JAMES BOUNDARIES AND σ-FRAGMENTED SELECTORS

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ABSTRACT. We study the boundary structure for w^*-compact subsets of dual Banach spaces. Being more precise, for a Banach space X, $0 < \varepsilon < 1$ and T a subset of the dual space X^* such that $\bigcup\{B(t,\varepsilon) : t \in T\}$ contains a James boundary for B_{X^*} we study different kind of conditions on T, besides T being countable, which ensure that

(S) $$X^* = \overline{\operatorname{span} T}^{\|\cdot\|}.$$

We analyze two different non separable cases where the equality (S) holds: (a) if $J : X \to 2^{B_{X^*}}$ is the duality mapping and there exists a σ-fragmented map $f : X \to X^*$ such that $B(f(x),\varepsilon) \cap J(x) \neq \emptyset$ for every $x \in X$, then (S) holds for $T = f(X)$ and in this case X is Asplund; (b) if T is weakly countably K-determined then (S) holds, X^* is weakly countably K-determined and moreover for every James boundary B of B_{X^*} we have $B_{X^*} = \overline{\operatorname{co}(B)}^{\|\cdot\|}$. Both approaches use Simons' inequality and ideas exploited by Godefroy in the separable case (*i.e.*, when T is countable). While proving (a) we prove that X is Asplund if, and only if, the duality mapping has an ε-selector, $0 < \varepsilon < 1$, that sends separable sets into separable ones. A consequence of the above is that the dual unit ball B_{X^*} is norm fragmented if, and only if, it is norm ε-fragmented for some fixed $0 < \varepsilon < 1$. Our analysis is completed by offering a characterization of those Banach spaces (non necessarily separable) without copies of ℓ^1 via the structure of the boundaries of w^*-compact sets of their duals. Several applications and complementary results are proved. Our results extend to the non separable case results by Rodé, Godefroy and Contreras-Payá.

1. INTRODUCTION

Given a Banach space X and a w^*-compact subset $K \subset X^*$, a James boundary for K is a subset B of K such that for every $x \in X$ there exists some $b \in B$ such that $b(x) = \sup\{k(x) : k \in K\}$. If K is moreover convex the classical James boundary $\operatorname{Ext} K$ of the set of the extreme points of K allows us to recover K through the equality $K = \overline{\operatorname{co}(\operatorname{Ext} K)}^{w^*}$. In general, James boundaries can be even disjoint of the set of extreme points. Therefore the mere idea of studying how properties of a given boundary are reflected on K has been a research field of continuous interest with applications to the theory of general Banach spaces, optimization, Fourier analysis, etc. Here is a non exhaustive list of papers and books dealing with these kind of problems [4, 5, 9, 12, 15, 16, 22, 23, 26, 28, 38, 40, 47]: the reference [13] offers an excellent survey about infinite dimensional convexity and in particular about integral representation theorems and boundaries.

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Our starting point is the following result:

Theorem 1.1 ([4, 13, 16]). *Let X be a Banach space, $0<\varepsilon<1$ and T a countable subset of X^* such that $\bigcup\{B(t,\varepsilon) : t\in T\}$ contains a James boundary for B_{X^*}. Then $X^*=\overline{\operatorname{span} T}^{\|\cdot\|}$ and therefore X^* is separable.*

Note that under the conditions in Theorem 1.1, once we know that X^* is separable a result by Rodé, see [40], can be used to obtain that for every James boundary B of B_{X^*} we have $B_{X^*}=\overline{\operatorname{co}(B)}^{\|\cdot\|_{X^*}}$: using Fonf and Lindenstrauss' terminology, [12], when the last equality holds we will say that B has property (S) –(S) stands for *strong*.

We aim in this paper to extend the results above to the non separable case (T will be then non countable) and thus answer a question raised in [38]. We envisage two different ways of extending the previous results to the non separable case:

✓ *Using ε-selectors for the duality mapping.* If $(X,\|\cdot\|)$ is a Banach space the duality mapping $J: X\to 2^{B_{X^*}}$ is defined at each $x\in X$ by

$$J(x):=\{x^*\in B_{X^*} : x^*(x)=\|x\|\}.$$

Our main result here, Theorem 4.1, states that if $f: X\to X^*$ is a σ-fragmented map such that $B(f(x),\varepsilon)\cap J(x)\neq\emptyset$ for every $x\in X$ then $X^*=\overline{\operatorname{span} f(X)}^{\|\cdot\|}$ and in this case X is Asplund.

✓ *Using descriptive properties of T.* Our result here, Theorem 5.1, says that if $T\subset X^*$ is weakly countably K-determined and there exists $0<\varepsilon<1$ such that $\bigcup\{B(t,\varepsilon) : t\in T\}$ contains a James boundary for B_{X^*}, then $X^*=\overline{\operatorname{span} T}^{\|\cdot\|}$ is weakly countably K-determined and every James boundary B of B_{X^*} has property (S).

Since every separable metric space is countably K-determined our second approach clearly extends Theorem 1.1 and its consequence above. The notion of σ-fragmented map, see Definition 1, is truly wide and in particular each function with countable range is σ-fragmented: thus our first approach also extends Theorem 1.1.

A brief description of the contents of the paper follows. Section 2 is devoted to the study of the notion of σ-fragmentability for single-valued and set-valued maps. We obtain a characterization of set-valued σ-fragmented maps via ε-selectors that are either piecewise barely constant or piecewise barely continuous, Theorem 2.1. σ-fragmented maps are precisely the uniform limits of piecewise barely constants maps, Corollary 2.2; pointwise cluster points of σ-fragmented maps are σ-fragmented, Proposition 2.3. The relationship between σ-fragmentability and networks is stated in one of the key results in this paper, Theorem 2.5, that leads to the other key result Theorem 2.8: the last result states how σ-fragmented maps send separable sets into separable ones.

In Section 3 we specialize the results of the previous section for the identity map from a Banach space equipped with its weak topology into itself with the norm metric. By doing so we exhibit several properties of σ-fragmented Banach spaces following the scheme presented in [31] for the renorming case.

In Section 4 we prove one of our main results already commented, Theorem 4.1. We also characterize Asplund spaces as those Banach spaces X for which the duality mapping J has an ε-selector, $0<\varepsilon<1$, $f: X\to X^*$ that sends separable subsets of X into separable subsets of X^*, Theorem 4.2 and Corollary 4.3. A consequence of the above is that the dual unit ball B_{X^*} is norm fragmented if, and only if, it is norm ε-fragmented for some fixed $0 < \varepsilon < 1$, Corollary 4.6. The section is closed with Theorem 4.7 and its Corollary where results as above are proved for any w^*-compact subset of X^* .

Section 5 starts with the proof of our Theorem 5.1 already presented above. Theorem 5.4 offers a characterization of those Banach spaces (non necessarily separable) without copies of ℓ^1 via the structure of the boundaries of w^*-compact sets of their duals and the topology γ on X^* of uniform convergence on bounded and countable sets of X. The paper is finished by giving our proof in Corollary 5.6 of the fact that for a dual Banach space X^* with property $\mathcal{C}$ all boundaries for B_{X^*} have property (S).

A bit of terminology: Most of our notation and terminology is standard otherwise it is either explained here or whenever it is needed: unexplained concepts and terminology can be found in our standard references for Banach spaces [5, 9] and topology [7, 29]. By letters $T, E, X, \ldots$ we denote topological spaces here. Sometimes our topological spaces are assumed to be metric and then the letters $d, \rho, \ldots$ denote the metrics on them. If (E, ρ) is a metric space, $x \in E$ and $\delta > 0$ we denote by $B_\rho(x, \delta)$ (or $B(x, \delta)$ if no confussion arises) the open ρ-ball centered at x of radius δ; if $A \subset E$ we write

$$\mathrm{diam}(A) := \sup\{d(x, y) : x, y \in A\}.$$

All our vector spaces $E, X, \ldots$ are assumed to be real. Sometimes E is assumed to be a normed space with the norm $\|\cdot\|$: the letter X is reserved to denote a Banach space. Given a subset S of a vector space, we write $\mathrm{co}(S)$, $\mathrm{aco}(S)$ and $\mathrm{span}(S)$ to denote, respectively, the convex, absolutely convex and linear hull of S. In the normed space $(E, \|\cdot\|)$ the unit ball $\{x \in E : \|x\| \leq 1\}$ is denoted by B_E. Thus the unit ball of E^* is B_{E^*}. If S is a subset of E^*, then $\sigma(E, S)$ denotes the weakest topology for X that makes each member of S continuous, or equivalently, the topology of pointwise convergence on S. Dually, if S is a subset of E, then $\sigma(E^*, S)$ is the topology for E^* of pointwise convergence on S. In particular $\sigma(E, E^*)$ and $\sigma(E^*, E)$ are the weak (w) and weak* (w*) topologies respectively. Of course, $\sigma(X, S)$ is always a locally convex topology and it is Hausdorff if and only if $X^* = \overline{\mathrm{span}\, S}^{\mathrm{w}^*}$ and similarly for $\sigma(X^*, S)$. Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle$ and $x^*(x)$ to denote the evaluation of x^* at x.

2. σ-FRAGMENTED MAPS

Our main tool to face the problems presented in the introduction is the notion of σ-fragmented map that was introduced in [26] in order to deal with selection problems. Since its introduction this notion has been used in different settings by different authors as for instance in [35]. In this section we will present a detailed study of σ-fragmented maps which is close in spirit to the properties studied for σ-continuous maps in connection with renorming properties of Banach spaces in [32].

Definition 1 ([26]). *Let f be a map from a topological space (T, τ) into a metric space (E, ρ). Let S be a subset of T. We say that $f|_S$ is* ρ-fragmented down to ε or ε-fragmented *for some $\varepsilon > 0$, if whenever C is a non-empty subset of S, there exists a τ-open subset V in T such that $C \cap V \neq \emptyset$ and $\rho - \mathrm{diam}(f(C \cap V)) < \varepsilon$:*

we simply use fragmented *instead of* ρ-fragmented *when* ρ *is understood. Given* $\varepsilon > 0$ *we say that* f *is* ε-σ-*fragmented if there exists countable family of subsets* $\{T_n^\varepsilon : n \in \mathbb{N}\}$ *that covers* T *such that* $f|_{T_n^\varepsilon}$ *is* ε-*fragmented for every* $n \in \mathbb{N}$.

The map f *is said to be* σ-fragmented *if it is* ε-σ-*fragmented for each* $\varepsilon > 0$.

For set-valued maps the corresponding notion of σ-fragmentability is below:

Definition 2 ([26])**.** *Let* F *be a set-valued map from a topological space* (T, τ) *into the subsets of a metric space* (E, ρ). *Let* S *be a subset of* T. *We say that* $F|_S$ *is* fragmented down to ε *for some* $\varepsilon > 0$, *if whenever* C *is a non-empty subset of* S, *there exists a* τ-*open subset* V *of* S *with* $V \cap C \neq \emptyset$ *and a subset* D *of* E *with* $\rho-\operatorname{diam}(D) < \varepsilon$ *such that* $F(t) \cap D \neq \emptyset$ *for every* $t \in V \cap C$. *Given* $\varepsilon > 0$ *we say that* F *is* ε-σ-*fragmented if there exists countable family of subsets* $\{T_n^\varepsilon : n \in \mathbb{N}\}$ *that covers* T *such that* $F|_{T_n^\varepsilon}$ *is* ε-*fragmented for every* $n \in \mathbb{N}$.

The set-valued map F *is said to be* σ-fragmented *if it is* ε-σ-*fragmented for each* $\varepsilon > 0$.

