Topology, measure theory and Banach spaces

B. Cascales

Universidad de Murcia

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The co-authors

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Throughout the lecture. . .

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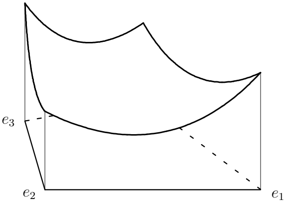

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The boundary problem (Godefroy)...extremal test for compactness

Let X Banach space, $B \subset B_{X^*}$ boundary and denote by $\tau_p(B)$ the topology defined on X by the pointwise convergence on B. Let H be a norm bounded and $\tau_p(B)$-compact subset of X.

Is H weakly compact?

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- ✓ Since $1 - g \geq 0$ we obtain $0 = |\mu|(\{y \in K : 1 - g(y) > 0\}) = |\mu|(K \setminus F)$
- ✓ Then for every $n \in \mathbb{N}$,

$$\int_K f_n d\mu = \int_F f_n d\mu = \int_F f_n(x) d\mu = f_n(x)$$

because μ is a probability itself.

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- ✓ **Bad news:** $\tau_p(B)$ is not compatible with $\langle X, X^* \rangle$.
- ✓ **Good news:** we can overcome the difficulties for many Banach spaces.

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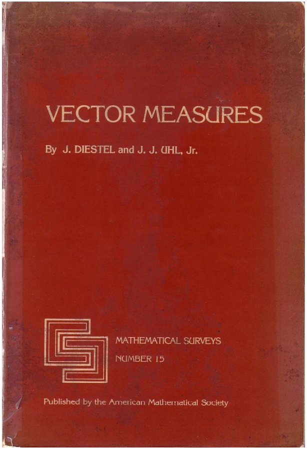

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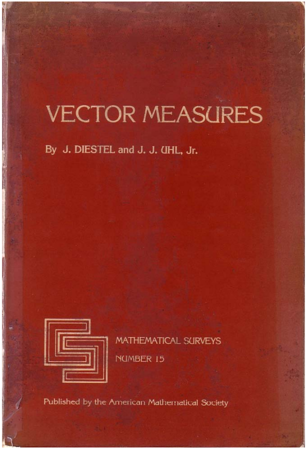

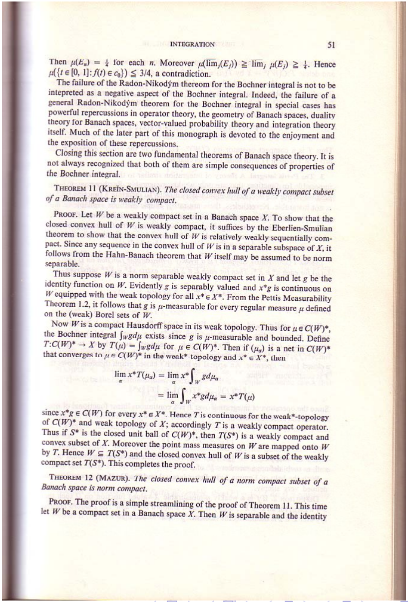

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The failure of the Radon-Nikodým thereom for the Bochner integral is not to be intepreted as a negative aspect of the Bochner integral. Indeed, the failure of a general Radon-Nikodým theorem for the Bochner integral in special cases has powerful repercussions in operator theory, the geometry of Banach spaces, duality theory for Banach spaces, vector-valued probability theory and integration theory itself. Much of the later part of this monograph is devoted to the enjoyment and the exposition of these repercussions.

Closing this section are two fundamental theorems of Banach space theory. It is not always recognized that both of them are simple consequences of properties of the Bochner integral.

THEOREM 11 (KREĬN-SMULIAN). *The closed convex hull of a weakly compact subset of a Banach space is weakly compact.*

PROOF. Let W be a weakly compact set in a Banach space X. To show that the closed convex hull of W is weakly compact, it suffices by the Eberlien-Smulian theorem to show that the convex hull of W is relatively weakly sequentially compact. Since any sequence in the convex hull of W is in a separable subspace of X, it follows from the Hahn-Banach theorem that W itself may be assumed to be norm separable.

