A new look at compactness via distances to function spaces

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Many classical results about compactness in functional analysis can be derived from suitable inequalities involving distances to spaces of continuous or Baire one functions: this approach gives an extra insight to the classical results as well as triggers a number of open questions in different exciting research branches. We exhibit here, for instance, *quantitative* versions of Grothendieck's characterization of weak compactness in spaces $C(K)$ and also of the Eberlein-Šmulyan and Krein-Šmulyan theorems. The above results specialized in Banach spaces lead to several equivalent measures of *non-weak compactness*. In a different direction we envisage a method to measure the distance from a function $f \in \mathbb{R}^X$ to $B_1(X)$ –space of Baire one functions on X– which allows us to obtain, when X is Polish, a quantitative version of the well known Rosenthal's result stating that in $B_1(X)$ the pointwise relatively countably compact sets are pointwise compact. Other results and applications are commented too.

Keywords: Eberlein-Grothendieck theorem, Krein-Smulyan theorem, oscillations, iterated limits, compactness, measures of non compactness, distances to function spaces, Rosenthal theorem, Baire one functions

1. Introduction

These are the written notes of a lecture with the same title delivered by the second named author at the *III International Course of Mathematical Analysis of Andalucía, Huelva, September 3-7, 2007.* We collect here results, mostly without proof, that mainly correspond to the papers.[1–4] A good deal of extra information about the subject can also be found in the Ph. D. dissertation by the second named author.[5]

In this *survey* we present recent *quantitative* versions of many of the classical compactness results in functional analysis and their relatives. As an example, and in order to fix ideas, one of the problems studied is illustrated and explained in the lines below. Take K a compact Hausdorff space and let $C(K)$ be the space of real-valued continuous functions defined on K. Look at $C(K)$ embedded in $\mathbb{R}^K$, let d be the metric of uniform convergence on $\mathbb{R}^K$ and take $H \subset \mathbb{R}^K$ a uniformly bounded set. If τ_p is the topology of pointwise convergence on $\mathbb{R}^K$, then Tychonoff's theorem says that $\overline{H}^{\mathbb{R}^K}$ is τ_p-compact. Therefore for H to being τ_p-relatively compact in $C(K)$ the only thing we should worry about is to have $\overline{H}^{\mathbb{R}^K} \subset C(K)$. Notice that if $\hat{d}$ is the *worst* distance from $\overline{H}^{\mathbb{R}^K}$ to $C(K)$ then $\hat{d} = 0$ if, and only, if $\overline{H}^{\mathbb{R}^K} \subset C(K)$. In general $\hat{d} \geq 0$ gives us a measure of non τ_p-compactness for H relative to $C(K)$.

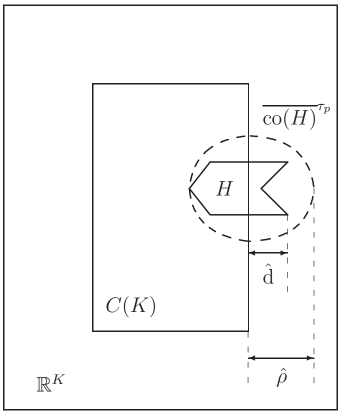

Figure 1

Hence the questions are: *a) Is there any way of computing $\hat{d}$? b) are there useful estimates involving $\hat{d}$ that are equivalent to qualitative properties of the sets H's?* The answer to *a)* has been known for a long time and is yes: the distance of a function $f \in \mathbb{R}^K$ to $C(K)$ can be computed in terms of the global oscillation of f on K, see section 2. Here is a first case in the spirit of *b)* that is illustrated through the Figure 1: if $\hat{\rho}$ is the *worst* distance from the closed convex hull $\overline{\text{co}(H)}^{\mathbb{R}^K}$ to $C(K)$, then it is proved that $\hat{d} \leq \hat{\rho} \leq 5\hat{d}$; – the constant 5 can be replaced by 2 for sets $H \subset C(K)$. Note that the above inequality is the quantitative version of the celebrated Krein-Šmulyan theorem about weak compactness of the closed convex hull of weakly compact sets in Banach spaces.

A bit of the history behind the classical results that we quantify follows. In 1940 Šmulyan[6] showed that weakly relatively compact subsets of a Banach space are weakly relatively sequentially compact. He also proved that if a Banach space E has w^*-separable dual then a subset H of E is weakly relatively countably compact if, and only if, H is weakly relatively sequentially compact. Dieudonné and Schwartz[7] extended this last result to locally convex spaces with a coarser metrizable topology. The converse of Šmulyan theorem was obtained by Eberlein[8] who proved that relatively countably compact sets are relatively compact sets for the weak topology of a Banach space. Grothendieck generalized these results to locally convex spaces that are quasicomplete for its Mackey topology: this result is based upon a similar one for spaces $(C(K), \tau_p)$ of continuous functions on a compact space K endowed with the pointwise convergence topology. Fremlin's notion of angelic space and some of its consequences can be used for proving the above results in a clever and clear way, see the book by Floret.[9] Orihuela[10] showed in 1987 that spaces $(C(X), \tau_p)$ with X a countably K-determined space (or even more general spaces) are angelic. Similarly, for spaces $(B_1(X), \tau_p)$ of Baire one functions on a Polish space with the pointwise convergence topology, Rosenthal showed that relatively countably compact sets are relatively compact. Bourgain, Fremlin and Talagrand[11] showed that in fact $(B_1(X), \tau_p)$ is angelic.

