MEASURABLE SELECTORS AND SET-VALUED PETTIS INTEGRAL IN NON-SEPARABLE BANACH SPACES

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ABSTRACT. Kuratowski and Ryll-Nardzewski's theorem about the existence of measurable selectors for multi-functions is one of the keystones for the study of set-valued integration; one of the drawbacks of this result is that separability is always required for the range space. In this paper we study Pettis integrability for multi-functions and we obtain a Kuratowski and Ryll-Nardzewski's type selection theorem without the requirement of separability for the range space. Being more precise, we show that any Pettis integrable multi-function $F : \Omega \to cwk(X)$ defined in a complete finite measure space (Ω, Σ, μ) with values in the family $cwk(X)$ of all non-empty convex weakly compact subsets of a general (non-necessarily separable) Banach space X always admits Pettis integrable selectors and that, moreover, for each $A \in \Sigma$ the Pettis integral $\int_A F\, d\mu$ coincides with the closure of the set of integrals over A of all Pettis integrable selectors of F. As a consequence we prove that if X is reflexive then every scalarly measurable multi-function $F : \Omega \to cwk(X)$ admits scalarly measurable selectors; the latter is also proved when (X^*, w^*) is angelic and has density character at most ω_1. In each of these two situations the Pettis integrability of a multi-function $F : \Omega \to cwk(X)$ is equivalent to the uniform integrability of the family $\{\sup x^*(F(\cdot)) : x^* \in B_{X^*}\} \subset \mathbb{R}^\Omega$. Results about norm-Borel measurable selectors for multi-functions satisfying stronger measurability properties but without the classical requirement of the range Banach space being separable are also obtained.

1. INTRODUCTION

Set-valued integration has its origin in the seminal papers by Aumann [2] and Debreu [9] and has been a very useful tool in areas like optimization and mathematical economics. The set-valued Pettis integral theory, which goes back to the monograph by Castaing and Valadier [7], has attracted recently the attention of several authors, see for instance [1, 5, 6, 11, 12, 15, 24, 44] and [45]. All these studies deal with multi-functions whose values are subsets of a Banach space X that is always assumed to be *separable*. The main reason for this limitation on X relies on the fact that an integrable multifunction should have integrable (measurable) selectors and the tool to find these measurable selectors has always been the well-known selection theorem of Kuratowski and Ryll-Nardzewski [29] that only works when the range space is separable. For a detailed account on measurable selection results and set-valued integration we refer the reader to the monographs [7, 27] and the survey [23].

Our main goal here is to show that most of the Pettis integral theory for multi-functions can be done without the restriction of separability on the range space. The extension from the separable case to the non-separable one is not so obvious and to do so we have to obtain a number of new measurable selection results for multi-functions in the non-separable case.

Throughout this paper (Ω, Σ, μ) is a complete finite measure space, X is a Banach space and $cwk(X)$ (resp. $ck(X)$) denotes the family of all convex weakly compact (resp. norm compact) non-empty subsets of X. We write $\delta^*(x^*, C) := \sup\{x^*(x) : x \in C\}$ for any

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bounded set $C \subset X$ and $x^* \in X^*$. A multi-function $F : \Omega \to cwk(X)$ is said to be *Pettis integrable* if

- ▶ $\delta^*(x^*, F)$ is integrable for each $x^* \in X^*$;
- ▶ for each $A \in \Sigma$, there is $\int_A F\, d\mu \in cwk(X)$ such that

$$\delta^*\Big(x^*, \int_A F\, d\mu\Big) = \int_A \delta^*(x^*, F)\, d\mu \quad \text{for every } x^* \in X^*.$$

Here the function $\delta^*(x^*, F) : \Omega \to \mathbb{R}$ is defined by $\delta^*(x^*, F)(\omega) = \delta^*(x^*, F(\omega))$.

The paper is organized as follows. In Section 2 we study Pettis integrable multi-functions via their selectors. Our Theorem 2.5 states that every Pettis integrable multi-function $F : \Omega \to cwk(X)$ admits indeed Pettis integrable selectors. Moreover, in this case, for each $A \in \Sigma$ the integral $\int_A F\, d\mu$ coincides with the *closure* of the set of integrals over A of all Pettis integrable selectors of F, Theorem 2.6. In the previous statement, the "closure" can be dropped provided that X^* is w^*-separable, Corollary 2.7. These results are the non-trivial extension of part of Theorem A below that is considered as the milestone result in the set-valued Pettis integral theory for separable Banach spaces.

Theorem A ([15, 44, 45] and [7, Chapter V, §4])**.** *Let X be a separable Banach space and $F : \Omega \to cwk(X)$ a multi-function. The following conditions are equivalent:*

- (i) *F is Pettis integrable.*
- (ii) *The family $W_F = \{\delta^*(x^*, F) : x^* \in B_{X^*}\}$ is uniformly integrable.*
- (iii) *The family W_F is made up of measurable functions and any scalarly measurable selector of F is Pettis integrable.*

In this case, for each $A \in \Sigma$ the integral $\int_A F\, d\mu$ coincides with the set of integrals over A of all Pettis integrable selectors of F.

To get ready for the proof of a full counterpart to Theorem A for non-separable Banach spaces we quote in Section 3 some known facts about the existence of countably additive selectors and the Orlicz-Pettis theorem for *multi-measures* which are due to Godet-Thobie [20], Costé [8] and Pallu de la Barrière [33]: new proofs for these results are included.

In Section 4 we discuss the possible extensions of Theorem A to the non-separable setting. The implications (i)⇒(ii) and (i)⇒(iii) hold without any assumption on X, Theorem 4.1 and Corollary 2.3. We show in Theorem 4.2 that the equivalence (i)⇔(iii) holds true if X has the following property: every *scalarly measurable* multi-function $F : \Omega \to cwk(X)$ (meaning that $\delta^*(x^*, F)$ is measurable for all $x^* \in X^*$) admits a scalarly measurable selector. This condition, which we call *Scalarly Measurable Selector Property* with respect to μ, shortly *μ-SMSP*, is shared by many Banach spaces besides the separable ones, as explained a few lines below. On the other hand, to prove (ii)⇒(i) we have to require, in addition to the μ-SMSP, that X has the so-called Pettis Integral Property with respect to μ (shortly *μ-PIP*), Corollary 4.3. The last part of Section 4 is devoted to characterize Pettis integrability of multi-valued functions via single-valued ones and we pay particular attention to the the case of multi-functions with norm compact values.

In Section 5 we are concerned with the existence of "measurable" selectors for multi-functions $F : \Omega \to cwk(X)$ which satisfy one of the following measurability properties:

- (α) $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every closed half-space $M \subset X$ (equivalently, F is scalarly measurable).
- (β) $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every convex closed set $M \subset X$.
- (γ) $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every norm closed set $M \subset X$.

When X is separable, it is known that (α), (β) and (γ) are equivalent to the *Effros measurability* of F (i.e. the same property than (γ) but replacing "closed" by "open", cf. [7, Theorem III.37]). In this case, the selection theorem of Kuratowski and Ryll-Nardzewski, cf. [7, Theorem III.30], ensures that such an F admits a $\mathrm{Borel}(X, \mathrm{norm})$-measurable (hence strongly measurable) selector. In the non-separable case these measurability notions are not equivalent in general and the situation becomes more complicated. Subsection 5.1 is started with Theorem 5.1 by proving that reflexive Banach spaces have μ-SMSP. Beyond that, our Theorem 5.4 shows that many other Banach spaces have μ-SMSP: for instance, this happens if the dual space is w^*-angelic and has w^*-density character less than or equal to the *uncountable* cardinal number $\kappa(\mu)$, Example 5.5 – we recall that the class of Banach spaces having w^*-angelic dual is very large and contains all weakly Lindelöf determined spaces and, in particular, all weakly compactly generated ones. Amongst other things we provide in Theorem 5.15 a different proof of Valadier's result [43] saying that spaces with w^*-separable dual also have μ-SMSP. To this end we prove that a multi-function $F : \Omega \to cwk(X)$ is scalarly measurable if and only if $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every set $M \subset X$ which can be written as a finite intersection of closed half-spaces, Theorem 5.10. The paper is closed by Subsection 5.2 where we study the existence of $\mathrm{Borel}(X, \mathrm{norm})$-measurable selectors for multi-functions $F : \Omega \to cwk(X)$ satisfying (β). We prove, for instance, that such selectors always exist provided that X admits an equivalent locally uniformly rotund norm, Corollary 5.19: this improves a result by Leese [30] who obtained the same conclusion for multi-functions satisfying (γ) when X admits an equivalent uniformly rotund norm.

Terminology. Our unexplained terminology can be found in our standard references for multi-functions [7, 27], Banach spaces [16] and vector integration [13, 42].

The cardinality of a set Γ is denoted by $\mathrm{card}(\Gamma)$. The cardinality of $\mathbb{N}$ (resp. $\mathbb{R}$) is denoted by $\aleph_0$ (resp. $\mathfrak{c}$). The symbol ω_1 stands for the first uncountable ordinal.

Our topological spaces $(T, \mathfrak{T})$ are always assumed to be Hausdorff. The density character of $(T, \mathfrak{T})$, denoted by $\mathrm{dens}(T, \mathfrak{T})$ or simply by $\mathrm{dens}(T)$, is the minimal cardinality of a dense set in T.

All vector spaces here are assumed to be real. Given a subset S of a vector space, we write $\mathrm{co}(S)$ and $\mathrm{span}(S)$ to denote, respectively, the convex and linear hull of S. By letters X and Y we always denote Banach spaces. B_Y is the closed unit ball of Y and Y^* stands for the topological dual of Y. Given $y^* \in Y^*$ and $y \in Y$, we write either $\langle y^*, y\rangle$ or $y^*(y)$ to denote the evaluation of y^* at y. The weak (resp. weak*) topology on Y (resp. Y^*) is denoted by w (resp. w^*). Given a non-empty set Γ (resp. a compact topological space K), we write $\ell_\infty(\Gamma)$ (resp. $C(K)$) to denote the Banach space of all bounded (resp. continuous) real-valued functions on Γ (resp. K), equipped with the supremum norm.

A function $f : \Omega \to Y$ is said to be *scalarly measurable* if, for each $y^* \in Y^*$, the composition $\langle y^*, f\rangle := y^* \circ f : \Omega \to \mathbb{R}$ is measurable. By a result of Edgar [14], f is scalarly measurable if and only if it is $\mathrm{Baire}(Y, w)$-measurable. Recall also that f is said to be *Pettis integrable* if

(i) $y^* \circ f$ is integrable for every $y^* \in Y^*$;
(ii) for each $A \in \Sigma$, there is an element $\int_A f \, d\mu \in Y$ such that

$$\langle y^*, \int_A f \, d\mu\rangle = \int_A y^* \circ f \, d\mu \quad \text{for every } y^* \in Y^*.$$

A function $f : \Omega \to Y$ is *strongly measurable* if it is the μ-a.e. limit of a sequence of simple functions or, equivalently, if it is $\mathrm{Borel}(Y, \mathrm{norm})$-measurable (or just scalarly measurable) and there is $E \in \Sigma$ with $\mu(\Omega \setminus E) = 0$ such that $f(E)$ is separable, cf. [13, Theorem 2, p. 42].