Thus suppose W is a norm separable weakly compact set in X and let g be the identity function on W. Evidently g is separably valued and x^*g is continuous on W equipped with the weak topology for all $x^* \in X^*$. From the Pettis Measurability Theorem 1.2, it follows that g is μ-measurable for every regular measure μ defined on the (weak) Borel sets of W.

Now W is a compact Hausdorff space in its weak topology. Thus for $\mu \in C(W)^*$, the Bochner integral $\int_W g\,d\mu$ exists since g is μ-measurable and bounded. Define $T: C(W)^* \to X$ by $T(\mu) = \int_W g\,d\mu$ for $\mu \in C(W)^*$. Then if (μ_α) is a net in $C(W)^*$ that converges to $\mu \in C(W)^*$ in the weak* topology and $x^* \in X^*$, then

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PROOF. The proof is a simple streamlining of the proof of Theorem 11. This time let W be a compact set in a Banach space X. Then W is separable and the identity

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$$\lim_\alpha x^*T(\mu_\alpha) = \lim_\alpha x^* \int_W g\,d\mu_\alpha$$
$$= \lim_\alpha \int_W x^*g\,d\mu_\alpha = x^*T(\mu)$$

since $x^*g \in C(W)$ for every $x^* \in X^*$. Hence T is continuous for the weak*-topology of $C(W)^*$ and weak topology of X; accordingly T is a weakly compact operator. Thus if S^* is the closed unit ball of $C(W)^*$, then $T(S^*)$ is a weakly compact and convex subset of X. Moreover the point mass measures on W are mapped onto W by T. Hence $W \subseteq T(S^*)$ and the closed convex hull of W is a subset of the weakly compact set $T(S^*)$. This completes the proof.

Krein-Smulyan type result

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Proof.- Fix μ a Radon probability on $(H, \tau_p(B))$, find a barycenter for μ?

$$\text{find } x_\mu \in X \text{ with } x^*(x_\mu) = \int_H x^*|_H d\mu, \text{ for every } x^* \in X^*?$$

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$(b_n^*)_n$ in B is independent on H if there are $s < t$ such that

$$\left(\bigcap_{n\in P} \{w \in H : b_n^*(w) < s\}\right) \cap \left(\bigcap_{n\in Q} \{w \in H : b_n^*(w) > t\}\right)$$

for every disjoint finite sets $P, Q \subset \mathbb{N}$.

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- ✓ $B|_H := \{x^*|_H : x^* \in B\} \subset C(H, \tau_p(B))$ and $B_{X^*}|_H = \overline{B|_H}^{\tau_p(H)}$;
- ✓ if $B|_H$ does not have independent sequences (Rosenthal) then $\overline{B|_H}^{\tau_p(H)}$ is made up of μ-measurable functions for each μ;
- ✓ indeed, $B|_H$ as above has Bourgain property with respect to μ.

Bourgain property... a bit of history

Definition

We say that a family $\mathscr{F} \subset \mathbb{R}^{\Omega}$ has **Bourgain property** if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \ldots, B_n \subset A$, $B_i \in \Sigma$, with $\mu(B_i) > 0$ such that for every $f \in \mathscr{F}$

$$\inf_{1 \leq i \leq n} |\cdot| \operatorname{diam}(f(B_i)) < \varepsilon.$$

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The property of Bourgain

- The notion wasn't published by Bourgain.