In recent years, several *quantitative* counterparts for some other classical results have been proved by different authors. These new versions strengthen the original theorems and lead to new problems and applications in topology and analysis: see, for instance,[12–16]

A bit of terminology: by letters $T, X, Y, \ldots$ we denote sets or completely regular topological spaces; (Z, d) is a metric space (Z if d is tacitly assumed); $\mathbb{R}$ is considered as a metric space endowed with the metric associated to the absolute value $|\cdot|$. The space Z^X is equipped with the product topology τ_p. We let $C(X, Z)$ denote the space of all Z-valued continuous functions on X, and let $B_1(X, Z)$ denote the space of all Z-valued functions of the first Baire class (Baire one functions), i.e. pointwise limits of Z-valued continuous functions. When $Z = \mathbb{R}$, we write, as usual, $C(X)$ and $B_1(X)$ for $C(X, \mathbb{R})$ and $B_1(X, \mathbb{R})$, respectively.

If $\emptyset \neq A \subset (Z, d)$ we write $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$. For A and B nonempty subsets of (Z, d), we consider the *usual distance* between A and B given by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

and the *Hausdorff non-symmetrized distance* from A to B defined by

$$\hat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

In Z^X we deal with the *standard supremum metric* given for arbitrary functions $f, g \in Z^X$ by

$$d(f, g) = \sup_{x \in X} d(f(x), g(x))$$

that is allowed to take the value $+\infty$. If $\mathcal{F} \subset Z^X$ is some space of functions we consequently define $d(f, \mathcal{F})$ and $\hat{d}(A, \mathcal{F})$ for sets $A \subset Z^X$; the spaces of functions $\mathcal{F}$ that we will consider are $C(X, Z)$ and $B_1(X, Z)$.

By $(E, \|\cdot\|)$ we denote a real Banach space (or simply E if $\|\cdot\|$ is tacitly assumed). Finally, B_E stands for the closed unit ball in E, E^* for the dual space of E and E^{**} for the bidual space of E; w is the weak topology of a Banach space and w^* is the weak* topology in the dual.

2. Distance to spaces of continuous functions

We start with the proof for the formula (1) below that gives us the distance of a function $f \in \mathbb{R}^X$ to the space of continuous functions $C(X)$. Next result is used in the proof that we provide for Theorem 2.2.

Theorem 2.1 ([17, Theorem 12.16]). *Let X be a normal space and let $f_1 \leq f_2$ be two real functions on X such that f_1 is upper semicontinuous and f_2 is lower semicontinuous. Then, there exists a continuous function $f \in C(X)$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in X$.*

Theorem 2.2. *Let X be a normal space. If $f \in \mathbb{R}^X$, then*

$$d(f, C(X)) = \frac{1}{2}\operatorname{osc}(f) \tag{1}$$

where

$$\operatorname{osc}(f) = \sup_{x \in X} \operatorname{osc}(f, x) = \sup_{x \in X} \inf\{\operatorname{diam} f(U) : U \subset X \text{ open}, x \in U\}.$$

Proof.

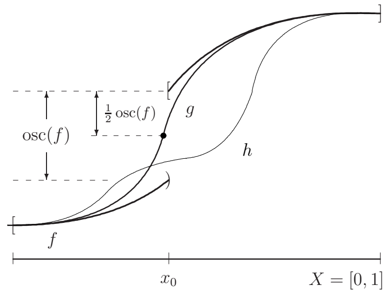

Figure 2

We prove first that $\frac{1}{2}\operatorname{osc}(f) \leq d(f, C(X))$. If $d(f, C(X))$ is infinite, the inequality clearly holds. Suppose that $\rho = d(f, C(X))$ is finite. Fix $x \in X$ and $\varepsilon > 0$. Take $g \in C(X)$ such that $d(f, g) \leq \rho + \varepsilon/3$. Since g is continuous at x, there is an open neighborhood U of x such that $\operatorname{diam}(g(U)) < \varepsilon/3$. Then, if $y, z \in U$,

$$d(f(y), f(z)) \leq d(f(y), g(y)) + d(g(y), g(z)) + d(g(z), f(z)) < 2\rho + \varepsilon.$$

Thus $\operatorname{osc}(f, x) < 2\rho + \varepsilon$ for each $\varepsilon > 0$. We conclude that $\operatorname{osc}(f, x) \leq 2\rho$ for every $x \in X$ and so the inequality $\frac{1}{2}\operatorname{osc}(f) \leq d(f, C(X))$ is established.