2. Set-valued Pettis integral and selectors

In order to prove our main result in this section stating that any Pettis integrable multi-function admits Pettis integrable selectors, Theorem 2.5, we need some previous work.

Recall first that a function $\varphi : X^* \to \mathbb{R}$ is said to be positively homogeneous if $\varphi(\alpha x^*) = \alpha\varphi(x^*)$ for every $\alpha > 0$ and $x^* \in X^*$. φ is said to be subadditive if

$\varphi(x^*+y^*) \leq \varphi(x^*)+\varphi(y^*)$ for all pairs $(x^*,y^*) \in X^* \times X^*$. φ is said to be sublinear if it is both positively homogeneous and subadditive. We note that if $C \in cwk(X)$ then the map $x^* \mapsto \delta^*(x^*,C)$ is a sublinear functional in X^* that is $\tau(X^*,X)$-continuous. Here $\tau(X^*,X)$ stands for the *Mackey topology* on X^*, that is, the topology of uniform convergence on weakly compact subsets of X, cf. [28, §21.4]. Recall that, by the Mackey-Arens theorem, $\tau(X^*,X)$ is the finest locally convex topology on X^* whose topological dual is X, hence the w^*-closure and the $\tau(X^*,X)$-closure of any convex set $C \subset X^*$ coincide, cf. [28, §21.4(2) and §20.8(6)].

Lemma 2.1. *Let $F : \Omega \to cwk(X)$ be a multi-function such that $\delta^*(x^*,F)$ is integrable for every $x^* \in X^*$. The following statements are equivalent:*

(i) *F is Pettis integrable.*
(ii) *For each $A \in \Sigma$, the mapping*

$$\varphi_A^F : X^* \to \mathbb{R}, \quad x^* \mapsto \int_A \delta^*(x^*,F)\, d\mu,$$

is $\tau(X^*,X)$-continuous.

Proof. The implication (i)⇒(ii) follows from the fact that

$$\delta^*\Big(x^*, \int_A F\, d\mu\Big) = \int_A \delta^*(x^*,F)\, d\mu \quad \text{for every } x^* \in X^*,$$

and the $\tau(X^*,X)$-continuity of the map $x^* \mapsto \delta^*(x^*, \int_A F\, d\mu)$. Conversely, assume that (ii) holds and fix $A \in \Sigma$. Since φ_A^F is a sublinear function, it is convex. This fact and the $\tau(X^*,X)$-continuity of φ_A^F allow us to deduce that for every $t \in \mathbb{R}$ the set $\{x^* \in X^* : \varphi_A^F(x^*) \leq t\}$ is convex and $\tau(X^*,X)$-closed, hence w^*-closed. Therefore φ_A^F is w^*-lower semicontinuous and [7, Theorem II-16] applies to provide us with a non-empty convex, closed and bounded set $C \subset X$ such that $\varphi_A^F(x^*) = \delta^*(x^*,C)$ for every $x^* \in X^*$. Finally, the fact that φ_A^F is $\tau(X^*,X)$-continuous can be applied again to conclude that C is weakly compact. Indeed, the set $U := \{x^* \in X^* : \varphi_A^F(x^*) < 1\} \cap \{x^* \in X^* : \varphi_A^F(-x^*) < 1\}$ is a $\tau(X^*,X)$-neighborhood of 0 and thus its polar $U^\circ = \{x \in X : |x^*(x)| \leq 1 \text{ for all } x^* \in U\}$ is weakly compact, [28, §21.4.1]. Since C is weakly closed and contained in U°, C is weakly compact as well. □

Observe that for every bounded set $C \subset X$ and every $x^* \in X^*$ we have

$$\inf\{x^*(x) :\ x \in C\} = -\delta^*(-x^*,C).$$

Lemma 2.2. *Let $F, G : \Omega \to cwk(X)$ be two multi-functions such that F is Pettis integrable, G is scalarly measurable and, for each $x^* \in X^*$, we have $\delta^*(x^*,G) \leq \delta^*(x^*,F)$ μ-a.e. Then G is Pettis integrable and $\int_A G\, d\mu \subset \int_A F\, d\mu$ for every $A \in \Sigma$.*

Proof. Given $x^* \in X^*$, we have $-\delta^*(-x^*,F) \leq \delta^*(x^*,G) \leq \delta^*(x^*,F)$ μ-a.e. and so $\delta^*(x^*,G)$ is integrable. Fix $A \in \Sigma$. The mapping φ_A^G is subadditive and satisfies $\varphi_A^G(x^*) \leq \varphi_A^F(x^*)$ for all $x^* \in X^*$, hence

$$|\varphi_A^G(x^*) - \varphi_A^G(y^*)| \leq |\varphi_A^F(x^*-y^*)| + |\varphi_A^F(y^*-x^*)|$$

for every $x^*, y^* \in X^*$. Since F is Pettis integrable, φ_A^F is $\tau(X^*,X)$-continuous and the previous inequality implies that φ_A^G is also $\tau(X^*,X)$-continuous. Since $A \in \Sigma$ is arbitrary, an appeal to Lemma 2.1 ensures that G is Pettis integrable. Moreover, for each $A \in \Sigma$ we have $\int_A G\, d\mu \subset \int_A F\, d\mu$, by the Hahn-Banach separation theorem and the fact that

$$\delta^*\Big(x^*, \int_A G\, d\mu\Big) = \int_A \delta^*(x^*,G)\, d\mu \leq \int_A \delta^*(x^*,F)\, d\mu = \delta^*\Big(x^*, \int_A F\, d\mu\Big)$$

for every $x^* \in X^*$. The proof is over. □

Given a multi-function $F : \Omega \to cwk(X)$ and $A \in \Sigma$ we write

$$IS_F(A) := \Big\{ \int_A f \, d\mu : \; f \text{ is a Pettis integrable selector of } F \Big\}.$$

Note that $IS_F(A)$ might be empty in general and that otherwise it is a convex subset of X. Next corollary says, in particular, that $IS_F(A) \subset \int_A F \, d\mu$ whenever F is Pettis integrable.

Corollary 2.3. *Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. If $f : \Omega \to X$ is a scalarly measurable selector of F, then f is Pettis integrable and*

$$\int_A f \, d\mu \in \int_A F \, d\mu \quad \textit{for every } A \in \Sigma.$$

Proof. Apply Lemma 2.2 to the multi-function $G(\omega) := \{f(\omega)\}$. □

To prove the main result of this section we also need the following lemma:

Lemma 2.4 ([43, Lemme 3])**.** *Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function. Fix $x_0^* \in X^*$ and consider the multi-function*

$$G : \Omega \to cwk(X), \quad G(\omega) := \{x \in F(\omega) : \; x_0^*(x) = \delta^*(x_0^*, F(\omega))\}.$$

Then G is scalarly measurable.

Theorem 2.5. *Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. Then F admits a Pettis integrable selector.*

Proof. Since $\int_A F \, d\mu \in cwk(X)$, we can find an *exposed point* $x_0 \in \int_A F \, d\mu$ (cf. [4, Theorem 3.6.1]), that is, there is some $x_0^* \in X^*$ such that $x_0^*(x_0) > x_0^*(x)$ for every $x \in \int_A F \, d\mu \setminus \{x_0\}$. Let us consider the multi-function

$$G : \Omega \to cwk(X), \quad G(\omega) := \{x \in F(\omega) : \; x_0^*(x) = \delta^*(x_0^*, F(\omega))\}.$$

By Lemma 2.4, G is scalary measurable. Since $G(\omega) \subset F(\omega)$ for every $\omega \in \Omega$ and F is Pettis integrable, an appeal to Lemma 2.2 ensures that G is Pettis integrable too, with $\int_\Omega G \, d\mu \subset \int_\Omega F \, d\mu$. Let $g : \Omega \to X$ be any selector of G. Clearly, g is also a selector of F. We will prove that g is scalarly measurable. Observe that

$$\begin{aligned}\delta^*\Big(x_0^*, \int_\Omega G \, d\mu\Big) &= \int_\Omega \delta^*(x_0^*, G) \, d\mu = \\ &= \int_\Omega \delta^*(x_0^*, F) \, d\mu = \delta^*\Big(x_0^*, \int_\Omega F \, d\mu\Big) = x_0^*(x_0) = \\ &= \int_\Omega (-\delta^*(-x_0^*, G)) \, d\mu = -\delta^*\Big(-x_0^*, \int_\Omega G \, d\mu\Big).\end{aligned}$$

It follows that $\int_\Omega G \, d\mu = \{x_0\}$. Given $x^* \in X^*$, we have $-\delta^*(-x^*, G) \leq \delta^*(x^*, G)$ and

$$\int_\Omega (-\delta^*(-x^*, G)) \, d\mu = x^*(x_0) = \int_\Omega \delta^*(x^*, G) \, d\mu,$$

hence $-\delta^*(-x^*, G) = \delta^*(x^*, G)$ μ-a.e. Therefore, $x^* \circ g = \delta^*(x^*, G)$ μ-a.e. and, in particular, $x^* \circ g$ is measurable. Since $x^* \in X^*$ is arbitrary, g is scalarly measurable. Finally, an appeal to Corollary 2.3 allows us to conclude that g is Pettis integrable. □

In our next result we establish that in fact any Pettis integrable multi-function admits a collection of Pettis integrable selectors which are dense in it (a kind of “generalized” *Castaing representation*).

Theorem 2.6. *Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. Then F admits a collection $\{f_\alpha\}_{\alpha < \mathrm{dens}(X^*, w^*)}$ of Pettis integrable selectors such that*

$$F(\omega) = \overline{\{f_\alpha(\omega) : \; \alpha < \mathrm{dens}(X^*, w^*)\}} \quad \textit{for every } \omega \in \Omega.$$

Moreover, $\int_A F \, d\mu = \overline{IS_F(A)}$ for every $A \in \Sigma$.

Proof. Notice first that $\kappa := \mathrm{dens}(X^*, w^*) = \mathrm{dens}(X^*, \tau(X^*, X))$. Fix a $\tau(X^*, X)$-dense set $\{x^*_\alpha : \alpha < \kappa\} \subset X^*$. For each $\alpha < \kappa$, the multi-function

$$L_\alpha : \Omega \to cwk(X), \quad L_\alpha(\omega) := \{x \in F(\omega) : \ x^*_\alpha(x) = \delta^*(x^*_\alpha, F(\omega))\},$$

is scalarly measurable by Lemma 2.4 and so Pettis integrable by Lemma 2.2. Then Theorem 2.5 applied to L_α ensures that there is a Pettis integrable selector $s_\alpha : \Omega \to X$ of L_α. Clearly, each s_α is also a selector of F. We claim that

$$F(\omega) = \overline{\mathrm{co}(\{s_\alpha(\omega) : \ \alpha < \kappa\})} \quad \text{for every } \omega \in \Omega.$$

Indeed, fix $\omega \in \Omega$ and set $C := \overline{\mathrm{co}(\{s_\alpha(\omega) : \alpha < \kappa\})} \subset F(\omega)$. Then $C \in cwk(X)$ and

$$\delta^*(x^*_\alpha, F(\omega)) \geq \delta^*(x^*_\alpha, C) \geq x^*_\alpha(s_\alpha(\omega)) = \delta^*(x^*_\alpha, F(\omega))$$

for every $\alpha < \kappa$. Since the set $\{x^*_\alpha : \alpha < \kappa\}$ is $\tau(X^*, X)$-dense in X^* and the maps $x^* \mapsto \delta^*(x^*, C)$ and $x^* \mapsto \delta^*(x^*, F(\omega))$ are $\tau(X^*, X)$-continuous we obtain the equality $\delta^*(x^*, F(\omega)) = \delta^*(x^*, C)$ for every $x^* \in X^*$ and, therefore, $F(\omega) = C$ as asserted. Observe that the collection $\{f_\alpha\}_{\alpha<\kappa}$ made up of all convex combinations of the s_α's with rational coefficients fulfills the required properties.