It is easily proved that in the above definitions of ε-fragmentability for f and F the sets "C's" can be taken to be closed without loss of generality. The easiest but non trivial examples of σ-fragmented maps are provided by the class of maps introduced in the following definition: we refer to [30] for the concept of barely continuous function.

Definition 3. *A map* f *from a topological space* (T, τ) *into a metric space* (E, ρ) *is said to be* barely continuous, (resp. barely constant) *if for every non-empty closed set* $A \subset T$ *the restriction* $f|_A$ *has at least one-point of continuity (resp. there exists an* τ-*open set* $W \subset T$ *such that* $W \cap A \neq \emptyset$ *and* $f|_{A \cap W}$ *is constant).*

We say that f *is piecewise barely continuous (resp. piecewise barely constant) if there exists a countable family of subsets* $\{T_n : n \in \mathbb{N}\}$ *that covers* T *such that* $f|_{T_n}$ *is barely continuous (resp. barely constant) for each* $n \in \mathbb{N}$.

Baire's Great Theorem establishes that a map f from a complete metric space T into a Banach space E is barely continuous if, and only if, f is the pointwise limit of a sequence of continuous functions, *i.e.*, f is a Baire one map, see [5, Theorem I.4.1]. It was proved in [26, Corollary 7] that a map f from a perfectly paracompact space T into a Banach space E is σ-fragmented with closed sets T_n^ε in Definition 1 if, and only if, it is a Baire one map. Corollary 7 in [26] is based upon the approximation result [26, Theorem 5] that is established there for σ-fragmented maps by closed sets T_n^ε: if we drop off the closedness of the T_n^ε's and only care about σ-fragmentability we can prove the following:

Theorem 2.1. *Let* F *be a set-valued map from a topological space* (T, τ) *into the subsets of a metric space* (E, ρ). *The following statements are equivalent:*

(i) F *is* σ-*fragmented;*
(ii) *for every* $\varepsilon > 0$ *there exists a piecewise barely constant map* $f_\varepsilon : T \to E$ *such that* $\rho - \operatorname{dist}(f_\varepsilon(t), F(t)) < \varepsilon$ *for every* $t \in T$.
(iii) *for every* $\varepsilon > 0$ *there exists a piecewise barely continuous map* $f_\varepsilon : T \to E$ *such that* $\rho - \operatorname{dist}(f_\varepsilon(t), F(t)) < \varepsilon$ *for every* $t \in T$.

Proof. (i)⇒(ii) Fix $\varepsilon > 0$. According to Definition 1 let us decompose T as $T = \bigcup_{n=1}^\infty T_n^\varepsilon$ in such a way that for each non-empty subset C of T_n^ε there exists an open subset V of T and a subset D of E with $\rho - \mathrm{diam}(D) < \varepsilon$, such that $V \cap C \neq \emptyset$ and

$$F(t) \cap D \neq \emptyset \quad \text{for every } t \in V \cap C.$$

Without any loss of generality we can assume that the sets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ are pairwise disjoint: now we will construct for every n a barely constant function $f_n^\varepsilon : T_n^\varepsilon \to E$ with

$$\rho - \mathrm{dist}(f_n^\varepsilon(t), F(t)) < \varepsilon \quad \text{for every } t \in T_n^\varepsilon.$$

Once the above has been proved, statement (ii) is satisfied if we define

$$f_\varepsilon(t) := f_n^\varepsilon(t) \text{ for } t \in T_n^\varepsilon, n \in \mathbb{N}.$$

Let us fix $n \in \mathbb{N}$ and let us construct f_n^ε. Since $F|_{T_n^\varepsilon}$ is ε-fragmented, an argument of transfinite induction provide us with well ordered families of open subsets $\{G_\gamma : \gamma < \Gamma_\varepsilon^n\}$ of T_n^ε together with subsets $\{D_\gamma : \gamma < \Gamma_\varepsilon^n\}$ of E with $\rho-\mathrm{diam}(D_\gamma) < \varepsilon$, $\gamma < \Gamma_\varepsilon^n$, such that for each $\mu < \Gamma_\varepsilon^n$ we have

$$M_\mu := G_\mu \setminus \left(\bigcup\{G_\gamma : \gamma < \mu\}\right) \neq \emptyset \text{ and } F(t) \cap D_\mu \neq \emptyset \text{ for every } t \in M_\mu$$

and

$$T_n^\varepsilon = \bigcup\{G_\gamma : \gamma < \Gamma_\varepsilon^n\}.$$

For each $\gamma < \Gamma_\varepsilon^n$ we pick a point y_γ in D_γ and define $f_n^\varepsilon(t) := y_\gamma$ whenever $t \in M_\gamma$. The map $f_n^\varepsilon : T_n^\varepsilon \to E$ is barely constant. Indeed, if for a non-empty subset A of T_n^ε we define γ_0 to be the first ordinal with $A \cap G_{\gamma_0} \neq \emptyset$ then we have that f_n^ε is constant on $A \cap G_{\gamma_0}$ because

$$A \cap G_{\gamma_0} \subset M_{\gamma_0} = G_{\gamma_0} \setminus \bigcup\{G_\beta : \beta < \gamma_0\}$$

and $f_n^\varepsilon(t) = y_{\gamma_0}$ for every $t \in M_{\gamma_0}$. On the other hand, since $\gamma < \Gamma_\varepsilon^n$ and $t \in M_\gamma$ imply that $f_n^\varepsilon(t) = y_\gamma$ with

$$y_\gamma \in D_\gamma, \rho - \mathrm{diam}(D_\gamma) < \varepsilon \text{ and } F(t) \cap D_\gamma \neq \emptyset,$$

we conclude that

$$\rho - \mathrm{dist}(f_n^\varepsilon(t), F(t)) < \varepsilon \text{ for every } t \in T_n^\varepsilon,$$

and the proof is over.

Being the implication (ii) $\Rightarrow$(iii) obvious we prove (iii) $\Rightarrow$(i). Fix $\varepsilon > 0$ and take $f_\varepsilon : T \to E$ such that we can decompose $T = \bigcup\{T_n^\varepsilon : n \in \mathbb{N}\}$ and $f_\varepsilon|_{T_n^\varepsilon}$ is barely continuous for each $n \in \mathbb{N}$ with

$$\rho - \mathrm{dist}(f_\varepsilon(t), F(t)) < \frac{\varepsilon}{3} \text{ for every } t \in T.$$

If C is a non-empty closed subset of T_n^ε, then there exists an open set V of T such that $V \cap C \neq \emptyset$ and $\rho - \mathrm{diam}(f_\varepsilon(V \cap C)) < \frac{\varepsilon}{3}$. If we define

$$D := \{y \in E : \rho - \mathrm{dist}(y, f_\varepsilon(V \cap C)) < \frac{\varepsilon}{3}\}$$

then we have that $\rho - \mathrm{diam}(D) < \varepsilon$ and $F(t) \cap D \neq \emptyset$ for every $t \in V \cap C$. The proof is over. □

When we deal with a single valued map the above result is the key to prove that the barely constant maps together with countable splitting and limit points produce all σ-fragmented maps.

Corollary 2.2. *A map f from a topological space (T,τ) into a metric space (E,ρ) is σ-fragmented if, and only if, there exists a sequence $\{f_n : T \to E,\ n \in \mathbb{N}\}$, of piecewise barely constant maps that uniformly converges to f.*

Proof. By Theorem 2.1 the map $f : T \to E$ is σ-fragmented if, and only if, we have that for $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, there exists a piecewise barely constant map $f_n : T \to E$ with $\rho(f_n(t), f(t)) < \frac{1}{n}$ for all $t \in T$. □

Next we show that σ-fragmentability is in fact preserved when taking pointwise cluster points of sequences of σ-fragmented maps:

Proposition 2.3. *Let f be a map from a topological space (T,τ) into a metric space (E,ρ). If there exists a sequence $\{f_n : T \to E : n = 1,2,\ldots\}$ of σ-fragmented maps with*

$$f(t) \in \overline{\{f_n(t) :\ n = 1,2,\ldots\}}^{\rho} \text{ for every } t \in T,$$

then f is σ-fragmented.

Proof. Let us fix $\varepsilon > 0$ and let us define the sets

$$S_n^\varepsilon := \big\{t \in T :\ \rho(f(t), f_n(t)) < \frac{\varepsilon}{3}\big\} \quad n = 1,2,\ldots$$

Clearly $T = \bigcup_{n=1}^\infty S_n^\varepsilon$ and for every n we can also decompose $T = \bigcup_{m=1}^\infty T_m^{n,\varepsilon}$ in such a way that $f_n|_{T_m^{n,\varepsilon}}$ is $\frac{\varepsilon}{3}$-fragmented for every $m = 1,2,\ldots$. Observe that we have

$$S_n^\varepsilon = \bigcup_{m=1}^\infty S_n^\varepsilon \cap T_m^{n,\varepsilon} \text{ and } T = \bigcup_{n,m=1}^\infty S_n^\varepsilon \cap T_m^{n,\varepsilon}.$$

Now, for every pair of $n,m \in \mathbb{N}$ if we take a non-empty subset C of $S_n^\varepsilon \cap T_m^{n,\varepsilon}$, then there exists a non-empty open set V of T with $V \cap C \neq \emptyset$ and $\rho - \operatorname{diam}(f_n(V \cap C)) < \frac{\varepsilon}{3}$; the last inequality and the fact that $C \subset S_n^\varepsilon$ leads to

$$\rho - \operatorname{diam}(f(V \cap C)) < \varepsilon,$$

when bearing in mind the definition of S_n^ε. The proof is over. □

For maps with values in a normed space $(E, \|\cdot\|)$ the term *σ-fragmented* will always refer to σ-fragmentability with respect to the given norm $\|\cdot\|$ in E.

Corollary 2.4. *Let f be a map from a topological space (T,τ) into the normed space $(E,\|\cdot\|)$. If there exists a sequence $\{f_n : T \to E : n = 1,2,\ldots\}$ of σ-fragmented maps with*

$$f(t) \in \overline{\{f_n(t) :\ n = 1,2,\ldots\}}^{\mathrm{w}} \text{ for every } t \in T,$$

then f is σ-fragmented.

Proof. It is easily checked that linear combinations of σ-fragmented maps are σ-fragmented. Hence the $\mathbb{Q}$-linear combinations of $\{f_n : n = 1,2,\ldots\}$ is a countable family $\{g_n : T \to E : n = 1,2,\ldots\}$ of σ-fragmented maps for which Hahn-Banach theorem, [9, Theorem 3.19], gives us that

$$f(t) \in \overline{\{g_n(t) :\ n = 1,2,\ldots\}}^{\|\cdot\|} \text{ for every } t \in T.$$

We apply now Proposition 2.3 and the proof is finished. □

The following definition can be found in [20].

Definition 4 (Hansell, [20]). *A family of subsets $\mathcal{E}$ in a topological space T is said to be* scattered *if $\mathcal{E}$ is disjoint and there exists a well ordering $\leq$ of $\mathcal{E}$ such that for every $E \in \mathcal{E}$ the set $\cup\{M \in \mathcal{E} : M \leq E\}$ is open relative to $\cup\mathcal{E}$. A family of subsets $\mathcal{E}$ is said to be σ-scattered if it can be decomposed into a countable union $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ such that every family $\mathcal{E}_n$ is scattered.*

Given a map $f : T \to E$ between topological spaces, we say that a family $\mathcal{B}$ of subsets of T is a *function base* for f if, whenever V is open in E, then $f^{-1}(V)$ is union of sets of $\mathcal{B}$, *i.e.*, $\mathcal{B}$ is a function base for f if it is a *network* in T for the topology given by $\{f^{-1}(V) : V \text{ is open in } E\}$, see [7, p. 170] for the notion of network. We recall that a family $\{F_i : i \in I\}$ in a topological space (T, τ) is said to be discrete if for every point $x \in T$ there exists a neighbourhood U of x such that U meets at most one member of the family $\{F_i : i \in I\}$, [7, p. 33]. Recall also that a family of subsets $\{D_j : j \in J\}$ of T is a refinement of the family $\{C_l : l \in L\}$ if $\bigcup_{j\in J} D_j = \bigcup_{l\in L} C_l$ and each D_j is contained in some C_l, [7, p. 165].