Let us prove now that $d(f, C(X)) \leq \frac{1}{2}\operatorname{osc}(f)$. We only have to prove the inequality when $\delta = \frac{1}{2}\operatorname{osc}(f)$ is finite. For $x \in X$ denote by $\mathcal{U}_x$ the family of open neighborhoods of x and define

$$\mathcal{V}_x := \{U \in \mathcal{U}_x : \operatorname{diam}(f(U)) < \operatorname{osc}(f) + 1\}.$$

Clearly $\mathcal{V}_x$ is a basis of neighborhoods for x and for each $U \in \mathcal{V}_x$, $f|_U$ is upper and lower bounded.

An easy computation gives us that

$$\begin{aligned} 2\delta \geq \operatorname{osc}(f,x) &= \inf_{U\in\mathcal{U}_x} \operatorname{diam}(f(U)) = \inf_{U\in\mathcal{V}_x} \operatorname{diam}(f(U)) \\ &= \inf_{U\in\mathcal{V}_x} \sup_{y,z\in U} (f(y)-f(z)) \\ &\geq \inf_{U,V\in\mathcal{V}_x} \sup_{y\in U, z\in V} (f(y)-f(z)) = \\ &= \inf_{U\in\mathcal{V}_x} \sup_{y\in U} f(y) - \sup_{U\in\mathcal{V}_x} \inf_{z\in U} f(z). \end{aligned}$$

If we define

$$f_1(x) := \inf_{U\in\mathcal{V}_x} \sup_{z\in U} f(z) - \delta$$

$$f_2(x) := \sup_{U\in\mathcal{V}_x} \inf_{z\in U} f(z) + \delta$$

then $f_1 \leq f_2$. It is easy to check that f_1 is upper semi-continuous and f_2 is lower semi-continuous. By Theorem 2.1, there is a continuous function $h \in C(X)$ such that

$$f_1(x) \leq h(x) \leq f_2(x)$$

for every $x \in X$. On the other hand, for every $x \in X$ we have

$$f_2(x) - \delta \leq f(x) \leq f_1(x) + \delta$$

and therefore

$$h(x) - \delta \leq f_2(x) - \delta \leq f(x) \leq f_1(x) + \delta \leq h(x) + \delta.$$

So $d(f,h) \leq \delta = \frac{1}{2}\operatorname{osc}(f)$ and this finishes the proof. □

A proof for the above result when X is a paracompact space and all functions are assumed to be bounded can be found in [18, Proposition 1.18]. We note that the validity of Theorem 2.2 characterizes normal spaces.

Corollary 2.1. *Let X be a topological space. The following statements are equivalent:*

(i) *X is normal,*
(ii) *for each $f \in \mathbb{R}^X$ there is $g \in C(X)$ such that $d(f,g) = \frac{1}{2}\operatorname{osc}(f)$,*
(iii) *$d(f, C(X)) = \frac{1}{2}\operatorname{osc}(f)$ for each function $f \in \mathbb{R}^X$.*

3. Distances to spaces of continuous functions on compact spaces

We aim now to estimate $\hat{d} = \hat{d}(\overline{H}^{\mathbb{R}^K}, C(K))$ using some other distinguished *quantities* that we shall define.

Let T be a topological space. For a subset A of T, $A^{\mathbb{N}}$ is considered as the set of all sequences in A and the set of all cluster points in T of a sequence $\varphi \in A^{\mathbb{N}}$ is denoted by $\mathrm{clust}_T(\varphi)$. Recall that $\mathrm{clust}_T(\varphi)$ is a closed subset of T that can be expressed as

$$\mathrm{clust}_T(\varphi) = \bigcap_{n \in \mathbb{N}} \overline{\{\varphi(m) : m > n\}}.$$

Definition 3.1. Let X be a topological space and (Z, d) a metric space. If H be a subset Z^X we define

$$\mathrm{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\mathrm{clust}_{\mathbb{R}^K}(\varphi), C(X, Z)).$$

If $K \subset X$ we write

$$\gamma_K(H) := \sup\{d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) : (f_m) \subset H, (x_n) \subset K\},$$

assuming the involved limits exist.

By definition we agree that $\inf \emptyset = +\infty$. Observe that if $H \subset C(X, Z)$ is a τ_p-relatively countably compact subset of $C(X, Z)$ then $\mathrm{ck}(H) = 0$. Also notice that $\gamma_K(H) = 0$ means in the language of[19] that H interchanges limits with K.

