilon^2/2$ $(0 < \varepsilon < \frac{1}{2})$. *Then there exist* $y \in S$ *and* $g \in S'$ *such that* $g(y) = 1$, $\|f-g\| \leqslant \varepsilon$ *and* $\|x-y\| < \varepsilon + \varepsilon^2$.

# A different way of writing BPB

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**Corollary. . . the way is oftentimes presented**

Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|x^*(x_0)| > 1 - \frac{\varepsilon^2}{4},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$|y^*(u_0)| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$

# Bishop-Phelps-Bollobás Property for operators

**Definition:** Acosta, Aron, García and Maestre, 2008

$(X, Y)$ is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any $\varepsilon > 0$ there are $\eta(\varepsilon) > 0$ such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that

$$\|T(x_0)\| > 1 - \eta(\varepsilon),$$

then there are $u_0 \in S_X$, $S \in S_{L(X,Y)}$ with

$$\|S(u_0)\| = 1$$

and

$$\|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| < \varepsilon.$$

1. $Y$ has *certain* almost-biorthogonal system $(X, Y)$ has BPBp any $X$;
2. $(\ell^1, Y)$ BPBp is characterized through a condition called AHSP: it holds for $Y$ finite dimensional, uniformly convex, $Y = L^1(\mu)$ for a $\sigma$-finite measure or $Y = C(K)$;
3. there is pair $(\ell^1, X)$ failing BPBp, but having BPp;
4. $(\ell^\infty_n, Y)$ has BPBp $Y$ uniformly convex no hope for $c_0$: $\eta(\varepsilon) = \eta(n, \varepsilon) \to 1$ with $n \to \infty$.

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4. $(\ell_n^\infty, Y)$ has BPBp $Y$ uniformly convex no hope for $c_0$: $\eta(\varepsilon) = \eta(n,\varepsilon) \to 1$ with $n \to \infty$.

**PROBLEM?**

No $Y$ infinite dimensional was known s.t. $(c_0, Y)$ has BPBP.

# Our main result

**Theorem (A. J. Guirao, V. Kadets and B. C. 2012)**

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T\colon X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

# The scope of our result

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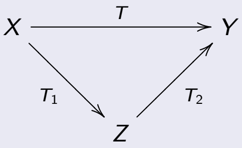
**Stegall, 1975**

An **operator** $T \in L(X,Y)$ is **Asplund**, if it factors through an Asplund space:




$Z$ is Asplund; $T_1 \in L(X,Z)$ and $T_2 \in L(Z,Y)$.

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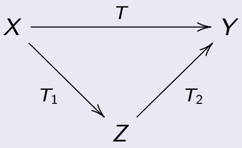


$Z$ is Asplund; $T_1 \in L(X, Z)$ and $T_2 \in L(Z, Y)$.

$T$ Asplund operator $\Leftrightarrow$ $T^*(B_{Y^*})$ is fragmented by the norm of $X^*$.

# The black box. . . fragmentability. . . I won't speak about it

## Asplund spaces: Namioka, Phelps and Stegall

Let $X$ be a Banach space. Then the following conditions are equivalent:

(i) $X$ is an Asplund space, *i.e.*, whenever $f$ is a convex continuous function defined on an open convex subset $U$ of $X$, the set of all points of $U$ where $f$ is Fréchet differentiable is a dense $G_\delta$-subset of $U$.

(ii) every $w^*$-compact subset of $(X^*, w^*)$ is fragmented by the norm;

(iii) each separable subspace of $X$ has separable dual;

(iv) $X^*$ has the Radon-Nikodým property.

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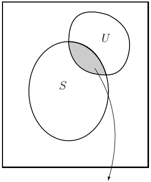


$\|\cdot\| - \mathrm{diam}(U \cap S) \le \varepsilon$

## Definition

$B_{X^*}$ is fragmented if for every $\varepsilon > 0$ and every non empty subset $S \subset B_{X^*}$ there exists a $w^*$-open subset $U \subset X$ such that $U \cap S \neq \emptyset$ and

$$\|\cdot\| - \mathrm{diam}(U \cap S) \le \varepsilon.$$

**Theorem (A. J. Guirao, V. Kadets and B. C. 2012)**

Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T : X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,\mathfrak{A})}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon$$

and

$$\|T - \widetilde{T}\| < 2\varepsilon.$$

**Corollary**

Let $T \in L(X, C_0(L))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in L(X, C_0(L))$ **weakly compact** with $\|S\| = 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

**Corollary**

$(X, C_0(L))$ has the BPBP for any Asplund space $X$ and any locally compact Hausdorff topological space $L$ ($X = c_0(\Gamma)$, **for instance**).