In order to prove the last assertion, fix $A \in \Sigma$. Using Corollary 2.3, we obtain that $\overline{IS_F(A)} \subset \int_A F \, d\mu$. On the other hand, for each $\alpha < \kappa$, the following holds:

$$x^*_\alpha\Big(\int_A s_\alpha \, d\mu\Big) = \int_A x^*_\alpha \circ s_\alpha \, d\mu = \int_A \delta^*(x^*_\alpha, F) \, d\mu = \delta^*\Big(x^*_\alpha, \int_A F \, d\mu\Big),$$

and so $\delta^*(x^*_\alpha, \overline{IS_F(A)}) \geq \delta^*(x^*_\alpha, \int_A F \, d\mu)$. Since $\{x^*_\alpha : \alpha < \kappa\}$ is $\tau(X^*, X)$-dense in X^*, the inequality $\delta^*(x^*, \overline{IS_F(A)}) \geq \delta^*(x^*, \int_A F \, d\mu)$ holds true for every $x^* \in X^*$ and we infer that $\int_A F \, d\mu \subset \overline{IS_F(A)}$. Therefore $\overline{IS_F(A)} = \int_A F \, d\mu$ and the proof is finished. □

It turns out that, *when X^* is w^*-separable, the sets $IS_F(A)$ are closed for any Pettis integrable multi-function $F : \Omega \to cwk(X)$*. The proof imitates that given in [15, Proposition 5.2] for a separable X and so we omit the details. Combining this fact with Theorem 2.6 we get the following result.

Corollary 2.7. *Suppose X^* is w^*-separable. Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. Then $\int_A F \, d\mu = IS_F(A)$ for every $A \in \Sigma$.*

3. Multi-measures and countably additive selectors

Given a sequence (C_n) in $cwk(X)$, the series $\sum_n C_n$ is said to be *unconditionally convergent* provided that for every choice $x_n \in C_n$, $n \in \mathbb{N}$, the series $\sum_n x_n$ is unconditionally convergent in X. In this case, the set

$$\sum_n C_n := \Big\{\sum_n x_n : \ x_n \in C_n \text{ for all } n \in \mathbb{N}\Big\}$$

also belongs to $cwk(X)$, see [6, Lemma 2.2]. Recall that the family $cwk(X)$, equipped with the Hausdorff metric h, is a complete metric space that can be isometrically embedded into the Banach space $\ell_\infty(B_{X^*})$ by means of the mapping

$$j : cwk(X) \to \ell_\infty(B_{X^*}), \quad j(C)(x^*) := \delta^*(x^*, C),$$

see e.g. [7, Chapter II]. It is known that a series $\sum_n C_n$ as above is unconditionally convergent if and only if the series $\sum_n j(C_n)$ is unconditionally convergent in $\ell_\infty(B_{X^*})$ (in this case, we have $j(\sum_n C_n) = \sum_n j(C_n)$), cf. [6, Lemma 2.3].

Definition 3.1. *A mapping $M : \Sigma \to cwk(X)$ is said to be a finitely additive (resp. countably additive) multi-measure if $M(A \cup B) = M(A) + M(B)$ whenever $A, B \in \Sigma$ are disjoint (resp. if for every disjoint sequence (E_n) in Σ the series $\sum_n M(E_n)$ is unconditionally convergent and $M(\bigcup_n E_n) = \sum_n M(E_n)$).*

Note that $M : \Sigma \to cwk(X)$ is a finitely (resp. countably) additive multi-measure if and only if the composition $j \circ M : \Sigma \to \ell_\infty(B_{X^*})$ is a finitely (resp. countably) additive measure. Therefore, if for $x^* \in X^*$ we define $\delta^*(x^*, M) : \Sigma \to \mathbb{R}$ by $A \mapsto \delta^*(x^*, M(A))$, then M is a finitely additive multi-measure if and only if $\delta^*(x^*, M)$ is finitely additive for every $x^* \in X^*$. For countably additive multi-measures the analogue characterization is also true, see Theorem 3.4, but requires some work that we present in this section: this result, due to Costé [8] and Pallu de la Barrière [33], can be seen as the set-valued version of the well-known fact that weakly countably additive vector measures are norm countably additive (Orlicz-Pettis theorem, cf. [13, Corollary 4, p. 22]).

From a technical point of view, the novelty of our approach to Theorem 3.4 relies mostly in the way of finding "finitely additive selectors" for finitely additive multi-measures, see Theorem 3.3, via a method of "linearization" of Lipschitz functions on Banach spaces that goes back to Pelczynski [34, p. 61].

Let $\mathrm{Lip}_0(X^*)$ be the Banach space of all Lipschitz functions $h : X^* \to \mathbb{R}$ satisfying $h(0) = 0$, equipped with the norm

$$\|h\|_{\mathrm{Lip}_0(X^*)} := \sup\Big\{ \frac{|h(x_1^*) - h(x_2^*)|}{\|x_1^* - x_2^*\|} :\ x_1^*, x_2^* \in X^*,\ x_1^* \neq x_2^* \Big\}.$$

Fix an *invariant mean* on X^* (considered as additive abelian group), that is, a linear mapping $\mathcal{I} : \ell_\infty(X^*) \to \mathbb{R}$ such that $\mathcal{I}(g) \geq 0$ whenever $g \geq 0$, $\mathcal{I}(1) = 1$ and $\mathcal{I}(g) = \mathcal{I}(g(\cdot + x^*))$ for every $g \in \ell_\infty(X^*)$ and every $x^* \in X^*$, cf. [25, Theorem 17.5]. It is known that we can define an operator $P : \mathrm{Lip}_0(X^*) \to X^{**}$ by the formula

$$\langle P(h), x^* \rangle := \mathcal{I}(h(\cdot + x^*) - h(\cdot)), \quad h \in \mathrm{Lip}_0(X^*),\ x^* \in X^*,$$

cf. [3, Proposition 7.5].

Lemma 3.2. *Let $C \in cwk(X)$. Then $\delta^*(\cdot, C) \in \mathrm{Lip}_0(X^*)$ and $P(\delta^*(\cdot, C)) \in C$.*

Proof. The first assertion is clear, since

$$|\delta^*(x_1^*, C) - \delta^*(x_2^*, C)| \leq \|x_1^* - x_2^*\| \cdot \sup\{\|x\| :\ x \in C\} \quad \text{for every } x_1^*, x_2^* \in X^*.$$

The proof of the second assertion is by contradiction. Suppose that $P(\delta^*(\cdot, C)) \notin C$. Since C is a convex w^*-closed subset of X^{**}, the Hahn-Banach separation theorem guarantees the existence of some $x^* \in X^*$ such that

$$(1) \qquad \langle P(\delta^*(\cdot, C)), x^* \rangle > \sup\{x^*(x) :\ x \in C\} = \delta^*(x^*, C).$$

On the other hand, we have $\delta^*(y^* + x^*, C) - \delta^*(y^*, C) \leq \delta^*(x^*, C)$ for every $y^* \in X^*$, and the properties of $\mathcal{I}$ yield

$$P(\delta^*(\cdot, C)) = \mathcal{I}(\delta^*(\cdot + x^*, C) - \delta^*(\cdot, C)) \leq \mathcal{I}(\delta^*(x^*, C)) = \delta^*(x^*, C),$$

which contradicts (1). The proof is over. □

We are now ready to deal with the aforementioned results about multi-measures.

Theorem 3.3 ([20], [8] and [33]). *Let $M : \Sigma \to cwk(X)$ be a finitely additive multi-measure. Then there is a finitely additive measure $m : \Sigma \to X$ such that $m(A) \in M(A)$ for every $A \in \Sigma$.*

Proof. Lemma 3.2 ensures that $\delta^*(\cdot, M(A)) \in \mathrm{Lip}_0(X^*)$ and

$$m(A) := P(\delta^*(\cdot, M(A))) \in M(A) \quad \text{for every } A \in \Sigma.$$

Since M is a finitely additive multi-measure and P is linear, m is finitely additive. □

For a given $x^* \in B_{X^*}$, let e_{x^*} denote the element of $B_{\ell_\infty(B_{X^*})^*}$ defined by the formula $e_{x^*}(\varphi) := \varphi(x^*)$.

Theorem 3.4 (Costé-Pallu de la Barrière). *Let $M : \Sigma \to cwk(X)$ be a mapping. The following statements are equivalent:*

(i) *M is a countably additive multi-measure.*
(ii) *$\delta^*(x^*, M)$ is countably additive for every $x^* \in X^*$.*
(iii) *$\delta^*(x^*, M)$ is countably additive for every $x^* \in X^*$ and there is a countably additive measure $m : \Sigma \to X$ such that $m(A) \in M(A)$ for every $A \in \Sigma$.*

Proof. The implication (i)⇒(ii) follows from the fact that $\delta^*(x^*, M) = \langle e_{x^*}, j \circ M \rangle$ for every $x^* \in B_{X^*}$.

Let us prove (ii)⇒(iii). By Theorem 3.3 there is a finitely additive measure $m : \Sigma \to X$ such that $m(A) \in M(A)$ for every $A \in \Sigma$. We claim that m is countably additive. To prove that it suffices to show that the composition $x^* \circ m$ is countably additive for every $x^* \in X^*$ and then appeal to the Orlicz-Pettis theorem, see [13, Corollary 4, p. 22]. Given $x^* \in X^*$, we have $-\delta^*(-x^*, M(A)) \leq (x^* \circ m)(A) \leq \delta^*(x^*, M(A))$ for every $A \in \Sigma$. Since both $-\delta^*(-x^*, M)$ and $\delta^*(x^*, M)$ are countably additive and $x^* \circ m$ is finitely additive, it follows that $x^* \circ m$ is countably additive, as claimed.

To finish we prove (iii)⇒(i). We will prove that the finitely additive measure $\nu := j \circ M : \Sigma \to \ell_\infty(B_{X^*})$ is countably additive. The proof is divided into two cases.