Bourgain property. . . a bit of history

for each x in E. Since the set $\{\langle f, x\rangle : \|x\| \leq 1\}$ contains no copy of the l_1-basis in $L_\infty(\Sigma, \mu)$ and the conditional expectation operator ξ is a contraction from $L_\infty(\Sigma, \mu)$ into $L_\infty(\Gamma, \mu)$, we may conclude that $T(B_E)$ contains no copy of the l_1-basis in $L_\infty(\Gamma, \mu)$. Consequently $T(B_E)$ is weakly precompact in $L_\infty(\Gamma, \mu)$ and there is a Pettis integrable kernel $g : (\Omega, \Gamma, \mu) \to E^*$ for the operator

$$T^* : L_1(\Gamma, \mu) \to E^*.$$

Then $\langle g, x\rangle = Tx = \xi(\langle f, x\rangle | \Gamma)$ a.e. for every x in E. Therefore

$$\int_B \langle g, x\rangle \, d\mu = \int_B \xi(\langle f, x\rangle | \Gamma) \, d\mu = \int_B \langle f, x\rangle \, d\mu$$

for every set B in Γ and hence $\int_B g \, d\mu = \int_B f \, d\mu$ for every set B in Γ. This shows that g is a Pettis conditional expectation of f for the σ-algebra Γ.

In view of Theorems 5 and 9, one can ask the following.

Question. If, in Theorem 9, we suppose that the set

$$\{\langle f, x\rangle : \|x\| \leq 1\}$$

is almost weakly precompact in $L_\infty(\mu)$, does f have a Pettis conditional expectation with respect to all sub-σ-algebras of Σ?

If the above were true, then any function satisfying the conditions of Theorem 5 would have a Pettis conditional expectation with respect to all Radon measurers on all sub-σ-algebras of the Borel σ-algebra of K.

IV. The Bourgain property

So far we have seen that the family $\{\langle f, x\rangle : \|x\| \leq 1\}$ plays a strong role in determining Pettis integrability for a bounded scalarly measurable function f from Ω into a dual space E^*. We continue this approach in this part, but, rather than viewing such families as subsets of $L_\infty(\mu)$, we now consider them simply as families of real-valued functions on Ω. A property of real-valued functions formulated by J. Bourgain [2] is the cornerstore of our discussion.

DEFINITION 10. Let (Ω, Σ, μ) be a measure space. A family Ψ of real-valued functions on Ω is said to have the *Bourgain property* if the following condition is satisfied: For each set A of positive measure and for each $\alpha > 0$, there is a finite collection F of subsets of positive measure of A such that for each function f in Ψ, the inequality $\sup f(B) - \inf f(B) < \alpha$ holds for some member B of F.

Definition

We say that a family $\mathscr{F} \subset \mathbb{R}^\Omega$ has **Bourgain property** if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \ldots, B_n \subset A$, $B_i \in \Sigma$, with $\mu(B_i) > 0$ such that for every $f \in \mathscr{F}$

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- The notion wasn't published by Bourgain.
- It appears in a paper by [RS85] and refers to handwritten notes by Bourgain.

Remarkable facts about Bourgain property

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Properties

- If $\mathscr{F} = \{f\}$, TFAE:
  - (i) (Bourgain property) For every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there is $B \in \Sigma$, $B \subset A$ with $\mu(B) > 0$ and $|\cdot|\text{diam}\, f(B) < \varepsilon$.
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- If $\mathscr{F}$ has Bourgain property, then $\mathscr{F}$ is made up of measurable functions.

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- $\mathscr{F}$ has Bourgain property and $f \in \overline{\mathscr{F}}^{\tau_p(\Omega)}$, then there is a sequence (f_n) in $\mathscr{F}$ that converges to f, μ-almost everywhere.

. . . back to Krein-Smulyan type result

Wish. . .

Take X Banach space and $B \subset B_{X^*}$ 1-norming (i.e. $\|x\| = \sup\{x^*(x) : x^* \in B\}$). For every norm bounded $\tau_p(B)$-relatively compact subset H of X its $\tau_p(B)$-closed convex hull $\overline{\mathrm{co}(H)}^{\tau_p(B)}$ is $\tau_p(B)$-compact.

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- ✓ if $B|_H$ has an independent sequence on $H \Rightarrow \beta\mathbb{N} \subset (B_{X^*}, w^*)$.