Theorem 3.1 ([1,2]). *Let K be a compact space and let H be a uniformly bounded subset of $C(K)$. We have*

$$\mathrm{ck}(H) \overset{(a)}{\leq} \hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)) \overset{(b)}{\leq} \gamma_K(H) \overset{(c)}{\leq} 2\,\mathrm{ck}(H).$$

Explanation of the proof.-. The details of the proof can be found in.[1,2] Here is a pretty short explanation of the ideas behind. Inequality (a) straightforwardly follows from the definitions involved. Inequality (c) uses the same kind of arguments than those used in the proof to show that if H is τ_p-relatively compact in $C(K)$ then H interchanges limits with K. Inequality (b) is much more involved than the other two: here the idea is to show that for every $x \in K$ and $f \in \overline{H}^{\mathbb{R}^K}$ the semi-oscillation

$$\mathrm{osc}^*(f, x) := \inf_U \{\sup_{y \in U} |f(y) - f(x)| : U \subset X \text{ open}, x \in U\}.$$

is at most $\gamma_K(H)$. Therefore, $\mathrm{osc}(f) \leq 2\gamma_K(H)$ and now Theorem 2.2 applies to finally obtain that $d(f, C(K)) \leq \gamma_K(H)$. Thus $d(\overline{H}^{\mathbb{R}^K}, C(K)) \leq \gamma_K(H)$ and (a) is proved. □

The following theorem is a quantitative version of the Krein-Šmulyan theorem: see next section for its consequences in Banach spaces.

Theorem 3.2 ([1]). *Let K be a compact topological space and let H be a uniformly bounded subset of $\mathbb{R}^K$. Then*

$$\gamma_K(H) = \gamma_K(co(H)) \tag{2}$$

and as a consequence for $H \subset C(K)$ we obtain that

$$\hat{d}(\overline{co(H)}^{\mathbb{R}^K}, C(K)) \leq 2\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)) \tag{3}$$

and if $H \subset \mathbb{R}^K$ is uniformly bounded then

$$\hat{d}(\overline{co(H)}^{\mathbb{R}^K}, C(K)) \leq 5\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)). \tag{4}$$

Explanation of the proof.-. The equality (2) is rather involved: the proof offered in[1] uses some ideas from the proof of the Krein-Smulyan theorem in Kelley-Namioka's book [20, Ch 5. Sec. 17]; we note that a version for Banach spaces, less general than the one here, was proved first in[12] using Ptak's combinatorial lemma. Inequality (3) easily follows from (2) and Theorem 3.1:

$$\hat{d}(\overline{\mathrm{co}(H)}^{\mathbb{R}^K}, C(K)) \leq \gamma_K(\mathrm{co}(H)) = \gamma_K(H) \leq 2\,\mathrm{ck}(H) \leq 2\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)).$$

When $H \subset \mathbb{R}^K$, we approximate H by some set in $C(K)$, then use inequality (3) and, after some computations with the sets, **5** appears as $5 = 2\times2+1$: see[1] for details. □

4. Distance to Banach spaces

The aim of this section is to specialize the result of the previous one in the case of Banach spaces: in order to do so we have to overcome some technicalities. If E is Banach space and H is a bounded subset of E and we consider the w^*-closure of H in E^{**} we can measure how far H is from being w-relatively compact in E using

$$k(H) = \hat{d}(\overline{H}^{w^*}, E) = \sup_{y \in \overline{H}^{w^*}} \inf_{x \in E} \|y - x\|.$$

Next theorem gives as a tool to export results obtained in the context of distances to spaces of continuous functions on a compact set to the context of Banach spaces.

Theorem 4.1 ([1]). *Let E be a Banach space and let B_{E^*} be the closed unit ball in the dual E^* endowed with the w^*-topology. Let $i : E \to E^{**}$ and $j : E^{**} \to \ell_\infty(B_{E^*})$ be the canonical embeddings. Then, for every $x^{**} \in E^{**}$ we have*

$$d(x^{**}, i(E)) = d(j(x^{**}), C(B_{E^*}))\,.$$

Explanation of the proof.-. The proof of this result goes along the proof we have given for Theorem 2.2 but instead of using Theorem 2.1 as a tool now the concourse of Hahn-Banach theorem is required: namely, it is used Theorem 21.20 in[21] that states that if $f_1 < f_2$ are two real-valued functions defined on B_{E^*} with f_1 concave and w^*-upper semicontinuous and f_2 convex and w^*-lower semicontinuous then there exist a w^*-continuous affine function h defined on B_{E^*} such that

$$f_1(x) < h(x) < f_2(x)$$

for every $x \in B_{E^*}$. See[1] for details. □

If we consider $\ell_\infty(B_{E^*})$ as a subspace of $(\mathbb{R}^{B_{E^*}}, \tau_p)$, then the natural embedding $j : (E^{**}, w^*) \to (\ell_\infty(B_{E^*}), \tau_p)$ is continuous. For a bounded set $H \subset E^{**}$, the closure $\overline{H}^{w^*}$ is w^*-compact and therefore the continuity of j gives us that $\overline{j(H)}^{\tau_p} = j(\overline{H}^{w^*})$. So

$$\begin{aligned} \hat{d}(\overline{j(H)}^{\tau_p}, C(B_{E^*}, w^*)) &= \hat{d}(j(\overline{H}^{w^*}), C(B_{E^*}, w^*)) \\ &= \sup_{z \in \overline{H}^{w^*}} d(j(z), C(B_{E^*}, w^*)) \\ &= \sup_{z \in \overline{H}^{w^*}} d(z, i(E)) = \hat{d}(\overline{H}^{w^*}, i(E)). \end{aligned} \tag{5}$$

