COMPACTOID FILTERS AND USCO MAPS

B. CASCALES AND L. ONCINA

ABSTRACT. The aim of this paper is to report in a short and self-contained way on the properties of compactoid and countably compactoid filters. We apply them to some questions in both topology and analysis such as the generation an extension of usco maps, the study of some properties of K-analytic spaces and the study of bounds for the weight of compact sets in spaces obtained through inductive operations.

1. INTRODUCTION

All our topologies, hence all our topological spaces, are assumed to be Hausdorff. We use the concept of filter, filter base, ultrafilter, net and subnet as introduced in [10, pp. 76-77] and [19, p. 65]. A filter in a topological space is said to be *compactoid* if every finer ultrafilter converges — see definition 1 below and [24, 8] for historical references. Compactoid filters generalize both convergent filters and compact sets. Compactoid filters have been widely applied in optimization, generalized differentiation, existence of upper semi-continuous compact-valued maps, etc. —see, for instance, [24, 8, 9, 21, 5] and the references therein. Sometimes compactoid filters have been taken to the general setting of pre-topologies and pseudo-topologies, [8]. Some other times, in some literature, results about filters in topological spaces, and their applications, have been presented without being aware that they were actually known results about compactoid filters.

We start in section 2 by gathering equivalent notions to the one of compactoid filter, theorem 2.1. The procedure to produce compactoid filters is standard as recalled in example 1 where we highlight a construction that appears in integration of functions with values in Banach spaces. A countably based filter which is compactoid for a given topology is still compactoid for any topology agreeing with the given one on compact sets, proposition 2.3. This last fact is not true for non countably based compactoid filters, example 2. A widely applicable characterization through sequences for a countable based filter to be compactoid is given in theorem 2.5.

The rest of the paper, organized in three more sections, is a batch of applications. Theorem 3.1 provides a simultaneous generation and extension result for usco maps, also involving minimality, that gathers and extends previous results in [5, 9] and [21]; a natural application is corollary 3.2 that relates minimal usco maps and continuous selectors; the natural application then is the characterization of spaces in Stegall class $\mathcal{S}$ given in corollary 3.3, which is folklore but hard to find in written literature. This completes section 3.

Section 4 deals with K-analytic spaces. We bring together in corollary 4.2 the tools to point out several different things: (a) A K-analytic space is analytic if there is a metric on the space metrizing all the compact sets, corollary 4.3 —we should mention that the question whether a K-analytic space with metrizable compacta has to be analytic is known to be undecidable, see [13]; (b) the Banach space of continuous functions on a compact space is weakly K-analytic provided it is K-analytic for the topology of pointwise convergence on a boundary, corollary 4.4; (c) K-analycity of a space of continuous functions with the pointwise convergence topology is characterized via the K-analycity of the space

Date: 27th September 2002.
2000 *Mathematics Subject Classification.* Primary 54C60, 54C20; Secondary 54A20.
Key words and phrases. filters, compact spaces, usco map, K-analytic spaces.
Both authors are supported by the research grant PB 98-0381 D.G.E.S. Spain.

of continuous functions on the Hewitt real-compactification of the underlying space, corollary 4.5.

Section 5 closes the paper with the study of usco maps defined on the product of directed sets, corollary 5.1, and its application, corollary 5.3, that offers bounds for the weight of compact sets in spaces obtained through inductive operations.

Our notation and terminology are standard. We take the books by Engelking, Kelley and Köthe, [10, 19] and [20], as our references for topology, Banach spaces and topological vector spaces. Our topological spaces are usually referred to by letters E, X, Y, $Z\dots$; compact spaces are denoted by $K, L, \dots$ Given a topological space Z we denote by $C(Z)$ the space of real continuous functions defined on Z; $\tau_p(Z)$ is the topology in $C(Z)$ of pointwise convergence on Z. When $(Y, \|\cdot\|)$ is a Banach space, B_Y denotes its closed unit ball, S_Y is its unit sphere and Y^* its (topological) dual space. For locally convex spaces $E, Y, \dots$ the (topological) dual is denoted, as usual, by $E', Y', \dots$. For both Banach and locally convex spaces the weak topology is denoted by w.

2. Compactoid and countably compactoid filters

If $\mathcal{F}$ is a filter in the topological space Y, its *cluster set* is the closed (maybe empty) set $C(\mathcal{F}) := \bigcap_{F\in\mathcal{F}} \overline{F}$. If $(y_i)_{i\in D}$ is a net in Y and for each $i \in D$ we set

$$R_i := \{y_j : j \in D, j \geq i\},$$

then $\mathcal{R} = (R_i)_{i\in D}$ is a filter base —we refer to $\mathcal{R}$ as the filter base associated to $(y_i)_{i\in D}$. The set of *cluster* points of $(y_i)_{i\in D}$ is by definition $C((y_i)_{i\in D}) := C(\mathcal{R})$. A point is a cluster point of $(y_i)_{i\in D}$ if, and only if, it is the limit of some subnet of $(y_i)_{i\in D}$. The following definition gathers the other terms used throughout this paper.