Particular case. Suppose $m(A) = 0$ for every $A \in \Sigma$. Take a disjoint sequence (A_n) in Σ. We will show first that the series $\sum_n \nu(A_n)$ is unconditionally convergent. This is equivalent to saying that the series of sets $\sum_n M(A_n)$ is unconditionally convergent. Fix $x_n \in M(A_n)$ for every $n \in \mathbb{N}$, and take a sequence $n_1 < n_2 < \dots$ in $\mathbb{N}$. Define $s_k = \sum_{i=1}^k x_{n_i}$ for every $k \in \mathbb{N}$. Note that

$$s_k = s_k + 0 \in \sum_{i=1}^k M(A_{n_i}) + M\Big(\Omega \setminus \bigcup_{i=1}^k A_{n_i}\Big) = M(\Omega) \quad \text{for every } k \in \mathbb{N}.$$

On the other hand, for each $x^* \in X^*$ the series $\sum_{i=1}^\infty x^*(x_{n_i})$ is convergent. Indeed, it suffices to bear in mind that

$$\sum_{i=1}^\infty |x^*(x_{n_i})| \leq \sum_{i=1}^\infty |\delta^*(x^*, M(A_{n_i}))| + \sum_{i=1}^\infty |\delta^*(-x^*, M(A_{n_i}))| < +\infty.$$

This ensures that the sequence (s_k) has at most one weak cluster point in X. Since (s_k) is contained in the weakly compact set $M(\Omega)$, it follows that the series $\sum_{i=1}^\infty x_{n_i}$ is weakly convergent. As the sequence $n_1 < n_2 < \dots$ is arbitrary, the Orlicz-Pettis theorem (cf. [13, Corollary 4, p. 22]) ensures that the series $\sum_n x_n$ is unconditionally convergent. This proves that the series $\sum_n \nu(A_n)$ converges unconditionally in $\ell_\infty(B_{X^*})$.

We claim now that $\sum_{n=1}^\infty \nu(A_n) = \nu(\bigcup_{n=1}^\infty A_n)$. Indeed, for each $x^* \in B_{X^*}$ we have

$$\Big(\sum_{n=1}^\infty \nu(A_n)\Big)(x^*) = \lim_{N\to\infty} \sum_{n=1}^N \nu(A_n)(x^*) = \lim_{N\to\infty} \sum_{n=1}^N \delta^*(x^*, M(A_n)) =$$
$$= \delta^*\Big(x^*, M\Big(\bigcup_{n=1}^\infty A_n\Big)\Big) = \nu\Big(\bigcup_{n=1}^\infty A_n\Big)(x^*).$$

The proof of the *Particular case* is finished.

General case. Define the mapping

$$M' : \Sigma \to cwk(X), \quad M'(A) = -m(A) + M(A).$$

It is clear that $\delta^*(x^*, M') = -x^* \circ m + \delta^*(x^*, M)$ for every $x^* \in X^*$. Note also that $0 \in M'(A)$ for every $A \in \Sigma$. The *Particular case* already proved ensures that the mapping $\nu' := j \circ M' : \Sigma \to \ell_\infty(B_{X^*})$ is a countably additive measure. On the other hand, the mapping $\nu'' : \Sigma \to \ell_\infty(B_{X^*})$ given by $\nu''(A)(x^*) := x^*(m(A))$ is obviously a countably additive measure. It follows that $\nu = \nu' + \nu''$ is countably additive, as required. □

For further information on the theory of multi-measures, we refer the reader to [23, Section 7], [27, Chapter 19] and the references therein.

4. Characterization of Pettis integrability for multi-functions

The aim of this section is to discuss the validity of Theorem A in the introduction within the setting of non-separable Banach spaces. Note that Corollary 2.3 gives us the extension to the non-separable case of (i)⇒(iii) in Theorem A.

With the help of the results about multi-measures isolated in Section 3 we start by proving Theorem 4.1 below that extends to the non-separable case the implication (i)⇒(ii) in Theorem A, see (d)⇒(e) in [15, Theorem 5.4]. Given $F : \Omega \to cwk(X)$ we write

$$W_F := \{\delta^*(x^*, F) : \ x^* \in B_{X^*}\} \subset \mathbb{R}^\Omega.$$

Recall that a family $\mathcal{H}$ of real-valued integrable functions defined on Ω is said to be *uniformly integrable* if it is bounded for $\|\cdot\|_1$ and for each $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{h\in\mathcal{H}} \int_E |h| \, d\mu \leq \varepsilon$ whenever $\mu(E) \leq \delta$.

Theorem 4.1. *Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. Define the indefinite Pettis integral of F by*

$$I_F : \Sigma \to cwk(X), \quad I_F(A) := \int_A F \, d\mu.$$

Then:

(i) *I_F is a countably additive multi-measure.*
(ii) *W_F is uniformly integrable.*

Proof. Clearly, $\delta^*(x^*, I_F)$ is countably additive for every $x^* \in X^*$ and we can apply Theorem 3.4 to conclude that I_F is a countably additive multi-measure. This proves (i).

We prove now statement (ii). The composition $\nu := j \circ I_F : \Sigma \to \ell_\infty(B_{X^*})$ is a countably additive vector measure that vanishes on all μ-null sets. Hence ν is μ-continuous, that is, $\lim_{\mu(A)\to 0} \|\nu\|(A) = 0$ (cf. [13, Theorem 1, p. 10]). On the other hand, observe that $\langle e_{x^*}, \nu\rangle(A) = \int_A \delta^*(x^*, F) \, d\mu$ for every $x^* \in B_{X^*}$ and every $A \in \Sigma$. In view of the above, the uniform integrability of W_F now follows from the fact that

$$\|\nu\|(A) \geq \sup_{x^*\in B_{X^*}} |\langle e_{x^*}, \nu\rangle|(A) = \sup_{x^*\in B_{X^*}} \int_A |\delta^*(x^*, F)| \, d\mu$$

for every $A \in \Sigma$. □

We turn our attention now to the implication (iii)⇒(i) in Theorem A for the non separable case: the proof below is inspired by some of the ideas in [15, Theorems 3.9 and 5.4]. We say that a Banach space X has the *Scalarly Measurable Selector Property* with respect to μ, shortly *μ-SMSP*, if every scalarly measurable multi-function $F : \Omega \to cwk(X)$ has a scalarly measurable selector.

Theorem 4.2. *Suppose X has the μ-SMSP. Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function such that every scalarly measurable selector of F is Pettis integrable. Then F is Pettis integrable.*

Proof. For any fixed $A \in \Sigma$ the set $\overline{IS_F(A)}$ is closed and convex. We prove now that $\overline{IS_F(A)} \in cwk(X)$. By James' theorem (cf. [17, §5]) we only have to prove that every $x^* \in X^*$ attains its supremum on $\overline{IS_F(A)}$. Fix $x^* \in X^*$ and consider the multi-function

$$G_{x^*} : \Omega \to cwk(X), \quad G_{x^*}(\omega) := \{x \in F(\omega) : \ x^*(x) = \delta^*(x^*, F(\omega))\}.$$

Since G_{x^*} is scalarly measurable (by Lemma 2.4) and X has the μ-SMSP, there is a scalarly measurable selector g_{x^*} of G_{x^*}. In particular, g_{x^*} is a selector of F and so it is Pettis integrable. Hence $\delta^*(x^*, F) = x^* \circ g_{x^*}$ is integrable. By the very definition, we have $\int_A g_{x^*} \, d\mu \in IS_F(A)$. We claim that

$$\sup\{x^*(x) : \ x \in \overline{IS_F(A)}\} = x^*\Big(\int_A g_{x^*} \, d\mu\Big).$$

Indeed, notice that for each Pettis integrable selector f of F we have

$$x^*\Big(\int_A g_{x^*}\,d\mu\Big) = \int_A x^*\circ g_{x^*}\,d\mu = \int_A \delta^*(x^*,F)\,d\mu \geq \\ \geq \int_A x^*\circ f\,d\mu = x^*\Big(\int_A f\,d\mu\Big),$$

hence

$$\sup\{x^*(x):\ x\in \overline{IS_F(A)}\} = \sup\{x^*(x):\ x\in IS_F(A)\} = x^*\Big(\int_A g_{x^*}\,d\mu\Big).$$

This proves that $\overline{IS_F(A)}$ is weakly compact. Moreover, the previous equality can be read as $\delta^*(x^*,\overline{IS_F(A)}) = \int_A \delta^*(x^*,F)\,d\mu$. It follows that F is Pettis integrable. □

Recall that the Banach space X is said to have the *μ-Pettis Integral Property* (shortly *μ-PIP*) if every scalarly measurable and scalarly bounded function $f:\Omega\to X$ is Pettis integrable. Here $f:\Omega\to X$ is said to be scalarly bounded if there is $M>0$ such that for each $x^*\in B_{X^*}$ we have $|x^*\circ f|\leq M$ μ-a.e. (the exceptional set depending on x^*). Equivalently, X has the μ-PIP if and only if the Pettis integrability of any function $f:\Omega\to X$ is equivalent to the fact that the family

$$Z_f = \{x^*\circ f:\ x^*\in B_{X^*}\}\subset \mathbb{R}^\Omega$$

is uniformly integrable.

Corollary 4.3. *Suppose X has the μ-SMSP and the μ-PIP. Let $F:\Omega\to cwk(X)$ be a multi-function. Then F is Pettis integrable if and only if W_F is uniformly integrable.*

Proof. It only remains to prove the "if" part. Observe that F is scalarly measurable. Each scalarly measurable selector f of F satisfies $-\delta^*(-x^*,F)\leq x^*\circ f\leq \delta^*(x^*,F)$ for all $x^*\in B_{X^*}$. Since W_F is uniformly integrable, the same holds for Z_f and thus f is Pettis integrable (because X has the μ-PIP). The result now follows from Theorem 4.2. □

The Banach space X has the PIP if it has the μ-PIP for any complete probability measure μ. The class of Banach spaces with the PIP is very large and contains, for instance, all spaces having Corson's property (C), see [42, Theorem 5-2-4], hence all weakly Lindelöf Banach spaces and all Banach spaces with w^*-angelic dual [35]. Recall that a topological space T is said to be *angelic* if each relatively countably compact set $C\subset T$ is relatively compact and, moreover, each point in the closure of C is the limit of a sequence in C.

The following cardinal number will be used in several examples that follow:

$$\kappa(\mu) = \min\{\mathrm{card}(\mathcal{E}):\ \mathcal{E}\subset\Sigma,\ \mu(E)=0 \text{ for every } E\in\mathcal{E},\ \mu^*(\cup\mathcal{E})>0\},$$

defined if there exist such infinite families $\mathcal{E}$ (this happens, for instance, if μ is not purely atomic). Here μ^* denotes the outer measure induced by μ. Notice that $\kappa(\mu)\geq\omega_1$. We point out that *the intersection of less than $\kappa(\mu)$ elements of Σ also belongs to Σ*, cf. [38, Lemma 4.4]. When $\kappa(\mu)$ cannot be defined, the intersection of any family of measurable sets is measurable and all our results involving $\kappa(\mu)$ are true without the restrictions on the cardinalities or density characters appearing in their statement. It is well known (cf. [40]) that Martin's Axiom implies the statement

"κ(Lebesgue measure on $[0,1]$) $=\mathfrak{c}$" (*Axiom M*).

The Banach space X has both the μ-SMSP and the PIP in each of the following cases:

- ► *X is separable*.
- ► *X is reflexive*, Theorem 5.1.
- ► *(X^*,w^*) is angelic and* $\mathrm{dens}(X^*,w^*)\leq\kappa(\mu)$, Example 5.5.
- ► *$X=Y^*$ has property (C) and* $\mathrm{dens}(Y)\leq\kappa(\mu)$, Example 5.6.