Similarly we have

$$d(\overline{j(H)}^{\tau_p}, C(B_{E^*}, w^*)) = d(\overline{H}^{w^*}, i(E)). \tag{6}$$

Definition 4.1. Let E be a Banach space and let H be a subset of E. We define:

$$\gamma(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m)_m \subset B_{E^*},\ (x_n)_n \subset H\},$$

assuming the involved limits exists,

$$\mathrm{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\mathrm{clust}_{E^{**},w^*}(\varphi), E)$$

and

$$\omega(H) := \inf\{\varepsilon > 0 : H \subset K_\varepsilon + \varepsilon B_E \text{ and } K_\varepsilon \subset X \text{ is } w\text{-compact}\}.$$

The function ω was introduced by de Blasi[22] as a measure of weak noncompactness that can be regarded as the counterpart for the weak topology of the classical Hausdorff measure of norm noncompactness. The function γ already appeared in[23] and in[24] with an a priori different definition: in the latter the sup is taken over all the sequences in the convex hull $\mathrm{co}(H)$ instead of sequences only in H; but by Theorem 3.2 $\gamma(H) = \gamma(\mathrm{co}(H))$ which says that our definition for γ is equivalent to the one given in.[24] The index k has been used in.[1,12,13] Whereas ω and γ are measures of weak noncompactness in the sense of the axiomatic definition given in[25] the function k fails to satisfy $\mathrm{k}(\mathrm{co}(H)) = \mathrm{k}(H)$, that is one of the properties required in order to be a measure of weak noncompactness in the sense of:[25] see[13,14] for counterexamples. Nonetheless, k as well as γ and ω does satisfy the condition $\mathrm{k}(H) = 0$ if, and only if, H is relatively weakly compact in E.

All the above quantities are related with each other.

Theorem 4.2 ([1,3]). *Let H be a bounded subset of a Banach space E. Then*

$$\mathrm{ck}(H) \le \mathrm{k}(H) \le \gamma(H) \le 2\,\mathrm{ck}(H) \le 2\,\mathrm{k}(H) \le 2\omega(H) \tag{7}$$

$$\gamma(H) = \gamma(\mathrm{co}(H)) \quad \textit{and} \quad \omega(H) = \omega(\mathrm{co}(H).$$

For any $x^{**} \in \overline{H}^{w^*}$, there is a sequence $(x_n)_n$ in H such that

$$\|x^{**} - y^{**}\| \le \gamma(H)$$

for any cluster point y^{**} of $(x_n)_n$ in E^{**}. Furthermore, H is relatively compact in (E, w) if, and only if, it is zero one (equivalently all) of the numbers $\mathrm{ck}(H), \mathrm{k}(H), \gamma(H)$ and $\omega(H)$.

Explanation of the proof.-. The first part of the Theorem uses the results stated in the previous section together with the equalities (5) and (6). For the second part, the approximation by sequences, again equalities (5) and (6) are used together now with [1, Proposition 5.2]. □

We point out that $\gamma(H) = \gamma(\mathrm{co}(H))$ and $\mathrm{k}(H) \le \gamma(H) \le 2\,\mathrm{k}(H)$ have also been established in:[12] note that inequalities (7) immediately imply

Krein-Smulyan theorem for Banach spaces that states that the closed convex hull of a weakly compact set is again weakly compact.

Recall that a topological space T is said to be *angelic* if, whenever H is a relatively countably compact subset of T, its closure $\overline{H}$ is compact and each element of $\overline{H}$ is a limit of a sequence in H: a good reference for angelic spaces is.[9] Inequalities (7) together with the approximation by sequences in Theorem 4.2 offers us a quantitative version of the angelicity of a Banach space endowed with its weak topology, Eberlein-Smulyan's theorem.

Corollary 4.1. *If E is a Banach space then (E, w) is angelic.*

In (7) the constants involved are sharp but sometimes the inequalities involved are equalities.

Theorem 4.3 ([3]). *If E is a Banach space with Corson property $\mathfrak{C}$, then for every bounded set $H \subset E$ we have* $\mathrm{ck}(H) = \mathrm{k}(H)$.

Recall that a Banach space E is said to have the Corson property $\mathfrak{C}$ if each collection of closed convex subsets of E with empty intersection has a countable subcollection with empty intersection: the class of Banach spaces with property $\mathfrak{C}$ is a wide class that contains the classes of Banach spaces which are Lindelöf for their weak topologies (in particular w-K-analytic Banach spaces) and also the class of Banach spaces with w^*-countably tight (in particular Banach spaces with w^*-angelic dual unit ball), see.[26] We note that equality $\mathrm{ck}(H) = \mathrm{k}(H)$ does not hold for general Banach spaces: see[3] for a counterexample.