Definition 1. *Let Y be a topological space, $\mathcal{F}$, $\mathcal{G}$ filters and $\mathcal{B}$ a filter base in Y.*

(i) *$\mathcal{F}$ is said to be compactoid in Y if every ultrafilter finer than $\mathcal{F}$ in Y converges to some point in Y;*
(ii) *$\mathcal{F}$ subconverges to a subset L in Y (denoted $\mathcal{F} \rightsquigarrow L$), if given any open subset V of Y with $L \subset V$ there exists $B \in \mathcal{F}$ such that $B \subset V$;*
(iii) *a net $(y_i)_{i\in D}$ is eventually in $\mathcal{F}$ (denoted by $(y_i)_{i\in D} \prec \mathcal{F}$) if given any $B \in \mathcal{F}$ there exists $i_0 \in D$ such that for every $i \geq i_0$ we have $y_i \in B$;*
(iv) *we say that the filter base $\mathcal{B}$ is compactoid, the net $(y_i)_{i\in D}$ is eventually in $\mathcal{B}$, etc. when the filter $\mathcal{F}$ generated by $\mathcal{B}$ is compactoid, $(y_i)_{i\in D}$ is eventually in $\mathcal{F}$, etc.;*
(v) *we say that $\mathcal{G}$ and $\mathcal{B}$ meet if $G \cap B \neq \emptyset$ whenever $G \in \mathcal{G}$ and $B \in \mathcal{B}$.*

The notion of compactoid filter as in (i) above appeared, amongst others, in [24, 8]. The notion of filter $\mathcal{F}$ that *subconverges* to L as in (ii) appeared in [24], [8] —with the terms $\mathcal{F}$ *semiconverges* to L– and [9] —with the terms $\mathcal{F}$ *is aimed at* L. It is easily seen that for the filter $\mathcal{F}$ the following properties hold:

$$C(\mathcal{F}) = \bigcap_{F\in\mathcal{F}} \overline{F} = \{y \in Y : y \text{ is cluster point of some net } (y_i)_{i\in D} \prec \mathcal{F}\} \qquad (1)$$

and

$$\text{if } L \subset Y \text{ is closed and } \mathcal{F} \rightsquigarrow L, \text{ then } C(\mathcal{F}) \subset L, \qquad (2)$$

when Y is regular. If L is compact non-empty, then (2) holds without assuming regularity on Y.

For compactoid filters the cluster sets are not empty. Much more is true. We collect first a number of properties spread out in the literature, [24, 8, 9], and show the equivalence between them.

Theorem 2.1. *Let Y be a topological space and $\mathcal{B}$ a filter base in Y. Consider the following statements:*

(i) *there is a non-empty compact subset L of Y such that $\mathcal{B} \rightsquigarrow L$ in Y;*

(ii) *$C(\mathcal{B})$ is non-empty compact and $\mathcal{B} \rightsquigarrow C(\mathcal{B})$ in Y;*
(iii) *for every open cover $\{O_s\}_{s\in S}$ of Y, there exists a finite subset S_0 of S and $B \in \mathcal{B}$ such that $B \subset \bigcup_{s\in S_0} O_s$;*
(iv) *for every filter $\mathcal{G}$ in Y that meets $\mathcal{B}$ we have $C(\mathcal{G}) \neq \emptyset$;*
(v) *every net $(y_i)_{i\in D} \prec \mathcal{B}$ has a cluster point in Y;*
(vi) *$\mathcal{B}$ is compactoid in Y.*

Then, (i) *and* (ii) *are equivalent and imply any of the conditions* (iii), (iv), (v), (vi) *which are equivalent between them. If moreover, Y is assumed to be regular, then all the conditions are equivalent.*

Proof. The equivalence between (i) and (ii) appears in [9, Proposition 2.1]. The implication (i)⇒(iii) is clear. Let us prove (iii)⇒(iv) by contradiction: assume that (iii) holds, that the filter $\mathcal{G}$ meets $\mathcal{B}$ and that $\bigcap_{G\in\mathcal{G}} \overline{G} = \emptyset$. Then $Y = \bigcup_{G\in\mathcal{G}} Y \setminus \overline{G}$. Since (iii) holds there are $G_1, \ldots, G_m \in \mathcal{G}$ and $B \in \mathcal{B}$ such that $B \subset \bigcup_{j=1}^m Y \setminus \overline{G_j}$. Thus

$$\emptyset \neq B \cap (\bigcap_{j=1}^m G_j) \subset (\bigcup_{j=1}^m Y \setminus \overline{G_j}) \cap (\bigcap_{j=1}^m G_j) = \emptyset,$$

which is a contradiction that finishes the proof for this implication.

We show now that (iv)⇒(iii) again by contradiction. So assume that (iii) does not hold and let $(O_i)_{i\in I}$ be a open cover of Y such that for any $F \subset I$ finite, and for any $B \in \mathcal{B}$, we have $B \cap (Y \setminus \bigcup_{i\in F} O_i) \neq \emptyset$. Call $A_F := Y \setminus \bigcup_{i\in F} O_i$ for each finite subset F of I. Thus $\mathcal{A} := \{A_F : F \subset I, F \text{ finite }\}$ is a filter base. Let $\mathcal{G}$ be the filter associated to $\mathcal{A}$. The filter $\mathcal{G}$ meets $\mathcal{B}$, and thus $\emptyset \neq \bigcap_{G\in\mathcal{G}} \overline{G} = \bigcap_{A\in\mathcal{A}} \overline{A}$. Now take $y \in \bigcap_{A\in\mathcal{A}} \overline{A}$. The point y must be in some O_i but at the same time $y \in A_{\{i\}} = Y \setminus O_i$, that is the contradiction we were looking for.

The implication (iv)⇒(v) is obvious: take a net $(y_i)_{i\in D} \prec \mathcal{B}$ and let $\mathcal{R} = (R_i)_{i\in D}$ be its associated filter base. The condition $(y_i)_{i\in D} \prec \mathcal{B}$ implies that $\mathcal{R}$ meets $\mathcal{B}$. Thus (v) follows from (iv).