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**Corollary**

Let $T \in L(X, A(\overline{\mathbb{D}}))$ **weakly compact** with $\|T\| = 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

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**Remark**

The theorem applies in particular to the ideals of finite rank operators $\mathscr{F}$, compact operators $\mathscr{K}$, $p$-summing operators $\Pi_p$ and of course to the weakly compact operators $\mathscr{W}$ themselves. To the best of our knowledge even in the case $\mathscr{W}(X,\mathfrak{A})$ the Bishop-Phelps property that follows is a brand new result.

# An idea of the proof for $\mathfrak{A} = C(K)$

## Theorem (R. M. Aron, O. Kozhushkina and B. C. 2011)

Let $T\colon X \to C(K)$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\widetilde{T} \in S_{L(X,C(K))}$ satisfying that

$$\|\widetilde{T}u_0\| = 1, \|x_0 - u_0\| \leq \varepsilon \quad \text{and} \quad \|T - \widetilde{T}\| < 2\varepsilon.$$

1. **Black box** provides a suitable open set $U \subset K$, $y^* \in S_{X^*}$ and $\rho < 2\varepsilon$ with
$$1 = |y^*(u_0)| = \|u_0\| \text{ and } \|x_0 - u_0\| < \varepsilon \;\&\; \|T^*(\delta_t) - y^*\| < \rho \;\forall t \in U$$
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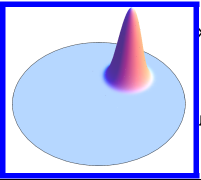

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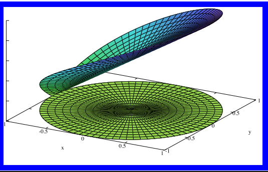

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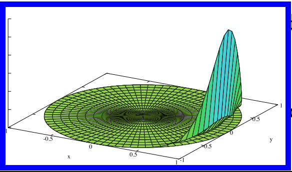

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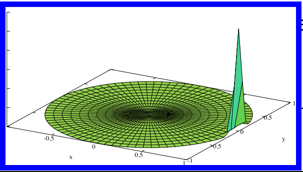


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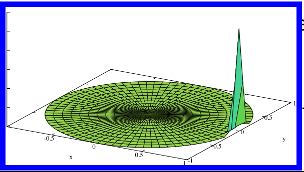


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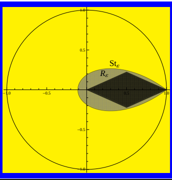


$$R_\varepsilon := \{z \in \mathbb{C} \colon |\mathrm{Re}(z) - 1/2| + (1/\sqrt{\varepsilon})|\mathrm{Im}(z)| \le 1/2\}$$

with main diagonal $[0, 1]$, that in its turn is contained in the Stolz's domain

$$\mathrm{St}_\varepsilon = \{z \in \overline{\mathbb{D}} \colon |z| + (1 - \varepsilon)|1 - z| \le 1\}.$$

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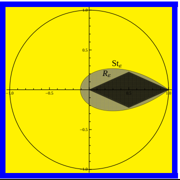


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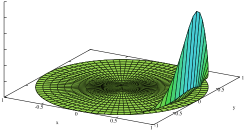


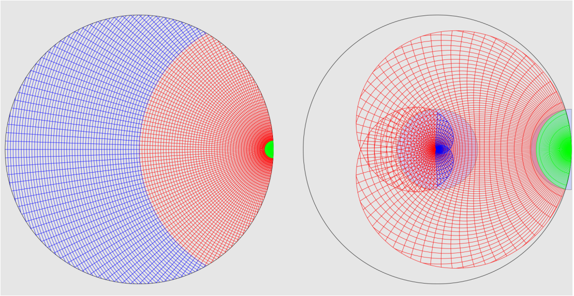

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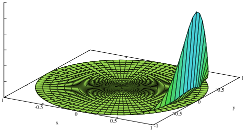

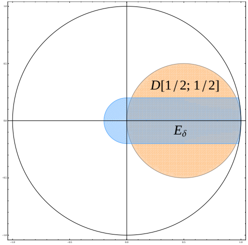

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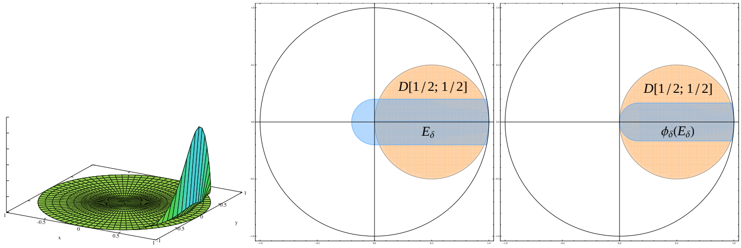