The Hausdorff measure of norm noncompactness is defined for bounded sets H of Banach spaces E as

$$\mathrm{h}(H) := \inf\{\varepsilon > 0 : H \subset K_\varepsilon + \varepsilon B_E \text{ and } K_\varepsilon \subset X \text{ is finite}\}.$$

A theorem of Schauder states that a continuous linear operator $T : E \to F$ is compact if, and only if, its adjoint operator $T^* : F^* \to E^*$ is compact. A quantitative strengthening of Schauder's result was proved by Goldenstein and Marcus (*cf.* [23, p. 367]) who established the inequalities

$$\frac{1}{2}\mathrm{h}(T(B_E)) \leq \mathrm{h}(T^*(B_{F^*})) \leq 2\mathrm{h}(T(B_E)). \tag{8}$$

For weak topologies Gantmacher established that the operator T is weakly compact if, and only if, T^* is weakly compact. Nonetheless, the corresponding quantitative version to (8) where h is replaced by ω fails for general

Banach spaces: Astala and Tylli constructed in [23, Theorem 4] a separable Banach space E and a sequence $(T_n)_n$ of operators $T_n : E \to c_0$ such that

$$\omega(T_n^*(B_{\ell^1})) = 1 \qquad \text{and} \qquad \omega(T_n^{**}(B_E^{**})) \le w(T_n(B_E)) \le \frac{1}{n}.$$

On the positive side there exists a quantitative version of Gantmacher result for γ and henceforth for k and ck.

Theorem 4.4 ([3]). *Let E and F be Banach spaces, $T : E \to F$ an operator and $T^* : F^* \to E^*$ its adjoint. Then*

$$\gamma(T(B_E)) \le \gamma(T^*(B_{F^*})) \le 2\gamma(T(B_E)).$$

As a combination of the result and the aforementioned Astala and Tylli's construction we obtain:

Corollary 4.2 ([3,23]). *The measures of weak noncompactness γ and ω are not equivalent, meaning, there is no $N > 0$ such that for any Banach space and any bounded set $H \subset E$ we have $\omega(H) \le N\gamma(H)$.*

The following result is a quantitative strengthening of the classical Grothendieck's characterization of weakly compact sets in spaces $C(K)$.

Theorem 4.5 ([3]). *Let K be a compact space and let H be a uniformly bounded subset of $C(K)$. Then we have*

$$\gamma_K(H) \le \gamma(H) \le 2\gamma_K(H).$$

Note that this result implies that such an H is uniformly bounded subset of $C(K)$, then H is relatively weakly compact (i.e. $\gamma(H) = 0$) if, and only if, H is relatively τ_p-compact (i.e. $\gamma(H) = 0$). It is worth mentioning that the proof we provided in[3] does not use the Lebesgue Convergence theorem as the classical proof of Grothendieck's theorem does: our proof relies on purely topological arguments.

5. Distances to continuous functions on countably *K*-determined spaces

For people just interested about results for spaces of continuous functions in non compact spaces X, it is possible to get rid of the constraints imposed in Theorem 3.1 and also deal with pointwise bounded sets $H \subset \mathbb{R}^X$ instead of uniformly bounded sets made up of continuous functions. To do so one needs to prove first the two technical lemmas that follow.

Lemma 5.1 ([2]). *Let X be a topological space, (Z, d) a metric space and H a relatively compact subset of the space (Z^X, τ_p). Then, for every relatively countably compact subset $K \subset X$ we have*

$$\gamma_K(H) \leq 2\big(\operatorname{ck}(H) + \hat{d}(H, C(X, Z))\big).$$

Lemma 5.2 ([2]). *Suppose that (Z, d) is a separable metric space and let X be a set. Given functions $f_1, \ldots, f_n \in Z^X$ and $D \subset X$ there is a countable subset $L \subset D$ such that for every $x \in D$*

$$\inf_{y \in L} \max_{1 \leq k \leq n} d(f_k(y), f_k(x)) = 0.$$

With the above two lemmas at hand and a long way of technical difficulties to overcome one arrives to the following two results that greatly extends Theorem 3.1.

Theorem 5.1 ([2]). *Let X be a countably K-determined space, (Z, d) a separable metric space and H a relatively compact subset of the space (Z^X, τ_p). Then, for any $f \in \overline{H}^{Z^X}$ there exists a sequence $(f_n)_n$ in H such that*

$$\sup_{x \in X} d(g(x), f(x)) \leq 2 \operatorname{ck}(H) + 2\hat{d}(H, C(X, Z)) \leq 4 \operatorname{ck}(H)$$

for any cluster point g of $(f_n)_n$ in Z^X.