We show that (v)⇒(vi). Let $\mathcal{U}$ be a ultrafilter finer than $\mathcal{B}$. Clearly $\mathcal{U}$ meets $\mathcal{B}$. Consider the directed set $\mathcal{U} \times \mathcal{B}$, where $(U, B) \succ (U', B')$ if, and only if, $U \subset U'$ and $B \subset B'$. Given $U \in \mathcal{U}$ and $B \in \mathcal{B}$, pick $y_{(U,B)} \in U \cap B$. The net $(y_{(U,B)})_{(U,B)\in\mathcal{U}\times\mathcal{B}}$ is eventually in $\mathcal{B}$ and therefore has a cluster point y in $C(\mathcal{B})$. But this net is also eventually in the ultrafilter $\mathcal{U}$ and therefore $y \in C(\mathcal{U})$. Consequently $\mathcal{U}$ converges to y.

The implication (vi)⇒(iv) is almost trivial too: take a filter $\mathcal{G}$ that meets $\mathcal{B}$ and consider the filter base $\mathcal{H} := \{G \cap B : G \in \mathcal{G}, B \in \mathcal{B}\}$ and let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{H}$. Then $\mathcal{U}$ is finer that both $\mathcal{B}$ and $\mathcal{G}$ and consequently (vi) implies $\emptyset \neq C(\mathcal{U}) \subset C(\mathcal{G})$. Thus (iv) holds.

Finally assume that Y is regular, and let us prove, for instance, (v)⇒(ii). We simply prove that $C(\mathcal{B})$ is compact —the reader will convince himself that $\mathcal{B} \rightsquigarrow C(\mathcal{B})$ too. To that end we will prove that any filter in $C(\mathcal{B})$ has a cluster point in $C(\mathcal{B})$. Let $\mathcal{A}$ be a filter in $C(\mathcal{B})$. Let us consider the filter base in Y

$$\theta(\mathcal{A}) = \{U \subset Y : U \text{ open}, A \subset U \text{ for some } A \in \mathcal{A}\}.$$

The filters $\theta(\mathcal{A})$ and $\mathcal{B}$ meet. Repeating the arguments in (v)⇒(vi) we produce a point y that belongs to $C(\theta(\mathcal{A})) \cap C(\mathcal{B})$. Now since Y is regular, for each $A \in \mathcal{A}$, we have $\overline{A} = \bigcap\{\overline{U} : U \text{ open}, U \supset A\}$, hence

$$y \in C(\theta(\mathcal{A})) = \bigcap\{\overline{U} : U \in \theta(\mathcal{A})\} = \bigcap\{\overline{A} : A \in \mathcal{A}\} = C(\mathcal{A})$$

and the proof is over.

□

Let us remark that the equivalence between (iii), (iv), (v) and (vi) could be done cyclically but we did prefer to establish (iii)⇔(iv) to clarify arguments that will shorten the proof of proposition 2.4.

Example 1. *The drill to produce compactoid filters is pretty standard. We pay attention in (iv) and (vi) below to an example that appears in vector integration.*

(i) Convergent nets. If $(y_i)_{i\in D}$ is a convergent net in Y, then its associated filter base $\mathcal{R} = (R_i)_{i\in D}$ is compactoid.

(ii) Filters containing a relatively compact set. A filter that contains a relatively compact set is compactoid.

(iii) Bounded filters in reflexive Banach spaces. Let Y be a reflexive Banach space and $\mathcal{F}$ a filter in Y that contains a norm bounded set. Then, $\mathcal{B}$ is compactoid in (Y, w): keep in mind that bounded sets in reflexive Banach spaces are w-relatively compact.

(iv) Limit sets of Riemann-Lebesgue integral sums. Suppose Y is a Banach space and (Ω, Σ, μ) is a complete probability space. For a given bounded function $f : \Omega \to Y$ (not necessarily measurable in any sense) we define a *Riemann-Lebesgue integral sum* as

$$S(f, \Pi, T) = \sum_{i=1}^{n} f(t_i)\mu(A_i),$$

where $\Pi = \{A_i\}_{i=1}^n$ is a partition of Ω by elements of Σ and $T = \{t_i\}_{i=1}^n$ is a collection of *sampling points*, i.e. $t_i \in A_i$ for $i = 1, 2, \ldots, n$. We endow $\{S(f, \Pi, T)\}_{(\Pi,T)}$ with a net structure by defining a partial order by the rule: $(\Pi_1, T_1) \succ (\Pi_2, T_2)$ if, and only if, Π_1 is finer than Π_2, meaning that every element of Π_1 is contained in some element of Π_2.

The set $I_\mu(f)$ of all cluster points of the net $\{S(f, \Pi, T)\}_{(\Pi,T)}$ in Y is called the *limit set* of the Riemann-Lebesgue integral sums of f, see [17]. $I_\mu(f)$, if not empty, plays the role of a generalized integral for f. It was proved in [17] that $I_\mu(f)$ is not empty, for every bounded function f, when μ is the Lebesgue probability in $\Omega = [0, 1]$ and Y is either reflexive or separable. This was extended to weakly compactly generated Banach spaces Y in [16].