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Theorem: Manjabacas, Vera and B.C., 1997

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...back to Krein-Smulyan type result

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Take X Banach space and $B \subset B_{X^*}$ 1-norming (i.e. $\|x\| = \sup\{x^*(x) : x^* \in B\}$). For every norm bounded $\tau_p(B)$-relatively compact subset H of X its $\tau_p(B)$-closed convex hull $\overline{\mathrm{co}(H)}^{\tau_p(B)}$ is $\tau_p(B)$-compact, assuming $\ell^1(c) \not\subset X$.

- ✓ if $B|_H$ does not have independent sequences (Rosenthal), then $B|_H$ has Bourgain property with respect to μ.
- ✓ $B_{X^*}|_H = \overline{B|_H}^{\tau_p(H)}$ has Bourgain property;
- ✓ we can write $T_\mu(x^*) = \int_H x^*|_H d\mu$ and $T_\mu \in X^{**}$;
- ✓ given any $A \subset B_{X^*}$ if $x^*|_H \in \overline{A|_H}^{\tau_p(H)}$, then there is a sequence $(x_n^*|_H)$ in $A|_H$ that converges to $x^*|_H$, μ-almost everywhere.
- ✓ then for each $A \subset B_{X^*}$, $T_\mu(\overline{A}^{w^*}) \subset \overline{T_\mu(A)}$;
- ✓ Grothendieck completeness$\Rightarrow$ $T_\mu = x_\mu \in X$ barycenter... goes on smoothly.
- ✓ if $B|_H$ has an independent sequence on $H \Rightarrow \beta\mathbb{N} \subset (B_{X^*}, w^*)$.

What we know about the boundary problem for X

Theorem: Manjabacas, Vera and B.C., 1997

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Corollary: Manjabacas, Vera and B.C., 1997

Let X be a Banach space such that $\ell^1(c) \not\subset X$ and B any boundary for B_{X^*}. If $H \subset X$ is norm bounded and $\tau_p(B)$-compact then H is weakly compact.

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A. S. Granero 2006

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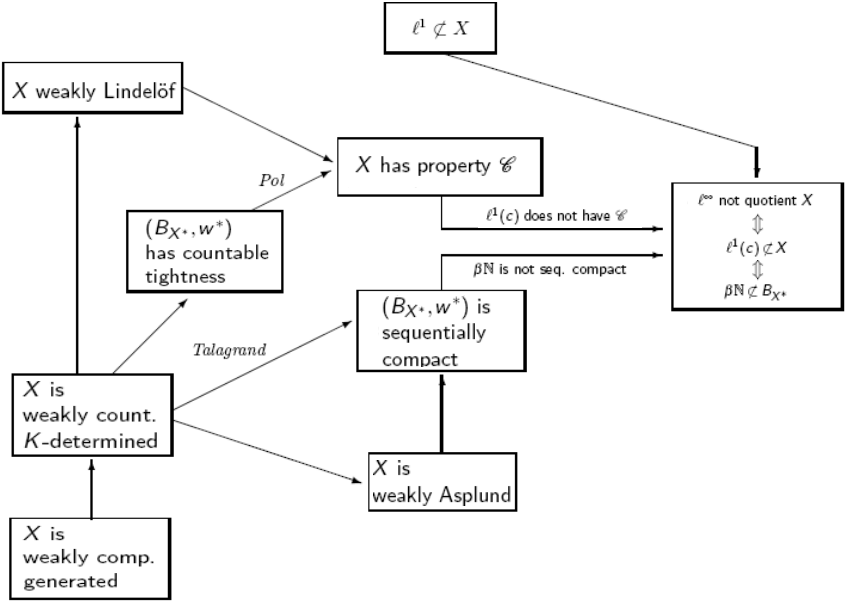
$\ell^1 \not\subset X$
X weakly Lindelöf
X has property $\mathscr{C}$
Pol
(B_{X^*}, w^*) has countable tightness
$\ell^1(c)$ does not have $\mathscr{C}$
ℓ^∞ not quotient X
$\Updownarrow$
$\ell^1(c) \not\subset X$
$\Updownarrow$
$\beta\mathbb{N} \not\subset B_{X^*}$
$\beta\mathbb{N}$ is not seq. compact
(B_{X^*}, w^*) is sequentially compact
Talagrand
X is weakly count. K-determined
X is weakly Asplund
X is weakly comp. generated