Theorem 5.2 ([2]). *Let X be a countably K-determined space, (Z, d) a separable metric space and H a relatively compact subset of the space (Z^X, τ_p). Then*

$$\operatorname{ck}(H) \leq \hat{d}(\overline{H}^{Z^X}, C(X, Z)) \leq 3 \operatorname{ck}(H) + 2\hat{d}(H, C(X, Z)) \leq 5 \operatorname{ck}(H).$$

Recall that a topological space X is said to be *countably K-determined* if there is a subspace $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and an upper semi-continuous set-valued map $T : \Sigma \to 2^X$ such that $T(\alpha)$ is compact for each $\alpha \in \Sigma$ and $T(\Sigma) := \bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$. Here the set-valued map T is called *upper semi-continuous* if for each $\alpha \in \Sigma$ and for any open subset U of X such that $T(\alpha) \subset U$ there exists a neighborhood V of α with $T(V) \subset U$. A good reference for countably K-determined spaces is[27] where they appear under the name *Lindelöf Σ-spaces*: notice that this class of spaces does properly contain the class of separable metric spaces and the class of K-analytic and (so) the σ-compact spaces.

We point out that the results above imply the main result in.[10]

Corollary 5.1 ([10]). *Let X be a countably K-determined space and (Z,d) a metric space. Then $C_p(X,Z)$ is an angelic space.*

Our Theorems 5.1 and 5.2, can be proved (same proofs and difficulty) in the more general setting of spaces X being *web-compact, quasi-Souslin, etc.* as studied in.[10] We also notice that this quite general results can be used to obtain some consequences in the setting of locally convex spaces.

Although there are examples showing that the constants are truly needed in the inequalities in Theorem 5.2, there are cases for which k = ck.

Lemma 5.3. *Let X be a first countable space, (Z,d) a metric space and H a pointwise relatively compact subset of (Z^X,τ_p). Then*

$$\sup_{f\in\overline{H}} \operatorname{osc}(f) = \sup_{\varphi\in H^{\mathbb{N}}} \inf\{\operatorname{osc}(f) : f \in \operatorname{clust}_{Z^X}(\varphi)\}. \tag{9}$$

For $Z=\mathbb{R}$ the equality (9) holds when X is countably tight.

The above lemma can be read as:

Proposition 5.1. *Let X be a metric space, E a Banach space and H a τ_p-relatively compact subset of E^X. Then*

$$\operatorname{ck}(H) \le \hat{d}(\overline{H}^{E^X}, C(X,E)) \le 2\operatorname{ck}(H).$$

In the particular case when $E=\mathbb{R}$ the space X can be taken normal and countably tight and we have

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, C(X)) = \operatorname{ck}(H).$$

6. Baire one functions

It is known that when E is a Banach space the uniform limits of Baire one functions are Baire one functions again. Hence, for a function $f\in E^X$ we have that $f\in B_1(X,E)$ if, and only if, $d(f,B_1(X,E))=0$. Consequently, for any subset $A\subset E^X$ we have $\hat{d}(A,B_1(X,E))=0$ if, and only if, $A\subset B_1(X,E)$. In this way, and similarly to the case of continuous functions, when $E=\mathbb{R}$ and $H\subset\mathbb{R}^X$ is pointwise bounded, the number $\hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X))$ gives us a measure of non τ_p-compactness of H relative to $B_1(X)$ –observe that $\hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X))=0$ implies that H is τ_p-relatively compact in $B_1(X)$. Henceforth, we might now pursue the study we already did for continuous functions but now dealing with Baire one functions. In order to do so the first difficulty to overcome is to answer to the following question:

Q. *Given $f \in Z^X$, is there any way to estimate the distance $d(f, B_1(X, Z))$?*

To effectively compute this distance we use the concept of fragmented and σ-fragmented map as introduced in.[28] Recall that for a given $\varepsilon > 0$, a metric space-valued function $f : X \to (Z, d)$ is ε-fragmented if for each non-empty subset $F \subset X$ there exists an open subset $U \subset X$ such that $U \cap F \neq \emptyset$ and $\operatorname{diam}(f(U \cap F)) \leq \varepsilon$. Given $\varepsilon > 0$, we say that f is ε-σ-fragmented by *closed sets* if there is a countable closed covering $(X_n)_n$ of X such that $f|_{X_n}$ is ε-fragmented for each $n \in \mathbb{N}$.

Definition 6.1. Let X be a topological space, (Z, d) a metric space and $f \in Z^X$ a function. We define:

$$\operatorname{frag}(f) := \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-fragmented}\},$$

$$\sigma\text{-frag}_c(f) := \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-}\sigma\text{-fragmented by closed sets}\},$$

where by definition, $\inf \emptyset = +\infty$.

The indexes frag and σ-frag$_c$ are related to each other as follows:

Theorem 6.1 ([4]). *Let X be a topological space and (Z, d) a metric space. If $f \in Z^X$ then the following inequality holds*

$$\sigma\text{-frag}_c(f) \leq \operatorname{frag}(f).$$

If moreover X is hereditarily Baire, then

$$\sigma\text{-frag}_c(f) = \operatorname{frag}(f).$$

With frag and σ-frag$_c$ one can estimate distances to $B_1(X, E)$.