For Y reflexive and (Ω, Σ, μ) atomless, $I_\mu(f)$ is a non-empty w-compact convex set that is the cluster set of a compactoid filter in (Y, w). Indeed, in this case, it has been proved in [6, Lemma 2.1] that for each (Π, T) the set

$$R(f, \Pi, T) = \{S(f, \Pi', T') : (\Pi', T') \succ (\Pi, T)\},$$

is convex, see [17, Lemma 2.2] for the original proof for the Lebesgue measure. Hahn-Banach's theorem applies now to obtain the equalities:

$$I_\mu(f) = \bigcap_{(\Pi,T)} \overline{R(f, \Pi, T)}^{\text{norm}} = \bigcap_{(\Pi,T)} \overline{R(f, \Pi, T)}^{w}.$$

Each $R(f, \Pi, T)$ is bounded. Hence, the filter base $(R(f, \Pi, T))_{(\Pi,T)}$ is compactoid in (Y, w) and $I_\mu(f)$ is its cluster set.

(v) Filters associated to usco maps. Recall that a multi-valued map $\psi : X \to 2^Y$ is usco if it is compact valued and upper semicontinuous, i.e. for every $x \in X$ the set $\psi(x)$ is compact non-empty and for every open set V in Y with $\psi(x) \subset V$ there is an open neighbourhood U of x in X such that $\psi(U) \subset V$. For each $x \in X$ fix $\mathcal{N}_x$ a neighbourhood base of x in X. Then ψ is upper semicontinuous if, and only if, $\psi(\mathcal{N}_x) \rightsquigarrow \psi(x)$ for every $x \in X$.

(vi) The integral of non-integrable bounded functions. For (Ω, Σ, μ) a complete atomless probability space and Y a reflexive Banach space write $\ell^\infty(\Omega, Y)$ to denote the Banach space of bounded functions from Ω to Y endowed with the supremum norm $\|\cdot\|_\infty$. Then the multi-valued map sending each $f \in \ell^\infty(\Omega, Y)$ to its limit set of Riemann-Lebesgue integral sums $I(f) := I_\mu(f)$ is a usco map with values in (Y, w). To see this, we will prove

that for each f we have $I(V_n) \rightsquigarrow I(f)$, where $V_n = \{g \in \ell^\infty(\Omega, Y) : \| g - f \|_\infty < 1/n\}$, $n \in \mathbb{N}$. This is proved using theorem 2.1 and showing that

$$I(V_1) \supset I(V_2) \supset \cdots \supset I(V_n) \supset \cdots$$

is a compactoid filter base in (Y, w) with $\bigcap_n \overline{I(V_n)}^w = I(f)$. Again, $(I(V_n))_n$ is compactoid because it contains a bounded set (in fact each $I(V_n)$ is bounded) and Y is reflexive. Indeed, for each $g \in V_n$ and (Π', T') we have $\| S(g, \Pi', T') - S(f, \Pi', T') \| \leq 1/n$. Consequently

$$R(g, \Pi, T) \subset R(f, \Pi, T) + (1/n)B_Y, \tag{3}$$

for every (Π, T). Thus $I(V_n) \subset (\| f \|_\infty + 1/n)B_Y$.

To convince the reader that $f \mapsto I(f)$ is w-usco we establish now that

$$\bigcap_n \overline{I(V_n)}^w \subset I(f). \tag{4}$$

Fix $\mathcal{N}_0$ a base of absolutely convex neighbourhoods of the origin in (Y, w). As we know that

$$I(f) = \bigcap_{(\Pi,T)} \overline{R(f, \Pi, T)}^w = \bigcap_{(\Pi,T)} \bigcap_{U \in \mathcal{N}_0} (R(f, \Pi, T) + U),$$

the inclusion (4) will be obtained when proving that for each (Π, T) and $U \in \mathcal{N}_0$ one has $\bigcap_n \overline{I(V_n)}^w \subset R(f, \Pi, T) + U$. Fix (Π, T) and $U \in \mathcal{N}_0$ and take $n \in \mathbb{N}$ such that $(1/n)B_Y \subset (1/4)U$. For each $g \in V_n$ the inclusion (3) implies

$$\begin{aligned} \overline{R(g, \Pi, T)}^w \subset R(g, \Pi, T) + (1/4)U &\subset R(f, \Pi, T) + (1/n)B_Y + (1/4)U \\ &\subset R(f, \Pi, T) + (1/2)U. \end{aligned}$$

Therefore $I(V_n) \subset R(f, \Pi, T) + (1/2)U$ and

$$\overline{I(V_n)}^w \subset I(V_n) + (1/2)U \subset R(f, \Pi, T) + U,$$

which implies $\bigcap_n \overline{I(V_n)}^w \subset R(f, \Pi, T) + U$ and our proof is over. □

Corollary 2.2. *Let $\mathcal{F}$ be a filter in the topological space Y and let $L \subset Y$ be a non-empty compact set. Consider the following statements:*

(i) *$\mathcal{F} \rightsquigarrow L$ in Y;*

(ii) *for every sequence $(y_n)_n \prec \mathcal{F}$ the set $\overline{\{y_n : n \in \mathbb{N}\}}$ is compact and the set of cluster points $C((y_n)_n)$ is contained in L.*

Then, (i) *always implies* (ii)*. If $\mathcal{F}$ has a countable base, then* (ii) *also implies* (i)*.*

Proof. Assume (i) holds and fix $(y_n)_n \prec \mathcal{F}$. Let $\mathcal{R} = (R_n)_n$ be the filter base associated to $(y_n)_n$. Since $\mathcal{R} \rightsquigarrow L$, theorem 2.1 applies to say that

$$C((y_n)_n) \text{ is non-empty compact and } \mathcal{R} \rightsquigarrow C((y_n)_n). \tag{5}$$

Property in (2) says that $C((y_n)_n) \subset L$. Take $\{O_s\}_{s \in S}$ an open cover of Y. Using (5) and condition (iii) in theorem 2.1 we can pick $n \in \mathbb{N}$ and a finite subset $S_0 \subset S$ such that $R_n = \{y_m : m \geq n\} \subset \bigcup_{s \in S_0} O_s$. This clearly implies that

$$\overline{\{y_n : n \in \mathbb{N}\}} = \{y_n : n \in \mathbb{N}\} \cup C((y_n)_n)$$

can be covered by finitely many O_s's because $C((y_n)_n)$ is compact.