Bourgain property & Birkhoff integrability

1. Given $H \subset X$ $\tau_p(B)$ compact and μ Radon probability we have studied (Pettis) integrability of $id : H \hookrightarrow X$ using Bourgain property of

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Using techniques of Pettis integration the known answer is: f is Pettis integrable... but in this case the outcome is in fact better.

Birkhoff definition

Let $f : \Omega \longrightarrow X$ be a function. If Γ is a partition of Ω into countably many sets (A_n) of Σ, the function f is called **summable** with respect to Γ if the restriction $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set of sums

$$J(f,\Gamma) = \left\{ \sum_n f(t_n)\mu(A_n) : t_n \in A_n \right\}$$

is made up of unconditionally convergent series.

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In this case, the **Birkhoff integral** $(B)\int_\Omega f \, d\mu$ of f is the only point in the intersection

$$\bigcap \{ \overline{\operatorname{co}(J(f,\Gamma))} : f \text{ is summable with respect to } \Gamma \}.$$

Birkhoff integrability: properties

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5. Birkhoff integrability has been historically ignored.

Our basic result

We characterize Birkhoff integrability via the property of Bourgain.

Bourgain property and Birkhoff integrability

Theorem (Rodriguez-B.C., 2005)

Let $f : \Omega \to X$ be a bounded function. TFAE:

(i) *f is Birkhoff integrable;*

(ii) *$Z_f = \{\langle x^*, f\rangle : \ x^* \in B_{X^*}\}$ has Bourgain property.*

Theorem (Rodriguez-B.C., 2005)

Let $f : \Omega \longrightarrow X$ be a function. TFAE:

(i) *f is Birkhoff integrable;*

(ii) *Z_f is uniformly integrable, Z_f has Bourgain property.*

Applications to URL integrable functions

Theorem (Rodriguez-B.C., 2005)

Let $f : \Omega \longrightarrow X$ be a function. TFAE:

(i) *f is Birkhoff integrable;*

(ii) *there is $x \in X$ satisfying: for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ for which f is summable and*

$$\|S(f,\Gamma,T) - x\| < \varepsilon \text{ for every choice } T \text{ in } \Gamma;$$

(iii) *there is $y \in X$ satisfying: for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ such that f is summable with respect to each countable partition Γ' finer than Γ and*

$$\|S(f,\Gamma',T') - y\| < \varepsilon \qquad \text{for every choice } T' \text{ in } \Gamma'.$$

In this case, $x = y = \int_\Omega f \, d\mu$.

Musiał question?

Zbl 0974.28007

Kadets, V.M.; Tseytlin, L.M.

On "integration" of non-integrable vector-valued functions.

Mat. Fiz. Anal. Geom. 7, No.1, 49-65 (2000)

Let μ be the Lebesgue measure on $[0,1]$ and X be a Banach space. A function $f:[0,1]\to X$ is called absolutely Riemann-Lebesgue integrable over a measurable set $A\subset[0,1]$ if there is $x\in X$ such that for every $\varepsilon>0$ there exists a measurable partition $\langle\Delta_i\rangle_{i=1}^{\infty}$ of A such that for every finer measurable partition $\langle\Gamma_j\rangle_{j=1}^{\infty}$ of A and arbitrary points $s_j\in\Gamma_j$ one has $\|\sum_j f(s_j)\mu(\Gamma_j)-x\|<\varepsilon$ and $\sum_j f(s_j)\mu(\Gamma_j)$ is absolutely convergent ($\langle\Gamma_j\rangle_{j=1}^{\infty}$ is finer than $\langle\Delta_i\rangle_{i=1}^{\infty}$ if each Δ_i is a union of some Γ_j's). In case of unconditional convergence one gets a definition of unconditionally Riemann-Lebesgue integrable function...