On the other hand, we will also see that X has the μ-SMSP whenever X^* is w^*-separable, Theorem 5.15. However, such an X does not have the μ-PIP in general. Indeed, Fremlin and Talagrand [18] showed that $\ell_\infty(\mathbb{N})$ fails the μ-PIP for certain pathological measure μ. They also proved that, at least under Axiom M, if B_{X^*} is w^*-separable for some equivalent norm on X (equivalently, X is isomorphic to a subspace of $\ell_\infty(\mathbb{N})$), then X has the PIP with respect to any *perfect* measure (for instance, a Radon finite measure on a topological space), cf. [42, Theorems 6-1-2 and 6-1-3].

We end up this section turning our attention to the following question, thoroughly studied in [5] and [6] within the setting of separable Banach spaces:

> *What is the relationship between the Pettis integrability of the multi-function $F : \Omega \to cwk(X)$ and that of the single-valued composition $j \circ F : \Omega \to \ell_\infty(B_{X^*})$?*

As in the separable case, see [6, Proposition 3.5], F is Pettis integrable whenever $j \circ F$ is. The proof of this fact given here is more direct.

Proposition 4.4. *Let $F : \Omega \to cwk(X)$ be a multi-function such that $j \circ F$ is Pettis integrable. Then F is Pettis integrable and*

$$j(I_F(A)) = \int_A j \circ F \, d\mu \quad \textit{for every } A \in \Sigma.$$

Proof. Since $j \circ F$ is Pettis integrable, the composition $\langle e_{x^*}, j \circ F \rangle = \delta^*(x^*, F)$ is integrable for every $x^* \in B_{X^*}$. Fix $A \in \Sigma$. The Pettis integrability of $j \circ F$ and the Hahn-Banach separation theorem ensure that

$$\int_A j \circ F \, d\mu \in \mu(A) \cdot \overline{\mathrm{co}((j \circ F)(A))},$$

cf. [13, proof of Corollary 8, p. 48]. Since $j(cwk(X))$ is a closed convex cone, we conclude that $\int_A j \circ F \, d\mu = j(C_A)$ for some $C_A \in cwk(X)$. Then

$$\int_A \delta^*(x^*, F) \, d\mu = \int_A \langle e_{x^*}, j \circ F \rangle \, d\mu = \langle e_{x^*}, \int_A j \circ F \, d\mu \rangle = \delta^*(x^*, C_A)$$

for every $x^* \in B_{X^*}$. This shows that F is Pettis integrable, with $j(I_F(A)) = \int_A j \circ F \, d\mu$ for every $A \in \Sigma$. □

It is known that the converse of Proposition 4.4 does not hold in general even for separable Banach spaces, see [5, Theorem 2.1]. However, it is valid under some additional assumptions on the given multi-function.

Proposition 4.5. *Let $F : \Omega \to cwk(X)$ be a multi-function such that $(j \circ F)(\Omega)$ is contained in a subspace of $\ell_\infty(B_{X^*})$ having w^*-angelic dual (this happens, for instance, if $F(\Omega)$ is separable for the Hausdorff distance). The following statements are equivalent:*

(i) *F is Pettis integrable;*
(ii) *W_F is uniformly integrable;*
(iii) *$j \circ F$ is Pettis integrable.*

Proof. The implication (i)⇒(ii) follows from Theorem 4.1 and (iii)⇒(i) from Proposition 4.4. Let us prove (ii)⇒(iii): let $Y \subset \ell_\infty(B_{X^*})$ be a subspace containing $(j \circ F)(\Omega)$ such that Y^* is w^*-angelic. Notice that the set $B := \{e_{x^*}|_Y : x^* \in B_{X^*}\} \subset B_{Y^*}$ is norming. The desired conclusion now follows by applying [6, Lemma 3.3] to the Y-valued function $j \circ F$, see the comments in [6, p. 552]. □

Recall that a convex, closed, bounded, non-empty set $C \subset X$ is norm compact if and only if the real-valued mapping given by $x^* \mapsto \delta^*(x^*, C)$ is w^*-continuous on B_{X^*}, cf. [31, Section 7]. Thus $j(ck(X)) \subset C(B_{X^*}) = C(B_{X^*}, w^*)$.

Proposition 4.6. *Suppose X^* is w^*-angelic. Let $F : \Omega \to cwk(X)$ be a multi-function with norm compact values such that W_F is uniformly integrable. Then F is Pettis integrable and $I_F(A)$ is norm compact for every $A \in \Sigma$.*

Proof. Fix $A \in \Sigma$. We claim that the mapping $\varphi_A^F : X^* \to \mathbb{R}$ given by $\varphi_A^F(x^*) = \int_A \delta^*(x^*, F)\, d\mu$ is w^*-continuous when restricted to B_{X^*}. Indeed, fix $B \subset B_{X^*}$ and take $x^* \in \overline{B}^{w^*}$. Since (X^*, w^*) is angelic, there is a sequence (x_n^*) in B converging to x^* in the w^*-topology. Given $\omega \in \Omega$, the set $F(\omega)$ is norm compact and so the mapping $\delta^*(\cdot, F(\omega))$ is w^*-continuous on B_{X^*}, hence $\delta^*(x_n^*, F(\omega)) \to \delta^*(x^*, F(\omega))$ as $n \to \infty$. Since W_F is uniformly integrable, an appeal to Vitali's convergence theorem ensures that

$$\varphi_A^F(x_n^*) = \int_A \delta^*(x_n^*, F)\, d\mu \to \int_A \delta^*(x^*, F)\, d\mu = \varphi_A^F(x^*) \quad \text{as } n \to \infty.$$

As $x^* \in \overline{B}^{w^*}$ is arbitrary, we conclude that $\varphi_A^F(\overline{B}^{w^*}) \subset \overline{\varphi_A^F(B)}$. Since this inclusion holds for any set $B \subset B_{X^*}$, the restriction $\varphi_A^F|_{B_{X^*}}$ is w^*-continuous, as claimed. Similarly, $\varphi_A^F|_{nB_{X^*}}$ is w^*-continuous for every $n \in \mathbb{N}$. Bearing in mind that φ_A^F is convex, an appeal to the Banach-Dieudonné theorem (cf. [16, Theorem 4.44]) ensures that φ_A^F is w^*-lower semicontinuous. By [7, Theorem II-16], there is a convex, closed, bounded, non-empty set $C \subset X$ such that $\varphi_A^F(x^*) = \delta^*(x^*, C)$ for every $x^* \in X^*$. The w^*-continuity of $\varphi_A^F|_{B_{X^*}}$ guarantees that C is norm compact and the proof is over. □

5. Measurable selectors

5.1. **Scalarly measurable selectors.** The first measurable selection results of this subsection follow from the existence of scalarly measurable selectors for Pettis integrable $cwk(X)$-valued functions, Theorem 2.5 above.

Theorem 5.1. *If X is reflexive, then it has the μ-SMSP.*

Proof. Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function. Since

$$\{\delta^*(x^*, F) : \; x^* \in X^*, \; \|x^*\| = 1\}$$

is a pointwise bounded family of measurable functions, we can find a countable partition $E_1, E_2, \dots$ of Ω in Σ and a sequence (M_n) of positive real numbers such that, for each $n \in \mathbb{N}$ and each $x^* \in X^*$ with $\|x^*\| = 1$, we have $|\delta^*(x^*, F)|_{E_n}| \le M_n$ μ-a.e. (cf. [32, Proposition 3.1]). Fix $n \in \mathbb{N}$ and consider the (constant) Pettis integrable multi-function $H_n : E_n \to cwk(X)$ given by $H_n(\omega) := M_n B_X$. Observe that for each $x^* \in X^*$ we have $\delta^*(x^*, F|_{E_n}) \le \delta^*(x^*, H_n)$ μ-a.e. From Lemma 2.2 it follows that $F|_{E_n}$ is Pettis integrable. By Theorem 2.5, we know that $F|_{E_n}$ admits a scalarly measurable selector $f_n : E_n \to X$. Define $f : \Omega \to X$ by $f(\omega) := f_n(\omega)$ if $\omega \in E_n$, $n \in \mathbb{N}$. Clearly, f is a scalarly measurable selector of F. □

Theorem 5.2. *Suppose X^* is w^*-angelic. Then every scalarly measurable multi-function $F : \Omega \to ck(X)$ admits a scalarly measurable selector.*

Proof. Again, since W_F is a pointwise bounded family of measurable functions, there is a countable partition $E_1, E_2, \dots$ of Ω in Σ and a sequence (M_n) of positive real numbers such that, for each $n \in \mathbb{N}$ and each $x^* \in B_{X^*}$, we have $|\delta^*(x^*, F)|_{E_n}| \le M_n$ μ-a.e. Given $n \in \mathbb{N}$, the previous inequality ensures that the family $W_{F|_{E_n}}$ is uniformly integrable and Proposition 4.6 can be applied to conclude that $F|_{E_n}$ is Pettis integrable. The proof finishes as in Theorem 5.1. □

At this point it is convenient to introduce the following terminology. Given a topological space T, we denote by $k(T)$ the collection of all compact non-empty subsets of T. Let $\mathcal{M}$ be a non-empty family of closed subsets of T. We say that a multi-function $F : \Omega \to k(T)$ is *$\mathcal{M}$-measurable* if $\{\omega \in \Omega : F(\omega) \cap M \neq \emptyset\} \in \Sigma$ for every $M \in \mathcal{M}$. Clearly, with this terminology, a multi-function $F:\Omega\to cwk(X)$ is scalarly measurable if and only if it is $\mathcal{M}$-measurable for $\mathcal{M}=$ collection of all closed half-spaces of X.

Lemma 5.3. *Let T be a topological space and $\mathcal{M}$ a non-empty family of closed subsets of T. Let $\gamma<\kappa(\mu)$ and, for each $\alpha<\gamma$, let $F_\alpha:\Omega\to k(T)$ be a $\mathcal{M}$-measurable multi-function. Suppose $F_\beta(\omega)\supset F_\alpha(\omega)$ for every $\beta<\alpha<\gamma$ and every $\omega\in\Omega$. Then:*

(i) *For each $\omega\in\Omega$, the set $F(\omega):=\bigcap_{\alpha<\gamma}F_\alpha(\omega)$ is compact and non-empty.*
(ii) *The multi-function $F:\Omega\to k(T)$ is $\mathcal{M}$-measurable.*

Proof. Given $\omega\in\Omega$, the net of compact non-empty sets $(F_\alpha(\omega))_{\alpha<\gamma}$ is decreasing and so it has compact non-empty intersection. In order to prove the second assertion, take $M\in\mathcal{M}$ and observe that, since $(F_\alpha(\omega)\cap M)_{\alpha<\gamma}$ is a decreasing net of compact sets, we have

$$\{\omega\in\Omega:\ F(\omega)\cap M\neq\emptyset\}=\bigcap_{\alpha<\gamma}\{\omega\in\Omega:\ F_\alpha(\omega)\cap M\neq\emptyset\}.$$

The $\mathcal{M}$-measurability of each F_α ensures that $\{\omega\in\Omega:F_\alpha(\omega)\cap M\neq\emptyset\}\in\Sigma$. Since $\mathrm{card}(\gamma)<\kappa(\mu)$, it follows that $\{\omega\in\Omega:\ F(\omega)\cap M\neq\emptyset\}\in\Sigma$. □

Our approach to the next theorem is inspired somehow by some of the ideas in the original proof of Valadier's result [43] saying that Banach spaces with w^*-separable dual always have the μ-SMSP (Theorem 5.15 below).