Theorem 6.2 ([4]). *Let X be a metric space and E a Banach space. If $f \in E^X$ then*

$$\frac{1}{2}\,\sigma\text{-frag}_c(f) \leq d(f, B_1(X, E)) \leq \sigma\text{-frag}_c(f).$$

In the case $E = \mathbb{R}$ we have the equality

$$d(f, B_1(X)) = \frac{1}{2}\,\sigma\text{-frag}_c(f).$$

Next result is a consequence of the two previous ones.

Corollary 6.1 ([4]). *If X is a hereditarily Baire metric space and $f \in \mathbb{R}^X$, then*

$$d(f, B_1(X)) = \frac{1}{2}\operatorname{frag}(f).$$

Note that the corollary above extends [15, Proposition 6.4.], where this result is only proved when X is Polish.

Bearing in mind the definitions involved one proves:

Lemma 6.1 ([4]). *Let X be a separable metric space, (Z,d) a metric space and H a pointwise relatively compact subset of (Z^X,τ_p). Then (closures are taken relative to τ_p),*

$$\sup_{f\in\overline{H}} \mathrm{frag}(f) = \sup_{\phi\in H^{\mathbb{N}}} \inf\{\mathrm{frag}(f) : f\in \mathrm{clust}(\phi)\}. \tag{10}$$

As we have done already in the case of continuous functions, we can study how far a set $H\subset E^X$ from being τ_p-relatively countably compact with respect to $B_1(X,E)$ using

$$\mathrm{ck}_{B_1}(H) := \sup_{\varphi\in H^{\mathbb{N}}} d(\mathrm{clust}_{Z^X}(\varphi), B_1(X,E)).$$

If we combine all the above, we can prove the following quantitative result about the difference between τ_p-relative compactness and τ_p-relative countable compactness with respect to $B_1(X,E)$. The particular case of $\mathrm{ck}(H)=0$ and $E=\mathbb{R}$ is the classic result due to Rosenthal.[29]

Theorem 6.3 ([4]). *Let X be a Polish space, E a Banach space and H a τ_p-relatively compact subset of E^X. Then*

$$\mathrm{ck}(H) \le \hat{d}(\overline{H}^{E^X}, B_1(X,E)) \le 2\,\mathrm{ck}(H).$$

In the particular case when $E=\mathbb{R}$ we have

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X)) = \mathrm{ck}(H).$$

7. Further studies

The very idea that *"qualitative"* properties can be derived from some *"inequalities"* is likely true for a great number of results. In our papers[4,30] there are more *"quantitative"* versions of classical results. We name some of them in the lines below.

In[4] we also obtain, with I. Namioka, a quantitative version of a Srivatsa's result that states that whenever X is metric any weakly continuous function $f\in E^X$ belongs to $B_1(X,E)$: our result here says that for an arbitrary $f\in E^X$ we have

$$d(f, B_1(X,E)) \le 2 \sup_{x^*\in B_{E^*}} \mathrm{osc}(x^*\circ f).$$

As a consequence it is proved that for functions in two variables $f : X\times K \to \mathbb{R}$, X complete metric and K compact, there exists a G_δ-dense set $D \subset X$ such that the oscillation of f at each $(x, k) \in D \times K$ is bounded by the oscillations of the *partial* functions f_x and f^k. We indeed prove using games, that if X is a σ-β-unfavorable space and K is a compact space, then there exists a dense G_δ-subset D of X such that, for each $(y, k) \in D \times K$,

$$\operatorname{osc}(f, (y, k)) \leq 6 \sup_{x\in X} \operatorname{osc}(f_x) + 8 \sup_{k\in K} \operatorname{osc}(f^k).$$

When the right hand side of the above inequality is zero we are dealing with separately continuous functions $f : X \times K \to \mathbb{R}$ and we obtain as a particular case some well-known results obtained by I. Namioka in the mid 1970's.

The first named author has studied in[30] the distances from the set of selectors $\operatorname{Sel}(F)$ of a set-valued map $F : X \to \mathcal{P}(E)$ to the space $B_1(X, E)$. To do so, the notion of d-τ-semioscillation of a set-valued map with values in a topological space (Y, τ) also endowed with a metric d is introduced. Being more precise it is proved that

$$d(\operatorname{Sel}(F), B_1(X, E)) \leq 2 \operatorname{osc}^*_w(F)$$

where $\operatorname{osc}^*_w(F)$ is the $\|\cdot\|$-w-semioscillation of F. In particular when F takes closed values and $\operatorname{osc}^*_w(F) = 0$ it is obtained that F has a Baire one selector: it should be pointed out that if F is weakly upper semicontinuous then $\operatorname{osc}^*_w(F) = 0$ and therefore these results strengthen a Srivatsa selection Theorem when F takes closed set.

More results along this line for other kind of spaces are foreseeable when studying distantes to spaces of measurable functions, to spaces of integrable functions, etc. We are making an effort in this direction right now: if the results obtained are worth-it, they will be published elsewhere.

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