Conversely, let $\mathcal{B}$ be a countable base for $\mathcal{F}$ and suppose that (ii) is satisfied. We can and do assume that $\mathcal{B}$ is written as a decreasing sequence $B_1 \supset B_2 \supset \cdots B_n \supset \cdots$ of non-empty sets. Given $V \subset Y$ open with $L \subset V$ we prove that for some $m \in \mathbb{N}$ we have $B_m \subset V$. If not, there is $y_n \in B_n \setminus V$ for each $n \in \mathbb{N}$. But $(y_n)_n \prec \mathcal{F}$ while $C((y_n)_n) \subset Y \setminus V$. Since $Y \setminus V$ is disjoint with L, this contradicts (ii) and the proof is over. □

Recall that a topological space Z is said to be a k-space when the following property holds: if a subset A of Z intersects each compact subset of Z in a closed set, then A is closed, see [19, page 230] and [10, Theorem 3.3.18 and page 204]. If (Y, τ) is a topological space then the family

$$\tau^k \text{of subsets of } Y \text{ with open intersections with all compact subspaces of } (Y, \tau), \tag{6}$$

is a topology on Y with the properties: (i) τ is coarser that τ^k; (ii) τ and τ^k have the same compact sets; (iii) (X, τ^k) is a k-space; (iv) $\tau = \tau^k$ if, and only if, (Y, τ) is a k-space.

Proposition 2.3. *Let τ and δ be two topologies on Y such that δ coincides with τ on all τ-compact sets. Let $\mathcal{B}$ be a countable filter base and let L be a non-empty τ-compact set of Y. If $\mathcal{B} \rightsquigarrow L$ in (Y, τ), then $\mathcal{B} \rightsquigarrow L$ in (Y, δ).*

Proof. Consider the k-space topology τ^k associated to τ by the rules in (6). The topologies τ^k and τ have the same compact subsets and τ^k is finer than both τ and δ. The proposition will follow if we establish that for the countable filter base $\mathcal{B}$ the condition $\mathcal{B} \rightsquigarrow L$ in (Y, τ) is equivalent to $\mathcal{B} \rightsquigarrow L$ in (Y, τ^k). Since τ and τ^k have the same compact sets, and τ^k is finer than τ, the previous equivalence is a straightforward consequence of corollary 2.2. □

Let us observe that the statement in proposition 2.3 does not remain true for non countable filter bases. To provide an example we need the following easy observation.

Remark 1. *Let Y be a set and let τ and δ be two comparable topologies on Y. The following statements are equivalent:*

(i) $\tau = \delta$*;*

(ii) τ *and* δ *have the same compactoid filters.*

Proof. We only have to take care of (ii)⇒(i) what is proved by contradiction. We prove that if τ is strictly coarser than δ then (ii) does not hold. Let $(y_i)_{i \in D}$ be a net converging to y in (Y, τ) that does not converge for δ. Take $U \subset Y$ an open δ-neighbourhood of y such that the set

$$J = \{j \in D : y_j \in Y \setminus U\} \tag{7}$$

is cofinal in $(D, \geq)$. Then the filter base $\mathcal{R} = \{R_j\}_{j \in J}$ associated to the net $(y_j)_{j \in J}$ clearly subconverges to $\{y\}$ in (Y, τ) but it is not compactoid in (Y, δ). Indeed, if $\mathcal{R}$ were compactoid for δ then

$$\emptyset \neq \bigcap_{j \in J} \overline{R_j}^{\delta} \subset \bigcap_{j \in J} \overline{R_j}^{\tau} = \{y\} \subset U,$$

which is a contradiction with the inclusion $\bigcap_{j \in J} \overline{R_j}^{\delta} \subset Y \setminus U$ that follows from the definition of J in (7). □

Example 2. *There is a set Y and two comparable topologies τ and δ on it with the same compact sets but with different families of compactoid filters.*

Proof. Take (Y, τ) any topological space that is not k-space (for instance the product of uncountable many copies $Y = \mathbb{R}^I$ of the real line, see [19, Problem J.(b) page 240]). Take $\delta = \tau^k$ the k-space topology associated to τ. Then τ has the same compact subsets than τ^k. Being τ strictly coarser than τ^k the remark 1 applies to tell us that some compactoid filter in (Y, τ) is not compactoid in (Y, δ). □

We have learnt two different things so far: *(a)* general compactoid filters and countably based compactoid filters can behave quite differently —proposition 2.3 and example 2; *(b)* sequences are enough to describe *subconvergence* to a compact set of countably based filters —corollary 2.2. We take our discussion about compactoid filters a step further: we pay attention now to properties determined by sequences.

Proposition 2.4. *Let Y be a topological space and $\mathcal{B}$ a filter base in Y. The following statements are equivalent:*

(i) *for every countable open cover $\{O_n\}_{n\in\mathbb{N}}$ of Y, there exists a finite subset N_0 of $\mathbb{N}$ and $B \in \mathcal{B}$ such that $B \subset \bigcup_{n\in N_0} O_n$;*
(ii) *for every countably based filter $\mathcal{G}$ in Y that meets $\mathcal{B}$, we have $C(\mathcal{G}) \neq \emptyset$.*

Each of the above equivalent conditions implies:

(iii) *every sequence $(y_n)_n \prec \mathcal{B}$ has a cluster point in Y.*

If moreover, $\mathcal{B}$ is countable then (i), (ii) *and* (iii) *are equivalent.*

Proof. For the equivalence (i)⇔(ii) just repeat the proof (iii)⇔(iv) in theorem 2.1 replacing $\mathcal{G}$ for its countable base and an arbitrary open cover by a countable one.