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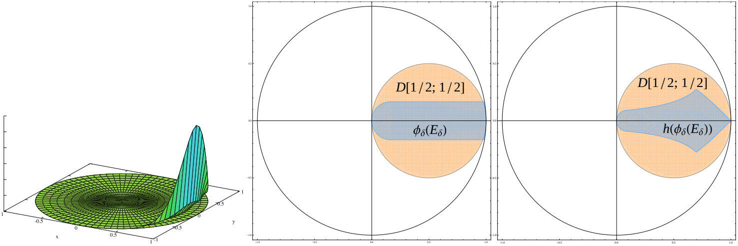

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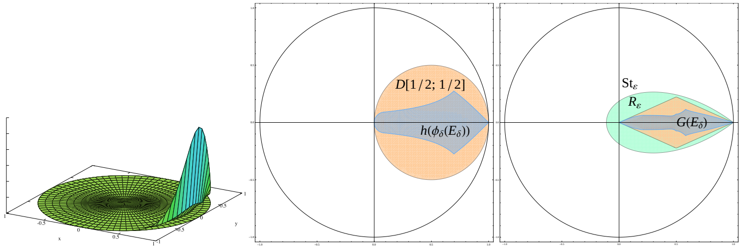

## Our Uryshon type lemma is suited for calculations with a computer

*Proof.* Let us fix $0 < \eta < \min\{\delta/6, 1/2\}$ and $n \in \mathbb{N}$, $n > \frac{2}{\eta}$. Let us define $U_1 := U$. We shall construct inductively a collection of points $\{t_j\}_{j=1}^{n}$, a decreasing finite sequence $\{U_j\}_{j=1}^{n+1}$ of open subsets of $U$, $t_j \in U_{j+1} \cap \Gamma_0$, and functions $\{f_j\}_{j=1}^{n} \subset A$, satisfying for any $j \in \{1, \dots, n\}$ the following conditions:

(i) $f_j(t_j) = 1$.
(ii) $|f_j(t)| < \frac{\eta}{2}$ for $t \in K \setminus U_j$.
(iii) $|f_j(t) - 1| < \frac{\eta}{2}$ for $t \in U_{j+1}$.

Indeed, Lemma 2.1 allows us to find a norm one function $f_1 \in A$ and a $t_1 \in U_1 \cap \Gamma_0$ such that $f_1(t_1) = 1$ and $|f_1(t)| < \frac{\eta}{2}$ for $t \in K \setminus U_1$. Define

$$U_2 = \{t \in K : |f_1(t) - 1| < \eta/2\}.$$

It is clear that $t_1 \in U_2 \cap \Gamma_0$ and $U_2 \subset U_1$. Now we can apply again Lemma 2.1 to $U_2$ to obtain $f_2$, $t_2$ and $U_3$. Proceeding inductively we obtain the expected functions $f_1, \dots, f_n$, points $t_1, \dots, t_n$ and sets $U = U_1 \supset U_2 \supset \cdots \supset U_{n+1}$ with $t_j \in U_{j+1} \cap \Gamma_0$, for $j = 1, \dots, n$.

Now, we define $f := \frac{1}{n}(f_1 + \cdots + f_n)$. Given $t \in K \setminus U$, it is clear that $t \in K \setminus U_j$ for $j = 1, \dots, n$, which implies that $|f_j(t)| < \frac{\eta}{2}$ for $j = 1, \dots, n$. Therefore, we have that

$$|f(t)| < \frac{\eta}{2} \text{ for any } t \in K \setminus U. \qquad (2.2)$$

Let us check that

$$\widehat{\mathrm{d}}(f(K), [0,1]) = \sup_{t \in K} \inf_{s \in [0,1]} |f(t) - s| \le \eta. \qquad (2.3)$$

. . .

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# We should finish with a problem...

## A reasonable question?

Does any pair of the kind $(C(K), C(L))$ have the BPBp for arbitrary compact spaces $K$ and $L$?

# We should finish with a problem. . . and recommendations

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## How to talk Mathematics

My recommendations amount to this: make it simple, organized and short. Make your lecture simple (special and concrete); be sure to prove something and ask something; prepare, in detail; organize the content and adjust to the level of the audience; keep it short, and, to be sure of doing so, prepare it so as to make it flexible. Remember that you are talking in order to attract the listeners to your subject and to inform them about it; and remember that less is more.

P.R. Halmos

http://www.math.northwestern.edu/graduate/Forum/HALMOS.html

# GRACIAS!