We note that next result already appeared in [20, Theorem 1.10] for the very particular case of f being the identity map $id : (T, \tau) \to (T, \rho)$ where ρ is a metric on T whose associated topology is finer than τ. Next theorem exhibits the relationship between the notion of σ-fragmented map and the previous concept of map with σ-scattered function base introduced by Hansell, see [19] and the references therein.

Theorem 2.5. *Let f be a map from a topological space (T, τ) into the metric space (E, ρ). The following statements are equivalent:*

(i) *f is σ-fragmented;*
(ii) *if $\{D_i : i \in I\}$ is a discrete family of subsets in (E, ρ) then the family $\{f^{-1}(D_i) : i \in I\}$ has a σ-scattered refinement;*
(iii) *f has a σ-scattered function base.*

Proof. (i)⇒(ii) Let $\{D_i : i \in I\}$ be a discrete family in (E, ρ) and let us define

$$D_{i,p} := \{x \in D_i : B_\rho(x, 1/p) \cap D_j = \emptyset \text{ for each } j \in I,\ j \neq i\}$$

for every positive integer p. We clearly have $D_i = \bigcup_{p=1}^{\infty} D_{i,p}$ for every $i \in I$ and the family $\{D_{i,p} : i \in I\}$ is $1/p$-discrete *–meaning that the distance between two different elements of the family is at least* $1/p$, for every $p = 1, 2, \dots$ Fix the positive integer p and let us use that f is $1/p - \sigma$-fragmented to produce the decomposition $T = \bigcup_{n=1}^{\infty} T_n^{1/p}$ such that, for every n we have a well ordered family of relatively open subsets $\{G_\gamma^{n,p} : \gamma < \Gamma_n^{1/p}\}$ on $T_n^{1/p}$ which covers $T_n^{1/p}$ and provide us with the scattered family

$$\{M_\mu^{n,p} := G_\mu^{n,p} \setminus \bigcup \{G_\beta^{n,p} : \beta < \mu\} \text{ for } \mu < \Gamma_n^{1/p}\}$$

such that

$$\rho - \operatorname{diam}(f(M_\mu^{n,p})) < \frac{1}{p} \text{ for every } \mu < \Gamma_n^{1/p}.$$

The $1/p$-discreteness of the family $\{D_{i,p} : i \in I\}$ implies that the set $M_\mu^{n,p}$ meets at most one member of the family $\{f^{-1}(D_{i,p}) : i \in I\}$. Thus the family formed by all the non void sets of the form

$$\{M_\mu^{n,p} \cap f^{-1}(D_{i,p})\},$$

for $i \in I$ and $\mu < \Gamma_{n,p}^{1/p}$ is scattered for fixed integers n and p because any subset of ordinals is a well ordered set.

All things considered we conclude that

$$\bigcup_{n,p=1}^{\infty} \{M_\mu^{n,p} \cap f^{-1}(D_{i,p}) : i \in I,\, \mu < \Gamma_n^{1/p}\}$$

is a σ-scattered refinement of the family $\{f^{-1}(D_i) : i \in I\}$ and (ii) is fulfilled.

(ii) $\Rightarrow$(iii) Stone's theorem, [7, Theorem 4.4.3], provides us with a σ-discrete base $\mathcal{B}$ for the metric topology in (E, ρ), that is, $\mathcal{B}$ is base for the topology and it can be split as $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ with each $\mathcal{B}_n$ is discrete. According to (ii) each $f^{-1}(\mathcal{B}_n)$ has a σ-scattered refinement, that is, there exist scattered families $\mathcal{E}_m^n$ in T with $\bigcup_{m=1}^{\infty} \mathcal{E}_m^n$ being a refinement of $f^{-1}(\mathcal{B}_n)$. Observe that $\bigcup_{n,m=1}^{\infty} \mathcal{E}_m^n$ is a σ-scattered refinement of $f^{-1}(\mathcal{B})$. Furthermore, we claim that $\bigcup_{n,m=1}^{\infty} \mathcal{E}_m^n$ is a function base of f. Indeed, given an open set $V \subset E$ and $x \in f^{-1}(V)$ there exists B in some $\mathcal{B}_n$ such that $x \in f^{-1}(B) \subset f^{-1}(V)$; from the equality

$$\bigcup_{B \in \mathcal{B}_n} f^{-1}(B) = \bigcup_{m} \bigcup_{C \in \mathcal{E}_m^n} C$$

and the facts that $\{f^{-1}(B) : B \in \mathcal{B}_n\}$ are disjoint together with $\bigcup_{m=1}^{\infty} \mathcal{E}_m^n$ being a refinement of $f^{-1}(\mathcal{B}_n)$ we infer that there exists C in some $\mathcal{E}_m^n$ such that

$$x \in C \subset f^{-1}(B) \subset f^{-1}(V),$$

and the proof for this implication is over.

(iii) $\Rightarrow$(i) Let $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ be a function base for f with $\mathcal{E}_n$ scattered family for every $n \in \mathbb{N}$, *i.e.*, $\mathcal{E}_n = \{E_\alpha^n : \alpha < \Gamma_n\}$ and $E_\alpha^n \subset U_\alpha^n \setminus \bigcup\{U_\beta^n : \beta < \alpha\}$ for some well ordered family of open sets $\{U_\alpha^n : \alpha < \Gamma_n\}$ in T. If we fix $T_n := \bigcup\{E_\alpha^n : \alpha < \Gamma_n\}$ and for every $\alpha < \Gamma_n$ we choose $t_\alpha^n \in E_\alpha^n$, then we can define $f_n(t) := f(t_\alpha^n)$ for every $t \in T_n$ when $t \in E_\alpha^n$. The function f_n is barely constant on T_n: if $A \subset T_n$ is non-empty and α_0 is the first ordinal with $A \cap U_{\alpha_0}^n \neq \emptyset$ then we have that

$$A \cap U_{\alpha_0}^n \cap T_n \subset E_{\alpha_0}^n \cap T_n,$$

and therefore $f_n(t) = f_n(t_{\alpha_0}^n)$ for every $t \in A \cap U_{\alpha_0}^n \cap T_n$. Now we extend f_n to the whole T by defining it as a constant function on $T \setminus T_n$ with a fix but arbitrary value on this set. Since $\mathcal{E}$ is a function base for f, we easily obtain that

$$f(t) \in \overline{\{f_n(t) : n = 1, 2, \dots\}}$$

for every $t \in T$: indeed, given $\varepsilon > 0$ there exist a positive integer m and $\beta < \Gamma_m$, such that $t \in E_\beta^m \subset f^{-1}(B_\rho(f(t), \varepsilon))$, thus $f_m(t) = f(t_\beta^m) \in B_\rho(f(t), \varepsilon)$. Now Proposition 2.3 applies to conclude that f is σ-fragmented and the proof is over. □

Remark 2.6. If the scattered function base for a map f can be constructed with sets which are difference of closed sets, then the map f enjoys properties which are close to measurability; we note that under this hypothesis f is in fact Borel measurable when the domain space is, for instance, a complete metric space or a Gulk'o compact, see [21, 20, 19].

We recall that a set $S \subset T$ is a Souslin-$\mathcal{F}$-set in the space (T, τ) if S is the result of the Souslin operation applied to closed sets of T, *i.e.*, for some collection of closed sets $\left\{F_{n_1,n_2,\ldots,n_k} : (n_1, n_2, \ldots, n_k) \in \mathbb{N}^{(\mathbb{N})}\right\}$, indexed in the set $\mathbb{N}^{(\mathbb{N})}$ of finite sequences of positive integers, we have

$$S = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} \{F_{\alpha|k}\},$$

where $\alpha|k := (n_1, n_2, ..., n_k)$ when $\alpha = (n_1, n_2, \ldots, n_k, \ldots) \in \mathbb{N}^{\mathbb{N}}$. Every Borel set in a metric space is a Souslin-$\mathcal{F}$-set, [41, Theorem 44]. A map between metric spaces is called analytic if the preimage of every open set is a Souslin-$\mathcal{F}$-set. Bearing in mind the seminal results by R. Hansell in [18] we can now state here the following:

Corollary 2.7. *Every analytic (in particular every Borel measurable) map from a complete metric space (T, d) into a metric space (E, ρ) is σ-fragmented.*

Proof. Theorem 3 of R. Hansell in [18] states that every analytic map from a complete metric space into a metric space has a σ-discrete function base. Since every discrete family of sets is clearly scattered, the conclusion in the corollary straightforwardly follows from Theorem 2.5, see also [20, Lemma 5.9]. □

An important property of Borel maps from complete metric spaces into metric spaces is that they send separable subsets of the domain into separable subsets of the range, see for instance [43, Theorem 4.3.8]. The fact that σ-fragmented maps enjoy the same property is proved in the next result whose proof has been taken from [32, Theorem 2.15], where the result has been used as an important tool for renormings in Banach spaces: we include the proof here for the sake of completeness.

Theorem 2.8. *Let (T, d) and (E, ρ) be metric spaces and $f : T \to E$ a σ-fragmented map. Then, for every $t \in T$ there exists a countable set $W_t \subset T$ such that*

$$f(t) \in \overline{\bigcup \{f(W_{t_n}) : n = 1, 2, \ldots\}}^{\rho}$$

whenever $\{t_n\}$ is a sequence converging to t in (T, d). In particular, $f(S)$ is separable whenever S is a separable subset of T.

Proof. Let $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ be the σ-scattered function base for the map f provided by Theorem 2.5, *i.e.*, $\mathcal{E}_n = \{E^n_\alpha : \alpha < \Gamma_n\}$ with

$$E^n_\alpha \subset U^n_\alpha \setminus \bigcup \{U^n_\beta : \beta < \alpha\}$$

and $\{U^n_\alpha : \alpha < \Gamma_n\}$ a well ordered family of open sets in T. For every $m \in \mathbb{N}$ we set

$$E^{n,m}_\alpha := \{t \in E^n_\alpha : d - \operatorname{dist}(t, T \setminus U^n_\alpha) \geq \frac{1}{m}\}.$$

We observe that for $t \in E^{n,m}_\alpha$ we have that

$$B_d\left(t, \frac{1}{2m}\right) \cap E^{n,m}_\beta = \emptyset \text{ for all } \beta \neq \alpha,\ \beta < \Gamma_n. \tag{2.1}$$

Indeed, for such a t we have $B_d\left(t, \frac{1}{2m}\right) \subset U^n_\alpha$ and thus $B_d\left(t, \frac{1}{2m}\right) \cap E^{n,m}_\beta = \emptyset$ for $\beta > \alpha$. On the other hand, if for some $\beta < \alpha$ we assume that there exists some $s \in B_d\left(t, \frac{1}{2m}\right) \cap E_\beta^{n,m}$ then we have that $B_d\left(s, \frac{1}{2m}\right) \subset U_\beta^n$ and in particular $t \in U_\beta^n$ that contradicts the fact that $t \in E_\alpha^{n,m} \subset E_\alpha^n \subset U_\alpha^n \setminus \bigcup\{U_\beta^n : \beta < \alpha\}$. Thus (2.1) is proved.