To have a proof for (ii)⇒(iii) it is enough to mimic (iv)⇒(v) in theorem 2.1. Assume now that $\mathcal{B}$ is countable and let us prove that (iii)⇒(ii). Take $\mathcal{G}$ a filter with a countable base $\mathcal{B}'$ that meets $\mathcal{B}$. We do assume that $\mathcal{B}$ and $\mathcal{B}'$ are respectively written as decreasing sequences $B_1 \supset B_2 \supset \cdots B_n \supset \cdots$ and $B'_1 \supset B'_2 \supset \cdots B'_n \supset \cdots$ of non-empty sets. Pick $y_n \in B_n \cap B'_n$ for each $n \in \mathbb{N}$. The sequence $(y_n)_n \prec \mathcal{B}$. Since (iii) holds $(y_n)_n$ has a cluster point y, but clearly $y \in \overline{B'_n}$ for every n. Hence $y \in \bigcap_{G\in\mathcal{G}} \overline{G}$ and (ii) is satisfied. □

Countable filter bases enjoying property (iii) in proposition 2.4 above are called in [5, Definition 1] relatively countably compact.

Definition 2. *Let Y be a topological space, $\mathcal{F}$ a filter in Y and $\mathcal{B}$ a filter base.*

(i) *$\mathcal{F}$ is said to be countably compactoid in Y if for every countable based filter $\mathcal{G}$ that meets $\mathcal{F}$, $C(\mathcal{G}) \neq \emptyset$, see* [8]*;*
(ii) *the* sequential cluster set *for $\mathcal{F}$ is defined as*

$$C_s(\mathcal{F}) := \{y \in Y : \text{ y is cluster point of some sequence } (y_n)_n \prec \mathcal{F}\};$$

see [5, 9]*;*
(iii) *we say that the filter base $\mathcal{B}$ is countably compactoid when the filter $\mathcal{F}$ generated by $\mathcal{B}$ is countably compactoid.*

We refer to [5, Main Lemma] for the proof of the lemma below.

Lemma 1. *Let Y be a topological space and $\mathcal{B}$ a countable filter base in Y. The following statements are equivalent:*

(i) *for every sequence $(y_n) \prec \mathcal{B}$ the set $\overline{\{y_n : n \in \mathbb{N}\}}$ is countably compact;*
(ii) *every sequence eventually in $\mathcal{B}$ has a cluster point in Y and $C_s(\mathcal{B})$ is countably compact.*

Kind of counterpart to theorem 2.1 but with the *countable* notions is the result that follows. The assumption we impose on Y is not very restrictive —see the applications in the subsequent sections— when compared, for instance, with the advantages of checking that a countable filter base is compactoid provided (iv) below holds.

Theorem 2.5. *Let Y be a topological space in which relatively countably compact sets are relatively compact. If $\mathcal{B}$ is a countable filter base in Y, then the following statements are equivalent:*

(i) *there is a non-empty compact subset L of Y such that $\mathcal{B} \rightsquigarrow L$ in Y;*
(ii) *for every countable open cover $\{O_n\}_{n\in\mathbb{N}}$ of Y, there exists a finite subset N of $\mathbb{N}$ and $B \in \mathcal{B}$ such that $B \subset \bigcup_{n\in N} O_n$;*
(iii) *$\mathcal{B}$ is countably compactoid in Y;*
(iv) *every sequence $(y_n)_n \prec \mathcal{B}$ has a cluster point in Y;*
(v) *$C_s(\mathcal{B})$ is non-empty countably compact and $\mathcal{B} \rightsquigarrow C_s(\mathcal{B})$ in Y;*
(vi) *$\mathcal{B}$ is compactoid in Y.*

Proof. (i)⇒(ii) is obvious. (ii)⇒(iii)⇒(iv) is proposition 2.4. Let us prove (iv)⇒(v). Take $(y_n)_n \prec \mathcal{B}$. If (iv) holds then the set $\{y_n : n \in \mathbb{N}\}$ is relatively countably compact. Indeed, given a sequence $(z_n)_n$ in $\{y_n : n \in \mathbb{N}\}$, then the set $\{m \in \mathbb{N} : y_m = z_n$ for some $n \in \mathbb{N}\}$ is either finite or infinite. In the first case $(z_n)_n$ has a subsequence that is constant, hence convergent. In the second case there are sequences of positive integers

$$n_1 < n_2 < \cdots < n_k < \ldots \quad m_1 < m_2 < \cdots < m_k < \ldots,$$

such that $z_{n_k} = y_{m_k}$, for every $k \in \mathbb{N}$. Thus $(z_{n_k})_k \prec \mathcal{B}$ and consequently $(z_n)_n$ has a cluster point in Y, which says that $\{y_n : n \in \mathbb{N}\}$ is relatively countably compact. The hypothesis on Y implies that $\overline{\{y_n : n \in \mathbb{N}\}}$ is compact and then lemma 1 applies to say that $C_s(\mathcal{B})$ is countably compact. The proof of $\mathcal{B} \rightsquigarrow C_s(\mathcal{B})$, that finishes the implication (iv)⇒(v), is similar to the last part of the proof in corollary 2.2. (v)⇒(i) is clear because $\mathcal{B} \rightsquigarrow \overline{C(\mathcal{B})}$ which is compact.