There are no results placing ARL and URL integrals among other known types of integrals such as Birkhoff's integral or generalized McShane's integral which have similar definitions (and it is relatively easy to see that URL integrable functions are also Birkhoff integrable).

The rest of the paper is devoted to the study...

Kazimierz Musiał (Wrocław)

Applications to dual spaces with WRNP

Definition

1. X^* has the weak Radon-Nikodým property;

3. for every complete probability space (Ω, Σ, μ) and for every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of σ-finite variation there is a Pettis integrable function $f : \Omega \longrightarrow X^*$ such that

$$\nu(E) = \int_E f \, d\mu$$

for every $E \in \Sigma$.

Applications to dual spaces with WRNP

Theorem: Musiał, Ryll-Nardzewski, Janicka and Bourgain

Let X be a Banach space. TFAE:

1. X^* has the weak Radon-Nikodým property;
2. X does not contain a copy of ℓ^1;
3. for every complete probability space (Ω, Σ, μ) and for every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of σ-finite variation there is a Pettis integrable function $f : \Omega \longrightarrow X^*$ such that

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The integral for a multifunction

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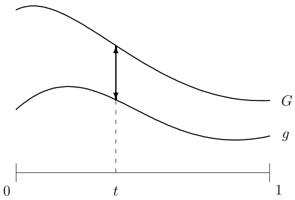

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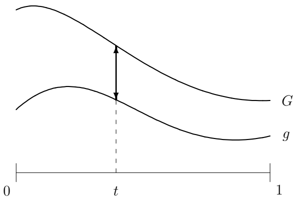

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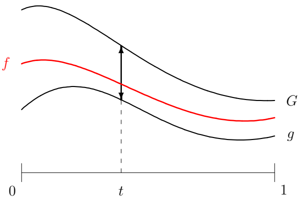

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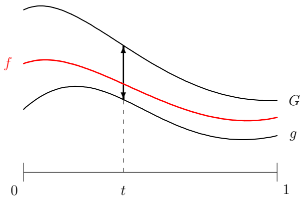

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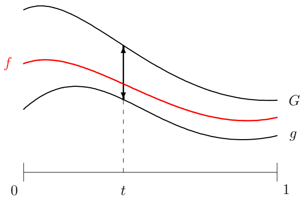

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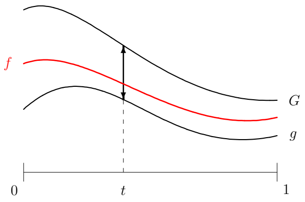

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1. Debreu, [Deb67], used the embedding technique dealing with Bochner integrability;
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3. We used the embedding technique with Birkhoff integrability: Rodriguez-B.C., 2004.

Kadets-Rodriguez-B.C., 2006

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Kadets-Rodriguez-B.C., 2006

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- ✓ . . . if and only if equivalent to the fact that $cwk(X)$ is separable when endowed with the Hausdorff distance.

... back to boundaries: Simons' techniques

Theorem: Orihuela, Muñoz, B.C., to appear

Let $J : X \to 2^{B_{X^*}}$ be the duality mapping

$$J(x) := \{x^* \in B_{X^*} : x^*(x) = \|x\|\}.$$

TFAE:

(i) X is Asplund, *i.e.*, X^* has RNP;

(ii) for some fixed $0 < \varepsilon < 1$, J has an ε-selector f that sends norm separable subsets of X into norm separable subsets of X^*;

(iii) for some fixed $0 < \varepsilon < 1$, dual unit ball B_{X^*} is norm ε-fragmented.

ε-selector: $d(f(x), J(x)) < \varepsilon$ for every $x \in X$

Two. . . three nice problems

1. The boundary problem in full generality (Godefroy).
2. Characterize Banach spaces X for which (B_{X^*}, w^*) is sequentially compact (Diestel).
3. Characterize Banach spaces X for which (B_{X^*}, w^*) is angelic.

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