Let us fix n and m in $\mathbb{N}$ and for every t in T and every $p \in \mathbb{N}$ such that $B_d(t, 1/p)$ meets at most one member of the family $\{E_\alpha^{n,m} : \alpha < \Gamma_n\}$, we choose one $t(n,m,p)$ in $B_d(t, 1/p) \cap \bigcup\{E_\alpha^{n,m} : \alpha < \Gamma_n\}$ when this intersection is non-empty. If we now define

$$W_t := \{t(n,m,p) : n, m, p \in \mathbb{N}\}$$

we claim that

$$f(t) \in \overline{\bigcup\{f(W_{t_k}) : k \in \mathbb{N}\}}^{\rho} \text{ whenever } \lim_{k\to\infty} d(t_k, t) = 0.$$

To prove that let us assume that $\lim_{k\to\infty} d(t_k, t) = 0$ and let us fix $\varepsilon > 0$. There exists an element of the function base $E_\alpha^n \subset \mathcal{E}_n$, $\alpha < \Gamma_n$, with $t \in E_\alpha^n$ and

$$E_\alpha^n \subset f^{-1}(B_\rho(f(t), \varepsilon)). \tag{2.2}$$

If m is a positive integer with $t \in E_\alpha^{n,m}$ we conclude that

$$B_d\left(t, \frac{1}{2m}\right) \cap E_\beta^{n,m} = \emptyset \text{ for all } \beta < \Gamma_n, \quad \beta \neq \alpha.$$

If k is a positive integer with $d(t_k, t) < \frac{1}{4m}$ then we have that

$$B_d\left(t_k, \frac{1}{4m}\right) \subset B\left(t, \frac{1}{2m}\right)$$

and that

$$t \in B_d\left(t_k, \frac{1}{4m}\right) \cap E_\alpha^{n,m}.$$

Consequently the ball $B_d\left(t_k, \frac{1}{4m}\right)$ only meets the set $E_\alpha^{n,m}$ of the family $\{E_\beta^{n,m} : \beta < \Gamma_n\}$. Thus $t_k(n,m,4m)$ is defined and we have that

$$t_k(n,m,4m) \in B_d\left(t_k, \frac{1}{4m}\right) \cap E_\alpha^{n,m} \subset E_\alpha^n,$$

the inclusion (2.2) allows us to conclude that $t_k(n,m,4m) \in f^{-1}(B_\rho(f(t), \varepsilon))$. Thus, we see that $t_k(n,m,4m) \in W_{t_k}$ and $\rho(f(t), f(t_k(n,m,4m))) < \varepsilon$ and the proof is over. $\square$

The σ-fragmentability property of maps is not only preserved by countable partitions. Indeed, if a map f is σ-fragmented when restricted to the sets of a scattered partition, then f is σ-fragmented. Being more precise we have the following result.

Proposition 2.9. *Let f be a map from a topological space (T, τ) into the metric space (E, ρ). If there exists a well ordered family of open sets $\{G_\gamma : \gamma < \Gamma\}$ covering T such that f is σ-fragmented restricted to every atom*

$$M_\gamma = G_\gamma \setminus \bigcup \{G_\beta : \beta < \gamma\},$$

then the map f is σ-fragmented on the whole of T.

Proof. Let us fix $\varepsilon > 0$ and let us split M_γ as

$$M_\gamma = \bigcup \left\{ M^n_{\gamma,\varepsilon} : n = 1, 2, \dots \right\},$$

in such a way that $f|_{M^n_{\gamma,\varepsilon}}$ is ε-fragmented. If we set $T^n_\varepsilon := \bigcup \left\{ M^n_{\gamma,\varepsilon} : \gamma < \Gamma \right\}$, then it is clear that $T = \bigcup \{ T^n_\varepsilon : n = 1, 2, ...\}$. We prove now that $f|_{T^n_\varepsilon}$ is ε-fragmented: if we fix some non-empty set $C \subset T^n_\varepsilon$ and we choose γ_0 to be the first ordinal such that $C \cap M_{\gamma_0}$ is non-empty, then

$$C \cap G_{\gamma_0} = C \cap (G_{\gamma_0} \setminus \bigcup \{ G_\beta : \beta < \gamma_0 \}) = C \cap M_{\gamma_0}.$$

On the other hand, since $C \subseteq T^n_\varepsilon$ we have that $\emptyset \neq C \cap M_{\gamma_0} = C \cap M^n_{\gamma_0,\varepsilon}$ and therefore the ε-fragmentabilility of $f|_{M^n_{\gamma_0,\varepsilon}}$ applies to provide us with an open set W in (T, τ) such that $W \cap C \cap M^n_{\gamma_0,\varepsilon}$ is non-empty and

$$\rho - \operatorname{diam}(f(W \cap C \cap M^n_{\gamma_0,\varepsilon})) < \varepsilon.$$

Since $W \cap C \cap G_{\gamma_0} = W \cap C \cap M^n_{\gamma_0,\varepsilon}$ we have that

$$\rho - \operatorname{diam}(f((W \cap G_{\gamma_0}) \cap C)) < \varepsilon,$$

and therefore $f|_{T^n_\varepsilon}$ is ε-fragmented. The proof is over. □

The previous results leads us to the following one: see [25, Theorem 4.1] for the identity map. We use the following terminology: a subset A of a metric space (E, ρ) is said to be ε-separable ($\varepsilon > 0$) if there exists some countable subset H in E such that

$$A \subset \bigcup \{ B_\rho(h, \varepsilon) : h \in H \}.$$

Proposition 2.10. *Let f be a map from a topological space (T, τ) into the metric space (E, ρ). If for every $\varepsilon > 0$ there exists a countable family of subsets $\{T^\varepsilon_n : n \in \mathbb{N}\}$ that covers T such that for every $n \in \mathbb{N}$ and every non-empty subset $C \subset T^\varepsilon_n$ there exists a τ-open subset V in T with $V \cap C$ non-empty and with $f(V \cap C)$ ε-separable, then the map f is σ-fragmented.*

Proof. For a fixed $\varepsilon > 0$ we shall construct a sequence

$$\{ f^\varepsilon_n : (T, \tau) \to (E, \rho) : n = 1, 2, ...\}$$

of σ-fragmented maps such that

$$\rho - \operatorname{dist}(f(t), \{ f^\varepsilon_n(t) : n = 1, 2, ...\}) < \varepsilon.$$

Therefore $f(t) \in \overline{\left\{ f^{1/p}_n(t) : n, p = 1, 2, ... \right\}}^\rho$ for every $t \in T$, and an appeal to Proposition 2.3 will ensure that f is σ-fragmented. For a fixed $\varepsilon > 0$ let us split $T = \bigcup_{n=1}^\infty T^\varepsilon_n$ satisfying the requirements in our hypothesis for the proposition. It is easily proved that in every T^ε_n we can produce a well ordered family of open sets $\left\{ G^n_\gamma : \gamma < \Gamma_{n,\varepsilon} \right\}$ such that

$$\bigcup \left\{ G^n_\gamma : \gamma < \Gamma_{n,\varepsilon} \right\} = T^\varepsilon_n$$

and for every atom

$$M^n_\gamma = G^n_\gamma \setminus \bigcup \left\{ G^n_\beta : \beta < \gamma \right\}$$

there exists a countable set H^n_γ in E such that

$$f(M^n_\gamma) \subset \bigcup \left\{ B_\rho(h, \varepsilon) : h \in H^n_\gamma \right\}.$$

If $H_\gamma^n = \{h_\gamma^n(j) : j = 1, 2, ...\}$ we define now the maps $f_{n,j}^\varepsilon : T_n^\varepsilon \to E$ given by

$$f_{n,j}^\varepsilon(t) := h_\gamma^n(j) \text{ if } t \in M_\gamma^n.$$

Since $f_{n,j}^\varepsilon$ is constant on the atoms of a well ordered family of open sets in T_n^ε, it is σ-fragmented on the whole piece T_n^ε after Proposition 2.9. We now define arbitrarily but constant the map $f_{n,j}^\varepsilon$ on $T \setminus T_n^\varepsilon$. The new $f_{n,j}^\varepsilon$ defined on the whole T is still σ-fragmented. On the other hand, it is clear that

$$\rho - \operatorname{dist}(f(t), \{f_{n,j}^\varepsilon(t) : j, n = 1, 2, \dots\}) < \varepsilon$$

for every $t \in T$, and the proof is over. □

We stress that the previous result for $f = id$ has been used in [24] where it is proved that $C_p(K)$ is σ-fragmented when K is a Rosenthal compacta of functions with countably many discontinuities at most.

3. σ-FRAGMENTED NORMED SPACES

Let $(E, \|\cdot\|)$ be a normed space and τ a topology on E coarser than the norm and H a subset of E. (H, τ) is said to be fragmented (resp. σ-fragmented) by the norm of E if the inclusion $i : (H, \tau) \to (E, \|\cdot\|)$ is fragmented (resp. σ-fragmented): when $\tau = \mathrm{w}$ we simply say that E is σ-fragmented, [25]. Recall that a subspace F of E^* is said to be *norming* if the function p of X given by

$$p(x) = \sup\{|x^*(x)| : x^* \in F \cap B_{E^*}\}$$

is a norm equivalent to $\|\cdot\|$; if this is the case then $\overline{F}^{\mathrm{w}^*} = E^*$.

Next result is the counterpart for σ-fragmentability of [31, Theorem 8] that has been proved for LUR renormings

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a normed space and $F \subset E^*$ a norming subspace. The following statements are equivalent:*

(i) *$(E, \sigma(E, F))$ is σ-fragmented by the norm;*
(ii) *the identity $id : E \to E$ is the uniform limit for the norm of a sequence of maps $\{I_n : E \to E, n = 1, 2, ...\}$ which are piecewise barely constant for the topology $\sigma(E, F)$;*
(iii) *there exists a sequence of maps $\{I_n : E \to E, n = 1, 2, ...\}$ which are piecewise barely constant for $\sigma(E, F)$ such that*
$$x \in \overline{\{I_n(x) : n = 1, 2, ...\}}^{\mathrm{w}} \text{ for every } x \in E;$$
(iv) *there exists a sequence of maps $\{I_n : E \to E, n = 1, 2, ...\}$ which are piecewise barely constant for $\sigma(E, F)$ such that*
$$x \in \overline{\operatorname{span}\{I_n(x) : n = 1, 2, ...\}}^{\|\cdot\|} \text{ for every } x \in E.$$

Proof. The equivalence between (i) and (ii) straightforwardly follows from Corollary 2.2. Clearly (ii) ⇒(iii). The rest of the proof uses ideas that already appeared in Corollary 2.4.

(iii) ⇒(iv) If $x \in \overline{\{I_n(x) : n = 1, 2, ...\}}^{\mathrm{w}}$, for every $x \in E$, then the Hahn Banach theorem, [9, Theorem 3.19], implies that

$$x \in \overline{\operatorname{span}\{I_n(x) : n = 1, 2, ...\}}^{\|\cdot\|} \text{ for every } x \in E$$

and thus condition (iv) is satisfied.

(iv) $\Rightarrow$(i) If condition (iv) is fulfilled then the set of all rational linear combinations of $\{I_n : E \to E, n = 1, 2, ...\}$ is a countable set of maps that we denote by $\{J_n : E \to E, n = 1, 2, ...\}$, which are σ-fragmented for the $\sigma(E, F)$-topology, and such that for every $x \in E$ we have that

$$x \in \overline{\{J_n(x) : n = 1, 2, ...\}}^{\|\cdot\|}.$$

Proposition 2.3 applies now to allow us to conclude that (i) is satisfied and the proof is over. □

4. σ-FRAGMENTED ε-SELECTORS FOR THE DUALITY MAPPING

This section is devoted to prove our main results in this paper. The notion below will be used repeatedly.