The implications (i)⇒(vi) and (vi)⇒(iii), that finish the proof, are, respectively, the implication (i)⇒(vi) and a particular case of (vi)⇒(iv) in theorem 2.1. ☐

The hypothesis on Y, in the theorem above, are satisfied in Dieudonné-complete spaces and also in angelic spaces. A topological space Y is angelic– Fremlin –if every relatively countably compact subset A of Y is relatively compact and its closure $\overline{A}$ is made up of the limits of sequences from A. Examples of angelic spaces include: spaces with coarser metrizable topologies, spaces $(C(K), \tau_p(K))$ for K compact, all Banach spaces with their weak topologies, etc., [12].

3. USCO MAPS

Recall that a usco map $\psi : X \to 2^Y$ is said to be minimal if $\psi = \Phi$ whenever the multi-valued map $\Phi : X \to 2^Y$ is usco and $\Phi(x) \subset \psi(x)$ for every $x \in X$. As an application of Zorn's lemma every usco map contains a minimal usco map.

Theorem 3.1. *Let X and Y be topological spaces, Y regular. For every $x \in X$ fix $\mathcal{N}_x$ a neighbourhood base for x in X. Let Z be a dense subset of X and $\varphi : Z \to 2^Y$ a multi-valued map satisfying*

$$\{\varphi(U \cap Z)\}_{U \in \mathcal{N}_x} \text{ is compactoid in } Y, \text{ for every } x \in X. \tag{8}$$

For each $x \in X$ define

$$\psi(x) = \bigcap \{\overline{\varphi(U \cap Z)} : U \in \mathcal{N}_x\}.$$

Then $\psi : X \to 2^Y$ is a usco multi-valued map and

$$\varphi(x) \subset \psi(x) \text{ for every } x \in Z. \tag{9}$$

The map ψ is "minimum" with respect to all usco maps from X to 2^Y which have property (9)*. Moreover,*

(i) *if φ is usco on Z, then $\varphi(x) = \psi(x)$ for every $x \in Z$;*
(ii) *if φ is minimal usco on Z, then ψ is minimal usco on X;*
(iii) *if φ is single-valued and continuous, then ψ is minimal usco and $\varphi(x) = \psi(x)$, for every $x \in Z$.*

In particular, when X is first countable and Y is such that relatively countably compact subsets are relatively compact, condition (8) *is satisfied if the following condition holds:*

$$\text{for every sequence } (x_n)_n \text{ in Z converging in } X, \text{ the set } \cup_n \varphi(x_n) \text{ is relatively compact in } Y. \tag{10}$$

Proof. For each $x \in X$, the set $\psi(x) = \bigcap\{\overline{\varphi(U \cap Z)} : U \in \mathcal{N}_x\}$ is compact non-empty in Y because $\{\varphi(U \cap Z)\}_{U\in\mathcal{N}_x}$ is compactoid. Let us show that ψ is upper semi-continuous. Take an open set V in Y with $\psi(x) \subset V$. Since the filter base $\{\varphi(U \cap Z)\}_{U\in\mathcal{N}_x}$ is compactoid and Y is regular, there exists $U \in \mathcal{N}_x$ such that $\overline{\varphi(U \cap Z)} \subset V$. Therefore, we have $\psi(U) \subset \overline{\varphi(U \cap Z)} \subset V$ and thus ψ is usco.

The inclusions in (9) are clearly satisfied. The fact that ψ is "minimum" with respect to all usco maps which have property (9) is pretty simple too: assume that $G : X \to 2^Y$ is usco, with $\varphi(x) \subset G(x)$ for $x \in Z$. Then

$$\psi(x) = \bigcap\{\overline{\varphi(U \cap Z)} : U \in \mathcal{N}_x\} \subset \bigcap\{\overline{G(U \cap Z)} : U \in \mathcal{N}_x\} \subset G(x),$$

for every $x \in X$, where the latter inclusion holds since G is usco.

Let us prove now (i), (ii) and (iii). If φ is usco on Z, then

$$\psi(x) = \bigcap\{\overline{\varphi(U \cap Z)} : U \in \mathcal{N}_x\} \subset \varphi(x), \text{ for every } x \in Z.$$

So $\varphi(x) = \psi(x)$ for every $x \in Z$ and (i) is established. Assume now that φ is minimal usco on Z and suppose that there is $G : X \to 2^Y$ usco with $G(x) \subset \psi(x)$ for all $x \in X$. For each $x \in Z$ we have $G(x) \subset \varphi(x) = \psi(x)$ after (i). Since φ is minimal usco on Z we obtain $G(x) = \psi(x) = \varphi(x)$ for all $x \in Z$. The minimality of ψ with respect to the usco maps satisfying (9) applies now to get $\psi(x) \subset G(x)$ for all $x \in X$. Therefore $\psi(x) = G(x)$ for all $x \in X$ and (ii) follows. It is clear that (iii) is a particular case of (ii).