Theorem 5.4. *Suppose there is a set $\Gamma\subset X^*$ satisfying the following properties:*

(i) $\mathrm{card}(\Gamma)\leq\kappa(\mu)$.
(ii) Γ *separates the points of X.*
(iii) *A function $f:\Omega\to X$ is scalarly measurable if and only if $x^*\circ f$ is measurable for every $x^*\in\Gamma$.*

Then X has the μ-SMSP.

Proof. Enumerate $\Gamma=\{x^*_\alpha:\alpha<\mathrm{card}(\Gamma)\}$. Fix a scalarly measurable multi-function $F:\Omega\to cwk(X)$. We divide the proof of the existence of a scalarly measurable selector of F into several steps.

Step 1. Define $F_0:=F$. We will construct by transfinite induction a family of scalarly measurable multi-functions $F_\alpha:\Omega\to cwk(X)$, with $\alpha<\mathrm{card}(\Gamma)$, such that

$$F_\alpha(\omega)=\bigcap_{\beta<\alpha}\{x\in F_\beta(\omega):\ x^*_\beta(x)=\delta^*(x^*_\beta,F_\beta(\omega))\}\quad\text{for all }\omega\in\Omega \tag{2}$$

for every $0<\alpha<\mathrm{card}(\Gamma)$. To this end, assume that $0<\gamma<\mathrm{card}(\Gamma)$ and that we have already constructed a family $(F_\alpha)_{\alpha<\gamma}$ of scalarly measurable multi-functions satisfying (2) for every $0<\alpha<\gamma$. Given $\alpha<\gamma$, Lemma 2.4 applies to conclude that the multi-function $G_\alpha:\Omega\to cwk(X)$ given by

$$G_\alpha(\omega):=\{x\in F_\alpha(\omega):\ x^*_\alpha(x)=\delta^*(x^*_\alpha,F_\alpha(\omega))\}$$

is scalarly measurable. Observe that $G_\beta(\omega)\supset G_\alpha(\omega)$ for every $\beta<\alpha<\gamma$ and every $\omega\in\Omega$. Since $\gamma<\mathrm{card}(\Gamma)\leq\kappa(\mu)$, Lemma 5.3 allows us to define a scalarly measurable multi-function $F_\gamma:\Omega\to cwk(X)$ by the formula $F_\gamma(\omega):=\bigcap_{\alpha<\gamma}G_\alpha(\omega)$. Obviously, F_γ satisfies (2) by construction.

Step 2. Given $\omega\in\Omega$, the net of weakly compact non-empty sets $(F_\alpha(\omega))_{\alpha<\mathrm{card}(\Gamma)}$ is decreasing and so $\bigcap_{\alpha<\mathrm{card}(\Gamma)}F_\alpha(\omega)\neq\emptyset$. In fact, this intersection contains only one point. Indeed, if $x_1,x_2\in\bigcap_{\alpha<\mathrm{card}(\Gamma)}F_\alpha(\omega)$, then

$$x^*_\beta(x_1)=\delta^*(x^*_\beta,F_\beta(\omega))=x^*_\beta(x_2)$$

for every $\beta < \mathrm{card}(\Gamma)$, and the fact that Γ separates the points of X implies $x_1 = x_2$. Therefore, there is a function $f : \Omega \to X$ such that

$$\bigcap_{\alpha<\mathrm{card}(\Gamma)} F_\alpha(\omega) = \{f(\omega)\} \quad \text{for every } \omega \in \Omega.$$

Step 3. Clearly, f is a selector of F. By assumption, in order to prove that f is scalarly measurable we only have to check that $x^*_\beta \circ f$ is measurable for every $\beta < \mathrm{card}(\Gamma)$. Indeed, take $\beta < \alpha < \mathrm{card}(\Gamma)$. Then $f(\omega) \in F_\alpha(\omega)$ and therefore $x^*_\beta(f(\omega)) = \delta^*(x^*_\beta, F_\beta(\omega))$ for every $\omega \in \Omega$. Since F_β is scalarly measurable, we conclude that $x^*_\beta \circ f$ is measurable. The proof is over. □

A well known result of Edgar, see [14, Theorem 2.3], states that the Baire σ-algebra of a locally convex space endowed with its weak topology is exactly the σ-algebra generated by all the elements of the topological dual. In particular, if $\Gamma \subset X^*$ is a set separating the points of X and $\sigma(X,\Gamma)$ denotes the topology on X of pointwise convergence on Γ, then $\mathrm{Baire}(X, \sigma(X,\Gamma))$ is just the σ-algebra on X generated by Γ. Thus, condition (iii) in Theorem 5.4 is equivalent to "f is $\mathrm{Baire}(X, \sigma(X,\Gamma))$-measurable". Bearing this in mind, observe that Theorem 5.4 ensures that X has the μ-SMSP in the following two cases:

Example 5.5. *(X^*, w^*) is angelic and $\mathrm{dens}(X^*, w^*) \leq \kappa(\mu)$.* By a result of Gulisashvili [21], when (X^*, w^*) is angelic, the equality $\mathrm{Baire}(X, \sigma(X,\Gamma)) = \mathrm{Baire}(X, w)$ holds for any set $\Gamma \subset X^*$ separating the points of X. A wide class of spaces having w^*-angelic dual is that of *weakly Lindelöf determined (WLD)* Banach spaces. This class contains all weakly compactly generated spaces (cf. [16, Chapters 11 and 12]) and for every WLD space X the equality $\mathrm{dens}(X^*, w^*) = \mathrm{dens}(X)$ holds. In particular, any weakly compactly generated Banach space with density character less than or equal to ω_1 has the μ-SMSP. For instance, this applies to $c_0(\omega_1)$, separable Banach spaces, etc.

Example 5.6. *$X = Y^*$ has property (C) and $\mathrm{dens}(Y) \leq \kappa(\mu)$.* Indeed, any norm dense set $\Gamma \subset Y$ separates the points of X and satisfies $\mathrm{Baire}(X, \sigma(X,\Gamma)) = \mathrm{Baire}(X, w^*)$. On the other hand, since X is a dual space having property (C), the equality $\mathrm{Baire}(X, w^*) = \mathrm{Baire}(X, w)$ holds, see [39, Corollary 3.10].

Next three lemmas are needed to prove Theorem 5.10.

Lemma 5.7. *Let $A \in cwk(X)$ and $x^*_0 \in X^*$ satisfying $\inf x^*_0(A) < b < \sup x^*_0(A)$ for some $b \in \mathbb{R}$. Let $x \in A$ such that $x^*_0(x) \geq b$. Then for every $\varepsilon > 0$ there is $y \in A$ such that $\|x - y\| \leq \varepsilon$ and $x^*_0(y) \in [b, \sup x^*_0(A)] \cap \mathbb{Q}$.*

Proof. Since $A \in cwk(X)$, we have $x^*_0(A) = [\inf x^*_0(A), \sup x^*_0(A)]$. There are two possibilities:

Case 1. Suppose $x^*_0(x) < \sup x^*_0(A)$. Fix $z \in A$ such that $x^*_0(z) = \sup x^*_0(A)$ and consider the mapping $\varphi : [0,1] \to [x^*_0(x), \sup x^*_0(A)]$ given by $\varphi(\theta) := x^*_0(\theta z + (1-\theta)x)$. We can choose $0 < \theta < \min\{\varepsilon/\|x - z\|, 1\}$ such that $\varphi(\theta) \in \mathbb{Q}$. Then the vector $y := \theta z + (1 - \theta)x$ satisfies the required properties.

Case 2. Suppose $x^*_0(x) = \sup x^*_0(A)$. Take $z \in A$ such that $x^*_0(z) = b$ and consider now the mapping $\varphi : [0,1] \to [b, \sup x^*_0(A)]$ given by $\varphi(\theta) := x^*_0(\theta z + (1-\theta)x)$. Choose $0 < \theta < \min\{\varepsilon/\|x - z\|, 1\}$ such that $\varphi(\theta) \in \mathbb{Q}$. Then $y := \theta z + (1 - \theta)x$ works. □

Lemma 5.8 ([43, Lemme 3] or [7, Proposition I-24])**.** *Let $C \in cwk(X)$, $x^*_0 \in X^*$ and $\alpha \in \mathbb{R}$. Suppose $H := \{x \in X : x^*_0(x) = \alpha\}$ intersects C. Then $C \cap H \in cwk(X)$ and*

$$\delta^*(x^*, C \cap H) = \inf\{\delta^*(x^* - \lambda x^*_0, C) + \lambda\alpha : \lambda \in \mathbb{Q}\} \quad \textit{for every } x^* \in X^*.$$

Lemma 5.9. *Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function and consider a measurable function $h : \Omega \to \mathbb{R}$. Fix $x^*_0 \in X^*$ and write*

$$L(\omega) := \{x \in X : x^*_0(x) \geq h(\omega)\} \quad \textit{for every } \omega \in \Omega.$$

Then $E := \{\omega \in \Omega : F(\omega) \cap L(\omega) \neq \emptyset\} \in \Sigma$ *and the multi-function*

$$G : E \to cwk(X), \quad G(\omega) := F(\omega) \cap L(\omega),$$

is scalarly measurable.

Proof. Clearly, the set $E = \{\omega \in \Omega : \delta^*(x_0^*, F(\omega)) \geq h(\omega)\}$ belongs to Σ. Note that $-\delta^*(-x_0^*, F(\omega)) = \inf x_0^*(F(\omega))$ for every $\omega \in \Omega$. The sets

$$E_1 := \{\omega \in E : \ \inf x_0^*(F(\omega)) \geq h(\omega)\}$$
$$E_2 := \{\omega \in E : \ \sup x_0^*(F(\omega)) = h(\omega)\}$$
$$E_3 := \{\omega \in E : \ \inf x_0^*(F(\omega)) < h(\omega) < \sup x_0^*(F(\omega))\}$$

belong to Σ and $E = E_1 \cup E_2 \cup E_3$. We have $G(\omega) = F(\omega)$ whenever $\omega \in E_1$, thus the restriction $G|_{E_1}$ is scalarly measurable. On the other hand, we also have

$$G(\omega) = \{x \in F(\omega) : \ x_0^*(x) = \delta^*(x_0^*, F(\omega))\} \quad \text{for every } \omega \in E_2,$$

hence Lemma 2.4 can be applied to conclude that $G|_{E_2}$ is scalarly measurable. In order to finish the proof it only remains to show that $G|_{E_3}$ is scalarly measurable as well.