Definition 5. *Let F be a set-valued map from a set T into the subsets of a metric space (E, ρ) and $\varepsilon > 0$. An ε-selector for F is a function $f : T \to E$ such that*

$$\rho - \operatorname{dist}(f(t), F(t)) < \varepsilon,$$

for every $t \in T$.

Each selector for F is clearly an ε-selector, for every $\varepsilon > 0$, but a given ε-selector is not necessarily a selector. Note that ε-selectors have appeared in Theorem 2.1. Sometimes ε-selectors are the first step when finding a real selector, see for instance [26, 28, 45].

Godefroy's result quoted in Theorem 1.1 can be rephrased in the following way that suites well our purposes.

Lemma 1. *Let $(X, \|\cdot\|)$ be a Banach space, $J : X \to 2^{B_{X^*}}$ the duality mapping and let f be an ε-selector of J, $0 < \varepsilon < 1$. If $Z \subset X$ is a subspace such that $f(Z)$ is separable for the norm of X^*, then*

$$Z^* = \overline{\operatorname{span} f(Z)|_Z}^{\|\cdot\|_{Z^*}} \tag{4.1}$$

and consequently Z^* is norm separable.

Proof. We start by observing that if $Z \subset X$ is a subspace and we consider the restriction map $f_Z : Z \to Z^*$ given by $f_Z(z) := f(z)|_Z$, $z \in Z$, then

$$\|\cdot\|_{Z^*} - \operatorname{dist}(f_Z(z), J_Z(z)) \leq \|\cdot\|_{X^*} - \operatorname{dist}(f(z), J(z)) < \varepsilon,$$

for every $z \in Z$, where $J_Z : Z \to 2^{B_{Z^*}}$ denotes the duality mapping in Z given at $z \in Z$ by $J_Z(z) := \{z^* \in B_{Z^*} : \langle z, z^* \rangle = \|z\|\}$. Hence f_Z is an ε-selector for J_Z and $f_Z(Z) = f(Z)|_Z \subset Z^*$ is norm separable. Therefore to prove the lemma there is no loss of generality if we assume that $X = Z$, what we do for the sake of simplicity when writing. Take a countable subset D of $f(X)$ for which $f(X) \subset \overline{D}^{\|\cdot\|_{X^*}}$. Observe that for every $x \in X$ we can pick some $x^*_x \in J(x)$ such that $\|x^*_x - f(x)\| < \varepsilon$. Fix $\delta > 0$ such that $\varepsilon' := \varepsilon + \delta < 1$ and for each $x \in X$ take d_x in D with $\|f(x) - d_x\| < \delta$. The set $B = \{x^*_x : x \in X\}$ is a James boundary for B_{X^*} for which we have $B \subset \bigcup\{B(d, \varepsilon') : d \in D\}$. We can apply now Godefroy's result, Theorem 1.1, to obtain that

$$X^* = \overline{\operatorname{span}\{D\}}^{\|\cdot\|_{X^*}} = \overline{\operatorname{span}\{f(X)\}}^{\|\cdot\|_{X^*}},$$

and the proof is over. □

Our main result below is proved using a reduction argument to separable Banach spaces similar to the one used in [26, Theorem 26] –argument that goes back to [8]. Nonetheless, we note that our situation here is more complicated that the one in [26, Theorem 26], because we deal here with maps which are only σ-fragmented instead of the Baire one maps used in [26]: to overcome the extra difficulties in our case we will make use of the precise description of how separable sets are sent into separable ones via σ-fragmented maps, see Theorem 2.8 above.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \to 2^{B_{X^*}}$ denote the duality mapping. If for some fixed $0 < \varepsilon < 1$ there exists a σ-fragmented ε-selector $f : X \to X^*$ of J, then we have that*

$$\overline{\operatorname{span} f(X)}^{\|\cdot\|_{X^*}} = X^*.$$

Proof. Take a linear form $g \in X^*$ and let us prove that $g \in \overline{\operatorname{span} f(X)}^{\|\cdot\|_{X^*}}$. The idea is to construct a subspace $Z \subset X$ satisfying the assumption in Lemma 1, *i.e.*, $f(Z) \subset X^*$ norm separable, in such a way that from the condition

$$g|_Z \in \overline{\operatorname{span}\{f(Z)|_Z\}}^{\|\cdot\|_{Z^*}} (= Z^*)$$

it follows that

$$g \in \overline{\operatorname{span}\{f(X)\}}^{\|\cdot\|_{X^*}}.$$

Let us find Z. For every $x \in X$ we use Theorem 2.8 to pick a countable subset W_x in X such that

$$f(x) \in \overline{\{f(W_{x_n}) : n = 1, 2, \ldots\}}^{\|\cdot\|_{X^*}},$$

whenever $(x_n)_n$ converges to x in the Banach space $(X, \|\cdot\|)$.

Let us choose $\{0\} \neq Z_1 \subset X$ to be a countable $\mathbb{Q}$-linear subspace. Define $D_1 := \bigcup\{W_x : x \in Z_1\}$ which is also a countable set and let us write

$$C_1 := \mathbb{Q} - \operatorname{span}\{f(W_x) : x \in D_1\} := \{h_{1,j} : j \in \mathbb{N}\}.$$

We now find vectors $\{v_{1,j} : j \in \mathbb{N}\} \subset B_X$ such that

$$\langle g - h_{1,j}, v_{1,j}\rangle \geq \|g - h_{1,j}\| - 1,$$

and we finally collect the above v's as $F_1 := \{v_{1,j} : j \in \mathbb{N}\}$.

An induction process produces, for every $n \in \mathbb{N}$, countable sets $C_n \subset X^*$, $Z_n, D_n \subset X$ and

$$F_n := \{v_{n,j} : j = 1, 2, \ldots, \} \subset B_X,$$

such that

(i) each Z_n is $\mathbb{Q}$-linear subspace with $Z_n \bigcup F_n \subset Z_{n+1}$;
(ii) $D_n := \bigcup\{W_x : x \in Z_n\}$;
(iii) if we enumerate $C_n := \mathbb{Q} - \operatorname{span}\{f(W_x) : x \in D_n\} := \{h_{n,j} : j \in \mathbb{N}\}$, then we have the inequalities

$$\langle g - h_{n,j}, v_{n,j}\rangle \geq \|g - h_{n,j}\| - \frac{1}{n} \text{ for every } j \in \mathbb{N}. \tag{4.2}$$

Indeed, if Z_i, D_i and F_i have been constructed for $1 \leq i \leq n$, then we define the countable sets

$$Z_{n+1} := \mathbb{Q} - \operatorname{span}\{Z_n \cup F_n\},$$

$$D_{n+1} := \cup\{W_x : x \in Z_{n+1}\},$$

and once we have enumerated

$$C_{n+1} := \mathbb{Q} - \operatorname{span}\{f(W_x) : x \in D_{n+1}\} := \{h_{n+1,j} : j \in \mathbb{N}\}$$

we simply find vectors

$$F_{n+1} := \{v_{n+1,j} : j = 1, 2, \dots\}$$

satisfying the corresponding inequality (4.2).

Let us define the subspace $Z := \overline{\bigcup\{Z_n : n = 1, 2, \dots\}}^{\|\cdot\|}$. Our construction tell us that $f(\bigcup_{n=1}^{\infty} D_n) = \bigcup\{f(W_x) : x \in \bigcup_{n=1}^{\infty} Z_n\}$. Given $z \in Z$ there exists a sequence $(z_m)_m$ in $\bigcup\{Z_n : n = 1, 2, \dots\}$ such that $\lim_m z_m = z$. Hence, by the choices we have made of the sets W's we conclude that

$$f(z) \in \overline{\bigcup\{f(W_{z_m}) : m = 1, 2, \dots\}}^{\|\cdot\|_{X^*}} \subset \overline{f\big(\cup\{D_n : n = 1, 2, \dots\}\big)}^{\|\cdot\|_{X^*}}.$$

In other words $f(Z) \subset \overline{f\big(\cup\{D_n : n = 1, 2, \dots\}\big)}^{\|\cdot\|_{X^*}}$ and we can apply Lemma 1 to Z to conclude that

$$g|_Z \in Z^* = \overline{\operatorname{span}\{f(Z)|_Z\}}^{\|\cdot\|_{Z^*}} \subset \overline{\operatorname{span} f\big(\cup\{D_n : n = 1, 2, \dots\}\big)|_Z}^{\|\cdot\|_{Z^*}}.$$

To finish we prove that the latter implies that $g \in \overline{\operatorname{span}\{f(X)\}}^{\|\cdot\|_{X^*}}$. Fix $\delta > 0$ and pick $h \in \operatorname{span} f\big(\cup\{D_n : n = 1, 2, \dots\}\big)$ such that

$$\|g|_Z - h|_Z\|_{Z^*} < \frac{\delta}{2}.$$

On one hand, since $D_j \subset D_{j+1}$, $j \in \mathbb{N}$, we can write

$$h = \sum_{i=1}^{p} q_i f(d_i)|_Z,\ q_i \in \mathbb{Q},\ d_i \in D_n, i = 1, 2, \dots, p,$$

for some $n \in \mathbb{N}$. On the other hand, since $h = \sum_{i=1}^{p} q_i f(d_i) \in C_n$ we have $h = h_{n,j}$ for some $j \in \mathbb{N}$ and without loss of generality we can and do assume that n is big enough to have $\delta > \frac{2}{n}$. All things considered we conclude that:

$$\begin{aligned}
\|g - h\|_{X^*} = \|g - h_{n,j}\|_{X^*} &\overset{(4.2)}{\leq} \frac{1}{n} + \langle g - h_{n,j}, v_{n,j}\rangle \leq \\
\frac{\delta}{2} + \langle g - h_{n,j}, v_{n,j}\rangle &\overset{v_{n,j}\in Z}{\leq} \frac{\delta}{2} + \|(g - h_{n,j})|_Z\|_{Z^*} \\
&= \frac{\delta}{2} + \|g|_Z - h|_Z\|_{Z^*} \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

Taking $\delta \to 0$ we have that $g \in \overline{\operatorname{span} f(X)}^{\|\cdot\|^{X^*}}$ and the proof is over. □

Another consequence of Lemma 1 is the following improvement of a result by C. Stegall: see, for instance, [5, Theorem I.5.9 and Remark I.5.11], [45, Corollary 9] and [47]. Unless otherwise explicitly stated the notions Baire one, Borel measurability, fragmentability, etc. when used for selectors $f : X \to X^*$ always refer to the norm in X and X^*.

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \to 2^{B_{X^*}}$ be the duality mapping. The following statements are equivalent:*

(i) *X is Asplund;*
(ii) *J has a Baire one selector;*

(iii) *J has a σ-fragmented selector;*
(iv) *for some $0 < \varepsilon < 1$, J has a σ-fragmented ε-selector;*
(v) *for some $0 < \varepsilon < 1$, J has an ε-selector that sends norm separable subsets of X into norm separable subsets of X^*.*

Proof. For the implication (i) ⇒(ii) we use that if X is Asplund, then Theorem 8 in [27] provides a Baire one selector for J. For the implication (ii) ⇒(iii) refer for instance to Proposition 2.3. The implication (iii) ⇒(iv) is clear and (iv) ⇒(v) follows from Theorem 2.8.

To finish, if we assume that (v) holds, then Lemma 1 applies to ensure that each separable subspace of $Z \subset X$ has separable dual Z^*. Therefore (i) holds and the proof is over. □

We note that if we simply want to prove that (i) ⇒(vi) in the above theorem there exists no need to use the full power of the Jayne-Rogers theorem, [27, Theorem 8]. Indeed, if X is an Asplund space, then Proposition 11 in [26] can be used to obtain that the duality mapping $J : X \to 2^{B_{X^*}}$ is σ- fragmented as a set-valued map: to obtain now a σ-fragmented ε-selector for J (*i.e.*, a selector as the one in (vi) in the Theorem above) we can just apply our Theorem 2.1 in this paper.

Corollary 4.3. *Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \to 2^{B_{X^*}}$ be the duality mapping. The following statements are equivalent:*

(i) *X is Asplund;*
(ii) *J has a Borel measurable selector;*
(iii) *J has an analytic selector.*

Proof. It is a consequence of Theorem 4.2 and Corollary 2.7. □

The equivalences in the above Corollary can be obtained also from [3, Theorem A], where the techniques used comes from vector integration.

Corollary 4.4. *Let X be a separable Banach space. The following statements are equivalent:*

(i) *the norm open subsets of B_{X^*} are Souslin-$\mathcal{F}$-sets in (B_{X^*}, w^*);*
(ii) *X^* is separable.*

Proof. Let us prove (i) ⇒(ii). Since X is separable, (B_{X^*}, w^*) is metrizable and thus the duality map $J : X \to 2^{B_{X^*}}$ has a selector $s : X \to B_{X^*}$ that is a Baire one map for the norm in X and the metrizable topology w* in B_{X^*}, see [27]. Since the norm open sets in B_{X^*} are Souslin-$\mathcal{F}$-sets in (B_{X^*}, w^*) the identity map from (B_{X^*}, w^*) to $(B_{X^*}, \|\cdot\|_{X^*})$ is analytic. Hence the composition of the identity with s is analytic too, and therefore s is a selector for J that is analytic for the norm topologies and X and X^*. Corollary 4.3 applies to conclude that X^* is separable.

The implication (ii) ⇒(i) follows from the fact that in dual separable spaces X^* we have the equality of Borel sets in (X, w^*) and Borel sets in $(X^*, \|\cdot\|_{X^*})$ because there exists an equivalent dual norm on X^* such that the weak* and the norm topologies coincide on the unit sphere, see [20, 36]. □

Remark 4.5. Let $(X, \|\cdot\|)$ be a separable Banach space with non-separable dual X^*. Although the duality mapping $J : X \to 2^{B_{X^*}}$ does not have a σ-fragmented ε-selector for the norm in X^*, $0 < \varepsilon < 1$, the arguments in the proof of the previous result tell us that J admits a selector $s : X \to B_{X^*}$ that is a Baire one map for the norm in X and the metrizable topology w* in B_{X^*}. Furthermore, Godefroy proved that for $\delta > 0$ there exists a subset Δ of the unit sphere of X, homeomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$, and such that

$$\|s(x) - s(y)\| > 1 - \delta$$

for every $x \neq y$ in Δ, [16]. It is unknown if the Cantor set Δ can be constructed in such a way that

$$\inf\{\|x^* - y^*\| : x^* \in J(x), y^* \in J(y)\} > 1 - \delta,$$

for every $x \neq y$ in Δ. This question is strongly related with Problem I.3 in [5] where it is asked if X^* is separable when the set of all support functionals of B_X at all points of Gâteaux differentiability of the norm of X is norm separable, see [16]. □

We prove next that if X is a Banach space then the unit ball (B_{X^*}, w^*) is fragmented by the norm of X^* if, and only if, it is ε-fragmented by some fixed $0 < \varepsilon < 1$. To complete the picture we relate these results with Szlenk index that is recalled in the lines that follow. Given the Banach space X and the w^*-compact set $K \subset X^*$ we define for any $\varepsilon > 0$

$$K'_\varepsilon := \{g \in K : \|\cdot\| - \operatorname{diam}(V \cap D) \geq \varepsilon, \quad \text{for all } w^* - \text{ open neigh. } V \text{ of } g\}.$$

Inductively we define for an ordinal α we set

$$K_\varepsilon^{\alpha+1} := (K_\varepsilon^\alpha)'$$

and if β is a limit ordinal

$$K_\varepsilon^\beta := \bigcap_{\alpha<\beta} K_\varepsilon^\alpha.$$

The Szlenk index of a Banach space X is defined as follows. For any $\varepsilon > 0$, let

$$S_z(X, \varepsilon) = \min\{\alpha : (B_{X^*})_\varepsilon^\alpha = \emptyset\}$$

if such an ordinal exists and ∞ otherwise. Let now set

$$S_z(X) = \sup_{\varepsilon>0}\{S_z(X,\varepsilon)\}.$$

A Banach space is Asplund if, and only if, $S_z(X) \neq \infty$. For a survey on the Szlenk index and its applications see [17].

Theorem 4.6. *The following conditions are equivalent for a Banach space X:*

(i) *X is an Asplund space;*
(ii) *there exists $0 < \varepsilon < 1$ such that (B_{X^*}, w^*) is ε-fragmented,* i.e.*, for every non-empty subset $C \subset B_{X^*}$ there exists some w^*-open set V in B_{X^*} such that $C \cap V \neq \emptyset$ and $\|\cdot\| - \operatorname{diam}(C \cap V) < \varepsilon$.*
(iii) *there exists $0 < \varepsilon < 1$ such that the duality mapping J is ε-σ-fragmented;*
(iv) *there exists some $0 < \varepsilon < 1$ with $S_z(X, \varepsilon) \neq \infty$.*

Proof. The implication (i) $\Rightarrow$ (ii) is classical: see for instance [34, 44]. The implicación (ii) $\Rightarrow$ (iii) follows from [26, Proposition 11]. To prove that (iii) $\Rightarrow$(i) we read again the proof of (i) $\Rightarrow$(ii) in Theorem 2.1 and observe that for our given ε and J we can construct a σ-fragmented map $f_\varepsilon : (X, \|\cdot\|) \to (X^*, \|\cdot\|_{X^*})$ such that $\|\cdot\| - \operatorname{dist}(f_\varepsilon(x), J(x)) < \varepsilon$ for every $x \in X$: now Theorem 4.2 completes the proof of (iii) $\Rightarrow$(i). The implication (i) $\Rightarrow$(iv) follows from the fact that X is

Asplund if and only if $S_z(X) \neq \infty$. To finish we prove that (iv) $\Rightarrow$ (ii). Let C be a non-empty w^*-closed subsets of B_{X^*}. Let α_0 be the first ordinal with

$$C \cap \left(B_{X^*} \setminus (B_{X^*})_\varepsilon^{\alpha_0}\right) \neq \emptyset.$$

By definition of the derivation process we know that α_0 must be a successor ordinal that we can write as $\alpha_0 = \beta_0 + 1$. So we have that

$$C \subset (B_{X^*})_\varepsilon^{\beta_0} \quad \text{but} \quad C \cap \left(B_{X^*} \setminus ((B_{X^*})_\varepsilon^{\beta_0})'\right) \neq \emptyset.$$

Take now some x^* in the latter non empty set. Then there is a w^*-open set $V \subset X^*$ with $x^* \in V$ and such that

$$\|\cdot\| - \operatorname{diam}(V \cap (B_{X^*})_\varepsilon^{\beta_0}) < \varepsilon.$$

Thus $W := \left(B_{X^*} \setminus ((B_{X^*})_\varepsilon^{\beta_0})'\right) \cap V$ is a w^*-open set with $C \cap W \neq \emptyset$ because $x^* \in C \cap W$ and

$$\|\cdot\| - \operatorname{diam}(C \cap W) < \varepsilon,$$

because $C \cap W \subset (B_{X^*})_\varepsilon^{\beta_0} \cap V$. □

We note that results in the same spirit than ours previous Theorem have been proved in [14, 10].

When we replace B_{X^*} by an arbitrary w^*-compact subset of X^* we can still obtain some of the equivalences in Theorem 4.2 and its consequence Theorem 4.6.

Theorem 4.7. *Let X be Banach space, let K be a w^*-compact subset of X^* and $F_K : X \to 2^K$ the set-valued map defined by*

$$F_K(x) := \{k \in K : k(x) = \sup\{g(x) : g \in K\}\}.$$

The following conditions are equivalent:

(i) *F_K has a Baire one selector;*
(ii) *F_K has a selector that sends norm separable subsets of X into norm separable subsets for X^*;*
(iii) *(K, w^*) is fragmented by the norm.*

Proof. For the proof (iii) $\Rightarrow$(i) we refer to [26, 27, 28]. Being the implication (i) $\Rightarrow$(ii) clear we only give a proof for (ii) $\Rightarrow$(iii). Assume (ii) holds and fix a selector $f : X \to K$ sending norm separable subsets of X into norm separable subsets of X^*. According to a result by Namioka, see [34, Theorem 3.4], to prove that (K, w^*) is fragmented by the norm we only have to prove that for each countable set $A \subset B_X$ the restriction $K|_A$ of K to A is separable when equipped with the metric d_A of uniform convergence on A. If we set $Y := \overline{\operatorname{span} A}^{\|\cdot\|}$, then $f(Y) \subset K$ is norm separable in X^*. If $r : X^* \to Y^*$ is the restriction map, then $r(K)$ is a w^*-compact subset of Y^* and $r(f(Y))$ is a James boundary for $r(K)$. Since $r(f(Y))$ is separable for the norm of Y^* Rode's result, see [40], applies to obtain that

$$r(K) \subset \overline{\operatorname{co}(r(K))}^{w^*} = \overline{\operatorname{co}(r(f(Y)))}^{\|\cdot\|_{Y^*}}.$$

Since $\overline{\operatorname{co}(r(f(Y)))}^{\|\cdot\|_{Y^*}}$ is norm separable we obtain that $(r(K), \|\cdot\|_{Y^*})$ is separable. Since $A \subset B_Y$, we conclude that $(K|_A, d_A)$ is a continuous image of $(r(K), \|\cdot\|_{Y^*})$. Hence $(K|_A, d_A)$ is separable and the proof is over. □

Remark 4.8. The implication (i)$\Rightarrow$(iii) in the Theorem above has been proved in [26, Theorem 26]; this implication has been proved again in [28, Theorem 7.1] where the extra hypothesis $\ell^1 \not\subset X$ is required and used and its necessity justified by the means of an example, see [28, Example 7.1]. We stress that the proviso $\ell^1 \not\subset X$ is unnecessary for these kind of results as our improvement (ii) $\Rightarrow$(iii) clearly shows once again. We note that the latter does not lead to any kind contradiction with Example 7.1 in [28] because this example is simply not correct. The bottom line in Example 7.1 is that it is claimed, and proved, that the attaining map $J : \ell^1 \to 2^{B_{\ell^\infty}}$ has a selector $f : (\ell^1, \|\cdot\|_1) \to (\ell^\infty, \|\cdot\|_\infty)$ that is Baire one. No such an f can be Baire one. Indeed, observe that if f is any given selector for J then $f(\ell^1)$ is a James boundary for B_{ℓ^∞}. It is easily checked that every James boundary for B_{ℓ^∞} must contain the set of all signs

$$D := \{(\varepsilon_n)_n : \varepsilon_n \in \{-1, 1\}, \text{ for every } n \in \mathbb{N}\}.$$

Since D is uncountable and the $\|\cdot\|_\infty$-distance of two different elements of D is 2, we conclude that D is not $\|\cdot\|_\infty$-separable. Hence $f(\ell^1)$, that contains D, is never $\|\cdot\|_\infty$-separable. Since $(\ell^1, \|\cdot\|_1)$ is separable f cannot be Baire because otherwise $f(\ell^1)$ must be $\|\cdot\|_\infty$-separable but it is not. $\square$

5. Approximation of boundaries by descriptive sets

Le us remember now the combinatorial principle that lies in James compactness theorem as it was found by S. Simons [42], and described in the famous lemma:

Lemma 2 (Simons)**.** *Let $(z_n)_n$ be a uniformly bounded sequence in $\ell^\infty(C)$ and let W be its convex hull. If B is a subset of C such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b \in B$ such that*

$$\sup\{\sum_{n=1}^\infty \lambda_n z_n(y) : y \in C\} = \sum_{n=1}^\infty \lambda_n z_n(b), \tag{5.1}$$

then

$$\sup_{b\in B}\{\limsup_{n\to\infty} z_n(b)\} \geq \inf\{\sup_C w : w \in W\}. \tag{5.2}$$

A topological space T is said to be *angelic* if, whenever A is a relatively countably compact subset of T, its closure $\overline{A}$ is compact and each element of $\overline{A}$ is a limit of a sequence in A: good references for angelic spaces are [11] and [37]. $C_p(T)$ stands for the space of real continuous functions endowed with the topology of pointwise convergence on T.