By Lemma 5.7, for each $\omega \in E_3$ we have

$$G(\omega) = \overline{\bigcup_{q \in I(\omega)} F(\omega) \cap \{x \in X : \ x_0^*(x) = q\}}^{\text{norm}}, \tag{3}$$

where $I(\omega) := \{q \in \mathbb{Q} : \ h(\omega) \leq q \leq \delta^*(x_0^*, F(\omega))\}$. Define

$$J(q) := \{\omega \in E_3 : \ h(\omega) \leq q \leq \delta^*(x_0^*, F(\omega))\} \in \Sigma$$

for every $q \in \mathbb{Q}$. Fix $x^* \in X^*$ and $a \in \mathbb{R}$, and write $W := \{x \in X : x^*(x) > a\}$. Given $q \in \mathbb{Q}$, Lemma 5.8 ensures that the multi-function $J(q) \to cwk(X)$ given by $\omega \mapsto F(\omega) \cap \{x \in X : x_0^*(x) = q\}$ is scalarly measurable, so the set

$$\{\omega \in J(q) : \ F(\omega) \cap \{x \in X : \ x_0^*(x) = q\} \cap W \neq \emptyset\}$$

belongs to Σ. Since W is open, equality (3) yields

$$\begin{aligned}
&\{\omega \in E_3 : \ G(\omega) \cap W \neq \emptyset\} = \\
&\quad = \Big\{\omega \in E_3 : \ \Big(\bigcup_{q \in I(\omega)} F(\omega) \cap \{x \in X : \ x_0^*(x) = q\}\Big) \cap W \neq \emptyset\Big\} = \\
&\quad\quad = \bigcup_{q \in \mathbb{Q}} \Big\{\omega \in J(q) : \ F(\omega) \cap \{x \in X : \ x_0^*(x) = q\} \cap W \neq \emptyset\Big\} \in \Sigma.
\end{aligned}$$

This shows that G is scalarly measurable. □

Let $\mathcal{M}^w$ be the collection of all finite intersections of closed half-spaces of X.

Theorem 5.10. *Let* $F : \Omega \to cwk(X)$ *be a multi-function. Then* F *is scalarly measurable if and only if* F *is* $\mathcal{M}^w$*-measurable.*

Proof. It only remains to check the "only if". We prove the following statement by induction on $n \in \mathbb{N}$:

(*) For each scalarly measurable multi-function $G : E \to cwk(X)$, where $E \in \Sigma$, the set $\{\omega \in E : G(\omega) \cap C \neq \emptyset\}$ belongs to Σ whenever C is the intersection of n closed half-spaces of X.

The case $n = 1$ follows directly from the scalar measurability. Assume $n > 1$ and the induction hypothesis. Fix a scalarly measurable multi-function $G : E \to cwk(X)$, where $E \in \Sigma$. Take $C := \bigcap_{i=1}^n \{x \in X : x_i^*(x) \geq a_i\}$, where $x_1^*, \dots, x_n^* \in X^*$ and $a_1, \dots, a_n \in \mathbb{R}$. Define $E' := \{\omega \in E : \delta^*(x_n^*, G(\omega)) \geq a_n\} \in \Sigma$ and consider the multi-function

$$G' : E' \to cwk(X), \quad G'(\omega) := G(\omega) \cap \{x \in X : \ x_n^*(x) \geq a_n\},$$

which is scalarly measurable by Lemma 5.9. Define $C' := \bigcap_{i=1}^{n-1}\{x \in X :\ x_i^*(x) \geq a_i\}$. Now, by induction hypothesis, the set

$$\{\omega \in E' :\ G'(\omega) \cap C' \neq \emptyset\} = \{\omega \in E :\ G(\omega) \cap C \neq \emptyset\}$$

belongs to Σ. The proof is over. □

The following lemma is a nice tool to get measurable selectors that will also be applied in the next subsection.

Lemma 5.11. *Let T be a topological space and $\mathcal{M}$ a non-empty family of closed subsets of T. Suppose $\mathcal{M}$ is closed under finite intersections. Let $g : T \to [0,\infty)$ be a function such that $g^{-1}([0,a]) \in \mathcal{M}$ for every $a \geq 0$. Let $F : \Omega \to k(T)$ be a $\mathcal{M}$-measurable multi-function. Then:*

(i) *For each $\omega \in \Omega$, the set*

$$G(\omega) := \{t \in F(\omega) :\ g(t) = \inf\{g(t') :\ t' \in F(\omega)\}\}$$

is compact and non-empty.

(ii) *The multi-function $G : \Omega \to k(T)$ is $\mathcal{M}$-measurable.*

Proof. Since $\mathcal{M}$ is made up of closed sets, g is lower semicontinuous and (i) follows straightforwardly bearing in mind that each $F(\omega)$ is compact and non-empty. We divide the proof of (ii) into several steps.

Step 1. Fix $n \in \mathbb{N}$. For each $m \in \mathbb{N}$ we define $A_{n,m} := g^{-1}([0, m/2^n]) \in \mathcal{M}$ and $B_{n,m} := \{\omega \in \Omega : F(\omega) \cap A_{n,m} \neq \emptyset\} \in \Sigma$. Clearly, $B_{n,m} \subset B_{n,m+1}$ for every $m \in \mathbb{N}$ and $\Omega = \bigcup_{m=1}^{\infty} B_{n,m}$. Define $C_{n,1} := B_{n,1}$ and $C_{n,m} := B_{n,m} \setminus B_{n,m-1}$ for every $m \geq 2$, so that $C_{n,1}, C_{n,2}, \dots$ is a countable partition of Ω in Σ. Consider the multi-function $F_n : \Omega \to k(T)$ defined by $F_n(\omega) := F(\omega) \cap A_{n,m}$ whenever $\omega \in C_{n,m}$. Then F_n is $\mathcal{M}$-measurable. Indeed, given $M \in \mathcal{M}$, note that $A_{n,m} \cap M \in \mathcal{M}$ for every $m \in \mathbb{N}$ and we have

$$\{\omega \in \Omega :\ F_n(\omega) \cap M \neq \emptyset\} = \\ = \bigcup_{m=1}^{\infty} \Big(C_{n,m} \cap \{\omega \in \Omega :\ F(\omega) \cap (A_{n,m} \cap M) \neq \emptyset\}\Big) \in \Sigma$$

since F is $\mathcal{M}$-measurable.

Step 2. Clearly, $C_{n,m} = C_{n+1,2m-1} \cup C_{n+1,2m}$ and $A_{n+1,2m-1} \subset A_{n+1,2m} = A_{n,m}$ for every $n, m \in \mathbb{N}$, by the very definitions. It follows that $F_{n+1}(\omega) \subset F_n(\omega)$ for every $\omega \in \Omega$ and every $n \in \mathbb{N}$. In view of Lemma 5.3, we can define a $\mathcal{M}$-measurable multi-function $H : \Omega \to k(T)$ by $H(\omega) := \bigcap_{n=1}^{\infty} F_n(\omega)$.

Step 3. Given $\omega \in \Omega$, note that a point $t \in F(\omega)$ does not belong to $G(\omega)$ if and only if $g(t') < m/2^n < g(t)$ for some $t' \in F(\omega)$ and some $n, m \in \mathbb{N}$, which is equivalent to saying that $\omega \in C_{n,m'}$ for some $1 \leq m' \leq m$ and $t \notin A_{n,m}$. It follows that $G(\omega) = H(\omega)$ for every $\omega \in \Omega$ and the proof is over. □

Lemma 5.12. *Let T be a topological space and $\mathcal{M}$ a non-empty family of closed subsets of T. Suppose $\mathcal{M}$ is closed under finite intersections. Let $\kappa < \kappa(\mu)$ be a cardinal and write $\mathcal{M}(\kappa)$ to denote the collection of all intersections of at most κ elements of $\mathcal{M}$. Then a multi-function $F : \Omega \to k(T)$ is $\mathcal{M}$-measurable if and only if it is $\mathcal{M}(\kappa)$-measurable.*

Proof. It only remains to prove the "only if". We will check that F is $\mathcal{M}(\kappa)$-measurable for every cardinal $\kappa < \kappa(\mu)$ by transfinite induction. Fix such a cardinal and assume that F is $\mathcal{M}(\kappa')$-measurable for every cardinal $\kappa' < \kappa$. Clearly, the conclusion follows automatically if κ is finite, since $\mathcal{M}$ is closed under finite intersections. So assume that κ is infinite. Take a family $\{M_\alpha : \alpha < \kappa\} \subset \mathcal{M}$ and define, for each ordinal $\beta < \kappa$, the set

$$N_\beta := \bigcap_{\alpha<\beta} M_\alpha \in \mathcal{M}(\mathrm{card}(\beta)),$$

so that $\{\omega \in \Omega : F(\omega) \cap N_\beta \neq \emptyset\} \in \Sigma$ by induction hypothesis. Given $\omega \in \Omega$, the net of compact sets $(F(\omega) \cap N_\beta)_{\beta<\kappa}$ is decreasing and, therefore, we have

$$\bigcap_{\beta<\kappa} \{\omega \in \Omega :\ F(\omega) \cap N_\beta \neq \emptyset\} = \Big\{\omega \in \Omega :\ F(\omega) \cap \Big(\bigcap_{\beta<\kappa} N_\beta\Big) \neq \emptyset\Big\}.$$

Observe that $\bigcap_{\beta<\kappa} N_\beta = \bigcap_{\alpha<\kappa} M_\alpha$. Since the intersection of less than $\kappa(\mu)$ elements of Σ also belongs to Σ and $\kappa < \kappa(\mu)$, we conclude that

$$\Big\{\omega \in \Omega :\ F(\omega) \cap \Big(\bigcap_{\alpha<\kappa} M_\alpha\Big) \neq \emptyset\Big\} \in \Sigma.$$

This shows that F is $\mathcal{M}(\kappa)$-measurable, as required. □

In the next two theorems we apply the previous work to present sufficient conditions on X to have the μ-SMSP. Recall that a norm $\|\cdot\|$ on X is said to be *strictly convex* if $x = x'$ whenever $x, x' \in X$ are such that $\|x\| = \|x'\| = 1$ and $\|x + x'\| = 2$.

Theorem 5.13. *If X admits an equivalent strictly convex norm with the property that $\mathrm{dens}(B_{X^*}, w^*) < \kappa(\mu)$, then X has the μ-SMSP.*

Proof. Write $\kappa := \mathrm{dens}(B_{X^*}, w^*)$. Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function. By Theorem 5.10 and Lemma 5.12, F is $\mathcal{M}^w(\kappa)$-measurable. Let $\|\cdot\|$ be an equivalent strictly convex norm with $\mathrm{dens}(B_{X^*}, w^*) < \kappa(\mu)$ and define $g : X \to [0, \infty)$ by $g(x) := \|x\|$. Observe that

$$g^{-1}([0, a]) = \bigcap_{x^* \in D} \{x \in X : |x^*(x)| \leq a\} \in \mathcal{M}^w(\kappa) \quad \text{for every } a \geq 0,$$

where $D \subset B_{X^*}$ is any w^*-dense set with $\mathrm{card}(D) = \kappa$. Given $\omega \in \Omega$, the set

$$G(\omega) := \{x \in F(\omega) :\ \|x\| = \inf\{\|x'\| :\ x' \in F(\omega)\}\}$$

contains only one point, say $f(\omega)$, because $F(\omega) \in cwk(X)$ and $\|\cdot\|$ is w-lower semicontinuous and strictly convex. Note that the function $f : \Omega \to X$ is a selector of F. We can now apply Lemma 5.11 (working with the topological space (X, w) and considering the family $\mathcal{M} = \mathcal{M}^w(\kappa)$) to conclude that $f^{-1}(C) \in \Sigma$ for every $C \in \mathcal{M}^w(\kappa)$, so that f is scalarly measurable. □

A norm $\|\cdot\|$ on X is called *locally uniformly rotund* (shortly *LUR*) if $\|x_n - x\| \to 0$ whenever the sequence (x_n) in X and $x \in X$ satisfy $\|x_n\| \to \|x\|$ and $\|x_n + x\| \to 2\|x\|$. Clearly, this property implies strict convexity. Many Banach spaces admit an equivalent LUR norm, for instance, the WLD ones, cf. [10, Corollary 1.10, p. 286]. For complete information about renormings in Banach spaces we refer the reader to [10], [19] and [46].