Lemma 3. *Let X be a Banach space, B a James boundary for B_{X^*}, $\varepsilon \geq 0$ and $T \subset X^*$ such that $B \subset \bigcup_{t\in T} B(t, \varepsilon)$. Assume that T has the property that for each $y^* \in X^*$ the compact subsets of $C_p(T \cup \{y^*\}, w)$ are angelic. The following statements hold:*

(i) *if $1 > \varepsilon$, then $X^* = \overline{\operatorname{span} T}^{\|\cdot\|}$;*
(ii) *if $\varepsilon = 0$, then $X^* = \overline{\operatorname{span} T}^{\|\cdot\|}$ and $B_{X^*} = \overline{\operatorname{co}(B)}^{\|\cdot\|}$.*

Proof. Statement (ii) straightforwardly follows from Theorem I.2 in [15]. Statement (i) can be proved exactly with the same ideas of Lemma 4 in [16]. The proof is by contradiction. Fix $\varepsilon < \varepsilon' < 1$. If $\overline{\operatorname{span} T}^{\|\cdot\|} \subsetneq X^*$, then there exist $x^{**} \in X^{**}$

with $\|x^{**}\| = 1$ and $x^{**}|_T = 0$. Take $y^* \in B_{X^*}$ such that $x^{**}(y^*) > \frac{1+\varepsilon'}{2}$. Consider the restrictions

$$B_X|_{T\cup\{y^*\}} \subset B_{X^{**}}|_{T\cup\{y^*\}} \subset C_p(T \cup \{y^*\}, w).$$

Since B_X is w^*-dense in $B_{X^{**}}$, our hypothesis ensure us of the existence of a sequence $(x_k)_k$ in B_X such that

$$\lim_k x^*(x_k) = x^{**}(x^*) = 0, \tag{5.3}$$

for every $x^* \in T$, and

$$\lim_k y^*(x_k) = x^{**}(y^*) > \frac{1+\varepsilon'}{2}.$$

From the last inequality, we can assume without loss of generality that

$$y^*(x_k) > \frac{1+\varepsilon'}{2} \text{ for every } k \in \mathbb{N}. \tag{5.4}$$

It follows from (5.3) and the inclusion $B \subset \bigcup_{t\in T} B(t,\varepsilon)$ that for every $b^* \in B$ we have

$$\limsup_{k\to\infty} b^*(x_k) \le \varepsilon.$$

Simon's inequality, Lemma 2, applied to $C := B_{X^*}$, B and the sequence $(x_k)_k$ ensures the existence of $x \in \mathrm{co}(\{x_k : k \in \mathbb{N}\})$ with $\|x\| < \varepsilon'$ but on the other hand we have that

$$\varepsilon' > \|x\| \ge y^*(x) \overset{(5.4)}{>} \frac{1+\varepsilon'}{2}.$$

The inequality $\varepsilon' > \frac{1+\varepsilon'}{2}$ implies that $\varepsilon' > 1$ and we reach a contradiction that finishes the proof. □

Recall that a topological space (T,τ) is said to be *countably K-determined (resp. K-analytic)* if there exists a upper-semicontinuous set-valued map $F : M \to 2^T$ for some separable metric space (resp. Polish space) M such that $F(M) = T$ and $F(m)$ is compact for each $m \in M$. Notice that this class of spaces does properly contain the classes of K-analytic and (so) the σ-compact spaces. The paper [46] is a milestone when speaking about Banach spaces which are countably K-determined when endowed with their weak topologies. The main result in [37] states that if T is a countably K-determined space then $C_p(T)$ is angelic.

Next theorem is the outcome of the previous preparation.

Theorem 5.1. *Let X be a Banach space, B a James boundary for B_{X^*}, $1 > \varepsilon \ge 0$ and $T \subset X^*$ such that $B \subset \bigcup_{t\in T} B(t,\varepsilon)$. If (T,w) is countably K-determined (resp. K-analytic) then:*

(i) $X^* = \overline{\mathrm{span}\, T}^{\|\cdot\|}$ *and X^* is weakly countably K-determined (resp. weakly K-analytic).*
(ii) *Every James boundary for B_{X^*} has property (S). In particular $B_{X^*} = \overline{\mathrm{co}(B)}^{\|\cdot\|}$.*

Proof. The equality $X^* = \overline{\mathrm{span}\, T}^{\|\cdot\|}$ in (i) follows from Lemma 3 bearing in mind that $T \cup \{y^*\}$ is countably K-determined for every $y^* \in X^*$ and therefore the space $C_p(T \cup \{y^*\})$ is angelic, [37]. If (T,w) is countably K-determined (resp.

K-analytic) then $\overline{\operatorname{span} T}^{\|\cdot\|}$ is again weakly countably K-determined (resp. K-analytic) by a result of [46] and thus (i) is proved. Statement (ii) follows from statement (ii) in Lemma 3 applied for $T = X^*$. □

We stress that weakly countably K-determined spaces are weakly Lindelöf. Furthermore if X is a Banach space such that X^* is weakly Lindelöf, then X is Asplund, see [6, Proposition 1.8]. In particular, *a fortiori* the Banach spaces we deal with in Theorem 5.1 are Asplund spaces. We might think that for an Asplund space X and any boundary B of B_{X^*} we must have that $B_{X^*} = \overline{\operatorname{co}(B)}^{\|\cdot\|}$. This is not true in general as the following example taken from [15] shows: let ω_1 be the first uncountable ordinal and let $X = C([0, \omega_1])$ be the space of continuous functions on $[0, \omega_1]$ equipped with the supremum norm. X is an Asplund space and if δ_α denotes the Dirac measure at α then the set $B := \{\pm\delta_\alpha : 0 \leq \alpha < \omega_1\}$ is a boundary for B_{X^*} for which $\delta_{\omega_1} \in B_{X^*} \setminus \overline{\operatorname{co}(B)}^{\|\cdot\|}$. The best positive results in this setting that we include below are due to Haydon and Godefroy.

Theorem 5.2 (Haydon,[23])**.** *Let X be a Banach space. The following statements are equivalent:*

(i) $\ell^1 \not\subset X$*;*

(ii) *for every w^*-compact convex subset C of X^*, $C = \overline{\operatorname{co}(\operatorname{Ext} C)}^{\|\cdot\|}$;*

(iii) *for every w^*-compact subset K of X^*, $\overline{\operatorname{co}(K)}^{w^*} = \overline{\operatorname{co}(K)}^{\|\cdot\|}$.*

Theorem 5.3 (Godefroy,[15])**.** *Let X be a separable Banach space. The following statements are equivalent:*

(i) $\ell^1 \not\subset X$*;*

(ii) *for every w^*-compact subset K of X^* and every James boundary B of K we have $\overline{\operatorname{co}(K)}^{w^*} = \overline{\operatorname{co}(B)}^{\|\cdot\|}$;*

(iii) *for every w^*-compact convex subset C of X^* and every James boundary B of C we have $C = \overline{\operatorname{co}(B)}^{\|\cdot\|}$.*

The example given on $C([0, \omega_1])$ after Theorem 5.1 shows that neither Theorem 5.2 holds for boundaries different from the *extreme points* –(i)⇒(ii) fails– nor Theorem 5.3 holds for all boundaries when X is not separable –(i)⇒(ii) fails–. Nonetheless, it is possible to have the best of the above two theorems for general Banach spaces and arbitrary James boundaries if we replace $\|\cdot\|$ in X^* by the topology γ of uniform convergence on bounded and countable subsets of X.

Theorem 5.4. *Let X be a Banach space. The following statement are equivalent:*

(i) $\ell^1 \not\subset X$*;*

(ii) *for every w^*-compact subset K of X^* and any James boundary B of K we have $\overline{\operatorname{co}(K)}^{w^*} = \overline{\operatorname{co}(B)}^{\gamma}$;*

(iii) *for every w^*-compact subset K of X^*, $\overline{\operatorname{co}(K)}^{w^*} = \overline{\operatorname{co}(K)}^{\gamma}$.*

Proof. (i) ⇒(ii) We have to prove that for each $\varepsilon > 0$, $x^* \in \overline{\operatorname{co}(K)}^{w^*}$ and $D \subset B_X$ bounded and countable there exists $b^* \in \operatorname{co}(B)$ such that

$$|x^*(d) - b^*(d)| < \varepsilon, \text{ for every } d \in D. \tag{5.5}$$

Define $Y := \overline{\operatorname{span} D}$. Set $r : X^* \to Y^*$ the restriction map. Then $r(B)$ is a boundary for the w^*-compact set $r(K) \subset Y^*$. Since r is linear and w^*-w^*-continuous we obtain that

$$x^*|_Y = r(x^*) \in r\big(\overline{\operatorname{co}(K)}^{w^*}\big) \subset \overline{\operatorname{co}(r(K))}^{w^*}.$$

Since Y is separable and does not contain ℓ^1, Theorem 5.3 applies to conclude that $\overline{\operatorname{co}(r(K))}^{w^*} = \overline{\operatorname{co}(r(B))}^{\|\cdot\|_Y}$. Therefore

$$x^*|_Y = r(x^*) \in \overline{\operatorname{co}(r(B))}^{\|\cdot\|_Y}$$

that clearly implies (5.5).

The implication (ii) ⇒(iii) is obvious. Our proof by contradiction for (iii) ⇒(i) is almost the same, with a little extra remark, as the proof for (iii) ⇒(i) in Theorem 5.2 as presented in [23, Theorem 3.3]. Assume that there exists an isomorphism onto its image $T : \ell^1 \to X$. Then the adjoint map $S := T^* : X^* \to (\ell^1)^*$ is onto. If we set $(e_n)_n$ to denote the canonical basis in ℓ^1 and we identify $(\ell^1)^* = \ell^\infty$ then S is nothing else but the map

$$\begin{aligned} S : X^* &\longrightarrow \ell^\infty \\ x^* &\to (x^*(Te_n))_n. \end{aligned}$$

Notice that S is w^*-w^*-continuous and also γ-$\|\cdot\|_\infty$-continuous. Take a w^*-compact subset $C \subset \ell^\infty$ such that

$$\overline{\operatorname{co}(C)}^{\|\cdot\|_\infty} \subsetneq \overline{\operatorname{co}(C)}^{w^*}, \tag{5.6}$$

see the proof of [23, Proposition 3.2]. We take now K a w^*-compact subset of X^* such that $S(K) = C$. Then we have that $\overline{\operatorname{co}(K)}^{\gamma} \subsetneq \overline{\operatorname{co}(K)}^{w^*}$, because otherwise the equality would imply

$$\begin{aligned} \overline{\operatorname{co}(C)}^{w^*} &= \overline{\operatorname{co}(S(K))}^{w^*} = \overline{S(\operatorname{co}(K))}^{w^*} = S\big(\overline{\operatorname{co}(K)}^{w^*}\big) = \\ &= S\big(\overline{\operatorname{co}(K)}^{\gamma}\big) \subset \overline{S(\operatorname{co}(K))}^{\|\cdot\|_\infty} = \overline{\operatorname{co}(C)}^{\|\cdot\|_\infty}, \end{aligned}$$

that contradicts (5.6) and finishes the proof. □

We note that implication (i)⇒(ii) in the last is indeed a James compactness theorem for the w^*-topology.

Corollary 5.5. *Let X be a Banach space such that $\ell^1 \not\subset X$. If $K \subset X^*$ is bounded, γ-closed, convex and for every $x \in X$ there exists some $k^* \in K$ such that*

$$k^*(x) = \sup\{y^*(x) : y^* \in K\}$$

then K is a w^*-compact subset of X^*.

Proof. Since K is a James boundary for $\overline{K}^{w^*}$, Theorem 5.4 applies to yield the equalities below that implies that K is w^*-compact:

$$\overline{\operatorname{co}(\overline{K}^{w^*})}^{w^*} = \overline{\operatorname{co}(K)}^{\gamma} = K. \quad \square$$

A Banach space X or more generally a convex subset M of X is said to have *property* $\mathcal{C}$ (after Corson) if each collection of relatively closed convex subsets of M with empty intersection has a countable subcollection with empty intersection. If (M, w) is Lindelöf, then M has property $\mathcal{C}$ since closed convex sets in X are also weak-closed. A good reference for property $\mathcal{C}$ is [39].

The following Lemma easily follows from [1, Lemma 9] and also from the main result in [39].

Lemma 4. *Let X be a Banach space. If X^* has property $\mathcal{C}$, then γ is stronger than the weak topology of X^*.*

When X^* has property $\mathcal{C}$ the above results lead to the following consequence.

Corollary 5.6. *Let X be a Banach space such that X^* has property $\mathcal{C}$. Then for every w^*-compact subset K of X^* and any James boundary B of K we have*

$$\overline{\operatorname{co}(K)}^{w^*} = \overline{\operatorname{co}(B)}^{\|\cdot\|}.$$

In particular every boundary for B_{X^*} has property (S).

Proof. On the one hand if X^* has property $\mathcal{C}$ then $\ell^1 \not\subset X$. On the other hand if X^* has property $\mathcal{C}$, Lemma 4 implies that the dual of (X^*, γ) is X^{**} and therefore for any convex set $C \subset X^*$ we have that $\overline{C}^\gamma = \overline{C}^{\|\cdot\|}$. The Corollary now follows from Theorem 5.4. □

Theorem 5.7. *Let X be a Banach space. Then for every w^*-compact weakly Lindelöf subset K of X^* and any James boundary B of K we have*

$$\overline{\operatorname{co}(K)}^{w^*} = \overline{\operatorname{co}(B)}^{\|\cdot\|}.$$

Proof. Theorem 4.5 in [2] ensures that

$$\overline{\operatorname{co}(K)}^{w^*} = \overline{\operatorname{co}(K)}^{\|\cdot\|_{X^*}}.$$

Now, we apply [2, Corollary 6.4] to obtain that $\overline{\operatorname{span} K}^{\|\cdot\|_{X^*}}$ contains $\overline{\operatorname{co}(K)}^{w^*}$ and it is a weakly Lindelöf determined Banach space, *i.e.*, its dual unit ball is Corson compact and in particular angelic when endowed with the w^* topology. Therefore, every element in the bidual unit ball $B_{X^{**}}$ is the limit in the topology of pointwise convergence on $\overline{\operatorname{co}(K)}^{\sigma(X^*,X)}$ of a sequence in B_X. The conclusion follows now from Theorem I.2 in [15]. □

References

[1] B. Cascales and I. Namioka, *The Lindelöf property and σ-fragmentability*, Fund. Math. **180** (2003), no. 2, 161–183. MR MR2064322 (2005b:54036)

[2] B. Cascales, I. Namioka, and J. Orihuela, *The Lindelöf property in Banach spaces*, Studia Math. **154** (2003), no. 2, 165–192. MR MR1949928 (2003m:54028)

[3] B. Cascales and A. J. Pallarés, *La propiedad de Radon-Nikodym en espacios de Banach duales*, Collect. Math. **45** (1994), no. 3, 263–270. MR 96i:46019

[4] M. Contreras and R. Payá, *On upper semicontinuity of duality mappings*, Proc. Amer. Math. Soc. **121** (1994), no. 2, 451–459. MR MR1215199 (94h:46017)

[5] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow, 1993. MR 94d:46012

[6] G. A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), no. 4, 663–677. MR 58 #7081

[7] R. Engelking, *General topology*, PWN—Polish Scientific Publishers, Warsaw, 1977, Translated from the Polish by the author, Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60]. MR 58 #18316b

[8] M. Fabian and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. **91** (1988), no. 2, 141–151. MR MR985081 (90b:46032)

[9] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 8, Springer-Verlag, New York, 2001. MR 2002f:46001

[10] M. Fabian, V. Montesinos, and V. Zizler, *Quantitative versions of asplund sets and weakly compact sets*, preprint, 2006.

[11] K. Floret, *Weakly compact sets*, Lecture Notes in Mathematics, vol. 801, Springer, Berlin, 1980, Lectures held at S.U.N.Y., Buffalo, in Spring 1978. MR 82b:46001

[12] V. P. Fonf and J. Lindenstrauss, *Boundaries and generation of convex sets*, Israel J. Math. **136** (2003), 157–172. MR MR1998108 (2004g:46021)

[13] V. P. Fonf, J. Lindenstrauss, and R. R. Phelps, *Infinite dimensional convexity*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 599–670. MR 2003c:46014

[14] J. Gilles, 2006.

[15] G. Godefroy, *Boundaries of a convex set and interpolation sets*, Math. Ann. **277** (1987), no. 2, 173–184. MR 88f:46037

[16] ———, *Some applications of Simons' inequality*, Serdica Math. J. **26** (2000), no. 1, 59–78. MR 2002c:46026

[17] ———, *The Szlenk index and its applications*, General topology in Banach spaces, Nova Sci. Publ., Huntington, NY, 2001, pp. 71–79. MR MR1901535 (2003b:46016)

[18] R. W. Hansell, *Borel measurable mappings for nonseparable metric spaces*, Trans. Amer. Math. Soc. **161** (1971), 145–169. MR MR0288228 (44 #5426)

[19] ———, *Descriptive topology*, Recent progress in general topology (Prague, 1991), North-Holland, Amsterdam, 1992, pp. 275–315. MR MR1229129

[20] ———, *Descriptive sets and the topology of nonseparable Banach spaces*, Serdica Math. J. **27** (2001), no. 1, 1–66. MR 2002e:46021

[21] R. W. Hansell and L. Oncina, *Generalized first class selectors for upper semi-continuous set-valued maps in Banach spaces*, Czechoslovak Math. J. **55(130)** (2005), no. 1, 145–155. MR MR2121662 (2005m:47102)

[22] R. Haydon, *An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson*, Quart. J. Math. Oxford (2) **27** (1976), no. 107, 379–385. MR MR0493264 (58 #12293)

[23] ———, *Some more characterizations of Banach spaces containing l_1*, Math. Proc. Cambridge Philos. Soc. **80** (1976), no. 2, 269–276. MR 54 #11031

[24] R. Haydon, A. Moltó, and J. Orihuela, *Spaces of functions with countably many discontinuities*, To appear Israel J. Math.

[25] J. E. Jayne, I. Namioka, and C. A. Rogers, *Topological properties of Banach spaces*, Proc. London Math. Soc. (3) **66** (1993), no. 3, 651–672. MR 94c:46041

[26] J. E. Jayne, J. Orihuela, A. J. Pallarés, and G. Vera, *σ-fragmentability of multivalued maps and selection theorems*, J. Funct. Anal. **117** (1993), no. 2, 243–273. MR 94m:46023

[27] J. E. Jayne and C. A. Rogers, *Borel selectors for upper semicontinuous set-valued maps*, Acta Math. **155** (1985), no. 1-2, 41–79. MR 87a:28011

[28] ———, *Selectors*, Princeton University Press, Princeton, NJ, 2002. MR MR1915965 (2003j:54018)

[29] J. L. Kelley, *General topology*, Springer-Verlag, New York, 1975, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27. MR 51 #6681

[30] E. Michael and I. Namioka, *Barely continuous functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), no. 10, 889–892. MR 55 #4094

[31] A. Moltó, J. Orihuela, S. Troyanski, and M. Valdivia, *Continuity properties up to a countable partition*, RACSAM **100** (2006), no. 1-2, 279–293.

[32] ———, *A non linear transfer technique for renorming*, To appear in Lectures Notes in Mathematics, 2006.

[33] M. Muñoz, *Índice de K-determinación de espacios topológicos y σ-fragmentabilidad de aplicaciones*, Ph.D. thesis, Universidad de Murcia, 2004.

[34] I. Namioka, *Radon-Nikodým compact spaces and fragmentability*, Mathematika **34** (1987), no. 2, 258–281. MR 89i:46021

[35] I. Namioka and R. Pol, *σ-fragmentability and analyticity*, Mathematika **43** (1996), no. 1, 172–181. MR 97d:54061

[36] L. Oncina, *The JNR property and the Borel structure of a Banach space*, Serdica Math. J. **26** (2000), no. 1, 13–32. MR 2001e:46024
[37] J. Orihuela, *Pointwise compactness in spaces of continuous functions*, J. London Math. Soc. (2) **36** (1987), no. 1, 143–152. MR 88f:46058
[38] A. Plichko, *On sequential property of banach spaces and duality*, preprint, 2003.
[39] R. Pol, *On a question of H. H. Corson and some related problems*, Fund. Math. **109** (1980), no. 2, 143–154. MR 82a:46022
[40] G. Rodé, *Superkonvexität und schwache Kompaktheit*, Arch. Math. (Basel) **36** (1981), no. 1, 62–72. MR 82j:46012
[41] C. A. Rogers, *Hausdorff measures*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1970 original, With a foreword by K. J. Falconer. MR MR1692618 (2000b:28009)
[42] S. Simons, *A convergence theorem with boundary*, Pacific J. Math. **40** (1972), 703–708. MR 47 #755
[43] S. M. Srivastava, *A course on Borel sets*, Graduate Texts in Mathematics, vol. 180, Springer-Verlag, New York, 1998. MR MR1619545 (99d:04002)
[44] C. Stegall, *The Radon-Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc. **206** (1975), 213–223. MR 51 #10581
[45] ———, *Functions of the first Baire class with values in Banach spaces*, Proc. Amer. Math. Soc. **111** (1991), no. 4, 981–991. MR 91k:26003
[46] M. Talagrand, *Espaces de Banach faiblement $\mathcal{K}$-analytiques*, Ann. of Math. (2) **110** (1979), no. 3, 407–438. MR 81a:46021
[47] M. Valdivia, *Boundaries of convex sets*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **87** (1993), no. 2-3, 177–183. MR MR1318858 (96a:46035)

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