As an application of the previous theorem we obtain:

Example 5.14. *$C([0, \omega_1])$ has the μ-SMSP whenever $\kappa(\mu) > \omega_1$.* Indeed, it is known that $C([0, \omega_1])$ admits an equivalent LUR (in particular, strictly convex) norm, because $[0, \omega_1]$ is a Valdivia compactum, cf. [10, Corollary 1.10, p. 286]. On the other hand, the dual unit ball of any equivalent norm on $C([0, \omega_1])$ has w^*-density character ω_1 (bear in mind that this space contains a subspace isomorphic to $c_0(\omega_1)$).

A similar argument allows us to give an alternative proof of the previously announced result of Valadier, see [43, Proposition 6].

Theorem 5.15 (Valadier)**.** *If X^* is w^*-separable, then X has the μ-SMSP.*

Proof. Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function. By Theorem 5.10 and Lemma 5.12, we know that F is $\mathcal{M}^w(\aleph_0)$-measurable. Fix a countable w^*-dense set $\{x_n^* : n \in \mathbb{N}\} \subset X^*$ and consider the operator

$$T : X \to \ell^2(\mathbb{N}), \quad T(x) := \Big(\frac{x_n^*(x)}{2^n}\Big).$$

Define $g : X \to [0,\infty)$ by $g(x) := \|T(x)\|_{\ell^2(\mathbb{N})}$. Since $B_{\ell^2(\mathbb{N})^*}$ is w^*-separable, we have $g^{-1}([0,a]) \in \mathcal{M}^w(\aleph_0)$ for every $a \geq 0$. Since g is a w-lower semicontinuous strictly convex norm on X (non necessarily equivalent to the original one!), the arguments in the proof of Theorem 5.13 (dealing now with the family of weakly closed sets $\mathcal{M}^w(\aleph_0)$) ensure that F admits a scalarly measurable selector. $\square$

It is well known that X admits an equivalent strictly convex norm whenever X^* is w^*-separable, cf. [10, Theorem 2.4, p. 46]. However, the fact that such an X has the μ-SMSP cannot be deduced, in general, from Theorem 5.13 above. Indeed, the Johnson-Lindentrauss space JL_2 has w^*-separable dual but, for any equivalent norm on JL_2, the corresponding dual unit ball is not w^*-separable, see [26, Example 1].

The technique used in the proof of Theorem 2.6 can be used to prove Theorem 5.16 below: the particular case of Banach spaces having w^*-separable dual was first proved by Valadier in [43, Proposition 7].

Theorem 5.16. *Suppose X has the μ-SMSP. Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function. Then there is a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(X^*,w^*)}$ of scalarly measurable selectors of F such that*

$$F(\omega) = \overline{\{f_\alpha(\omega) :\ \alpha < \mathrm{dens}(X^*,w^*)\}} \quad \textit{for every } \omega \in \Omega.$$

5.2. **Borel measurable selectors.** In this subsection we exploit Lemma 5.11 in order to find nice selectors for multi-functions with stronger measurability properties. It is convenient to recall first some facts concerning measurability in Banach spaces.

Let $\mathcal{M}^{nc}$ (resp. $\mathcal{M}^{cc}$) be the collection of all norm closed (resp. convex closed) subsets of X. Write $\sigma(\mathcal{M}^{cc})$ to denote the smallest σ-algebra on X containing $\mathcal{M}^{cc}$. In general, we have

$$\mathrm{Baire}(X,w) \subset \sigma(\mathcal{M}^{cc}) \subset \mathrm{Borel}(X,w) \subset \mathrm{Borel}(X,\mathrm{norm}).$$

All these σ-algebras coincide for separable X but some inclusions may be strict beyond the separable case. Talagrand [41] showed that $\mathrm{Borel}(\ell_\infty(\mathbb{N}),w) \neq \mathrm{Borel}(\ell_\infty(\mathbb{N}),\mathrm{norm})$ and Edgar [14] proved that the equality $\mathrm{Borel}(X,w) = \mathrm{Borel}(X,\mathrm{norm})$ holds whenever X admits an equivalent Kadec norm (i.e. a norm for which the weak and norm topologies coincide on the unit sphere; clearly, every LUR norm is Kadec). A result of Raja [36, Theorem 1.2] states that X admits an equivalent LUR norm if and only if every norm open set $U \subset X$ can be written as $U = \bigcup_{n=1}^\infty (C_n \setminus D_n)$, where $C_n, D_n \in \mathcal{M}^{cc}$ for every $n \in \mathbb{N}$; in this case, we have $\sigma(\mathcal{M}^{cc}) = \mathrm{Borel}(X,\mathrm{norm})$. On the other hand, it is known that $\mathrm{Baire}(X,w) \neq \sigma(\mathcal{M}^{cc})$ whenever X^* is not w^*-separable, cf. [22, Theorem 1.5.3], but also for $\ell_\infty(\mathbb{N})$ and the Johnson-Lindenstrauss spaces [26], see [37, Theorem 2.3].

Theorem 5.17. *Suppose X admits an equivalent strictly convex norm. Then every $\mathcal{M}^{cc}$-measurable multi-function $F : \Omega \to cwk(X)$ admits a $\sigma(\mathcal{M}^{cc})$-measurable selector.*

Proof. Fix an equivalent strictly convex norm $\|\cdot\|$ on X. Given $\omega \in \Omega$, the set

$$G(\omega) := \{x \in F(\omega) :\ \|x\| = \inf\{\|x'\| :\ x' \in F(\omega)\}\}$$

contains only one point $f(\omega)$ because $F(\omega) \in cwk(X)$ and $\|\cdot\|$ is w-lower semicontinuous and strictly convex. The function $f : \Omega \to X$ is a selector of F. Obviously, the mapping $g : X \to [0,\infty)$ given by $g(x) := \|x\|$ satisfies $g^{-1}([0,a]) \in \mathcal{M}^{cc}$ for every $a \geq 0$. We can apply Lemma 5.11 (working with the topological space (X,w) and taking $\mathcal{M} = \mathcal{M}^{cc}$) to conclude that f is $\sigma(\mathcal{M}^{cc})$-measurable. $\square$

In fact, under the same assumption we can say more:

Theorem 5.18. *Suppose X admits an equivalent strictly convex norm. Then every $\mathcal{M}^{cc}$-measurable multi-function $F : \Omega \to cwk(X)$ admits a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(X)}$ of $\sigma(\mathcal{M}^{cc})$-measurable selectors such that*

$$F(\omega) = \overline{\{f_\alpha(\omega) :\ \alpha < \mathrm{dens}(X)\}} \quad \textit{for every } \omega \in \Omega.$$

Proof. Fix a dense set $\{x_\alpha : \alpha < \kappa\} \subset X$, where $\kappa := \mathrm{dens}(X)$, and take an equivalent strictly convex norm $\|\cdot\|$ on X. Fix $\alpha < \kappa$. Since the multi-function $F_\alpha : \Omega \to cwk(X)$ given by $F_\alpha(\omega) := -x_\alpha + F(\omega)$ is $\mathcal{M}^{cc}$-measurable, a glance at the proof of Theorem 5.17 reveals that F_α admits a $\sigma(\mathcal{M}^{cc})$-measurable selector $g_\alpha : \Omega \to X$ with the property that

$$\|g_\alpha(\omega)\| = \inf\{\|x - x_\alpha\| : \ x \in F(\omega)\} \quad \text{for every } \omega \in \Omega. \tag{4}$$

Let us consider the $\sigma(\mathcal{M}^{cc})$-measurable selector $f_\alpha : \Omega \to X$ of F defined by the formula $f_\alpha(\omega) := g_\alpha(\omega) + x_\alpha$. We claim that the collection $\{f_\alpha\}_{\alpha<\kappa}$ fulfills the required property. Indeed, fix $\omega \in \Omega$ and $x \in F(\omega)$. Given $\varepsilon > 0$, there is $\alpha < \kappa$ such that $\|x - x_\alpha\| \le \varepsilon$, hence (4) yields

$$\|f_\alpha(\omega) - x\| \le \|g_\alpha(\omega)\| + \|x - x_\alpha\| \le 2\|x - x_\alpha\| \le 2\varepsilon.$$

As $x \in F(\omega)$ and $\varepsilon > 0$ are arbitrary, we get $F(\omega) = \overline{\{f_\alpha(\omega) : \alpha < \kappa\}}$. □

As we have mentioned at the beginning of the subsection, if X admits an equivalent LUR norm then $\sigma(\mathcal{M}^{cc}) = \mathrm{Borel}(X, \mathrm{norm})$. Bearing in mind that every LUR norm is strictly convex, from Theorem 5.18 we deduce the following corollary.

Corollary 5.19. *Suppose X admits an equivalent LUR norm. Let $F : \Omega \to cwk(X)$ be a $\mathcal{M}^{cc}$-measurable multi-function. Then F admits a collection of $\mathrm{Borel}(X, \mathrm{norm})$-measurable selectors $\{f_\alpha\}_{\alpha<\mathrm{dens}(X)}$ such that*

$$F(\omega) = \overline{\{f_\alpha(\omega) : \ \alpha < \mathrm{dens}(X)\}} \quad \textit{for every } \omega \in \Omega.$$

We stress that the previous corollary improves a result of Leese [30, Theorem 2], who proved the existence of $\mathrm{Borel}(X, \mathrm{norm})$-measurable selectors for $\mathcal{M}^{nc}$-measurable multi-functions when X admits an equivalent *uniformly rotund* norm.

Similar arguments to those of Theorems 5.17 and 5.18, now dealing with the norm topology of X, allow us to deduce the following result.

Theorem 5.20. *Suppose X admits an equivalent strictly convex norm. Let $F : \Omega \to ck(X)$ be a $\mathcal{M}^{nc}$-measurable multi-function. Then F admits a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(X)}$ of $\mathrm{Borel}(X, \mathrm{norm})$-measurable selectors such that*

$$F(\omega) = \overline{\{f_\alpha(\omega) : \ \alpha < \mathrm{dens}(X)\}} \quad \textit{for every } \omega \in \Omega.$$

Under such assumptions, the existence of at least one $\mathrm{Borel}(X, \mathrm{norm})$-measurable selector was first proved by Leese [30, Theorem 1].

To the best of our knowledge, the question below remains unanswered in full generality:
OPEN PROBLEM.- *Does every Banach space have the μ-SMSP for any μ?*

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