Let us prove the last part of the theorem. Fix $\mathcal{N}_x$ a countable neighbourhood base for every $x \in X$. We prove that if condition (10) holds then condition (8) holds for these countable bases $\mathcal{N}_x$. Since relatively countably compact subsets in Y are relatively compact, it suffices to prove that $\{\varphi(U \cap Z)\}_{U\in\mathcal{N}_x}$ is countably compactoid for every $x \in X$ after theorem 2.5. Take $(y_n)_n \prec \{\varphi(U \cap Z)\}_{U\in\mathcal{N}_x}$. We can and do assume that $\mathcal{N}_x$ is written as a decreasing sequence $U_1 \supset U_2 \supset \cdots \supset U_m \supset \cdots$ There are positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$y_n \in \varphi(U_k \cap Z) \text{ for } n_k \le n < n_{k+1}, k = 1, 2, \ldots$$

We choose now

$$x_n \in U_k \cap Z \text{ with } y_n \in \varphi(x_n) \text{ for } n_k \le n < n_{k+1}, k = 1, 2, \ldots$$

Clearly $(x_n)_{n\ge n_1}$ lives in Z and converges in X, so $\{y_n : n \ge n_1\} \subset \bigcup_{n=n_1}^{\infty} \varphi(x_n)$ is relatively compact and therefore our proof is concluded. □

Remark 2. *In the conditions of theorem 3.1, if φ satisfies* (8) *then,*

$$\textit{for any set } A \subset Z \textit{ relatively compact in } X,\ \varphi(A) \textit{ is relatively compact in } Y. \qquad (11)$$

In particular, when X is first countable and Y has relatively countably compact subsets which are relatively compact, conditions (8), (11) *and* (10) *for φ are equivalent.*

Proof. Assume that (8) holds and let $A \subset Z$ be relatively compact in X. Take a net $(y_i)_{i\in D}$ in $\varphi(A)$. Choose $(x_i)_{i\in D}$ in A such that $y_i \in \varphi(x_i)$ for all $i \in D$. Since A is relatively compact in X, the net $(x_i)_{i\in D}$ has a subnet $(x_j)_{j\in J}$ converging to a point $x \in X$. Since $(y_j)_{j\in J} \prec \{\varphi(U \cap Z)\}_{U\in\mathcal{N}_x}\}$, theorem 2.1 says that the net $(y_j)_{j\in J}$ has cluster points. Hence, $(y_i)_{i\in D}$ has also cluster points and $\varphi(A)$ is relatively compact. Therefore we have proved (8)⇒(11). The implication (11)⇒(10) clearly holds without extra assumptions, and the last part of theorem 3.1 contains the proof for (10)⇒(8) when X is first countable and relatively countably compact subsets of Y are relatively compact. □

The first part of the remark above strengthens Theorem 2.1 in [26] where it was proved that a subcontinuous multi-valued map $\varphi : X \to 2^Y$ has the property that if $K \subset X$ is compact then $\varphi(K)$ is relatively compact in Y –φ is subcontinuous if, and only if, whenever $(x_i)_{i\in D}$ is a convergent net in X and $(y_i)_{i\in D}$ is a net in Y with $y_i \in \varphi(x_i)$

then $(y_i)_{i\in D}$ has a convergent subnet. For $X = Z$ and $\varphi : X \to 2^Y$ the condition (8) is equivalent to φ being subcontinuous, see [8, Propostion 5.4].

Recall that a single-valued map $f : X \to Y$ is called a selector for the multi-valued map $\psi : X \to 2^Y$ if $f(x) \in \psi(x)$ for every $x \in X$.

Corollary 3.2. *Let X and Y be topological spaces, Y regular. Then the following statements are equivalent:*

(i) *every usco map $\psi : X \to 2^Y$ has a selector $f : X \to Y$ such that the set of points of continuity of f is dense in X;*
(ii) *every usco map $\psi : X \to 2^Y$ has a selector $f : X \to Y$ such that $f|_Z : Z \to Y$ is continuous for some dense set $Z \subset X$;*
(iii) *every minimal usco map is single-valued on a dense set.*

Proof. The implication (i)⇒(ii) simply reminds us that if Z is a set of points of global continuity for f then the restriction $f|_Z : Z \to Y$ is continuous at each point of Z.

Let us see how (ii)⇒(iii). Let $F : X \to 2^Y$ be minimal usco. Let $f : X \to Y$ be a selector for F, and Z the dense subset given in (ii). Since f is a selector of a usco map, it follows that for every $x \in X$ the filter base $\{f(U \cap Z)\}_{U\in\mathcal{N}_x}$ is compactoid. So we can apply theorem 3.1 to $f|_Z : Z \to Y$ and obtain $\psi : X \to 2^Y$ minimal usco satisfying condition (9), that is,

$$\{f(x)\} \subset \psi(x), \text{ for every } x \in Z. \tag{12}$$

Since ψ is “minimum” with respect to all usco map satisfying (12) and f is a selector for F we obtain that $\psi(x) \subset F(x)$ for every $x \in X$. Hence $\psi(x) = F(x)$ for all $x \in X$, because of the minimality of F. On the other hand $\psi(x) = \{f(x)\}$ for $x \in Z$ after (iii) in theorem 3.1 and so F is single-valued on Z.

To finish we prove (iii)⇒(i). Let $\psi : X \to 2^Y$ be usco and let $F : X \to 2^Y$ be a minimal usco map such that $F(x) \subset \psi(x)$ for every $x \in X$. By hypothesis, F is single-valued on a dense subset Z of X. Now define $f : X \to Y$ as

$$f(x) := \begin{cases} \text{the only point in } F(x) & \text{if } \quad x \in Z; \\ \text{an arbitrary point in } F(x) & \text{if } \quad x \in X \setminus Z. \end{cases}$$

Then f is a selector for ψ that is continuous at each point of Z. □

Definition 3. *A completely regular topological space Y is said to be in Stegall class $\mathcal{S}$ if whenever X is a Baire space and $\psi : X \to 2^Y$ is a minimal usco map, there exists a residual subset Z of X such that $\psi(x)$ is a singleton for every $x \in Z$.*

For a good compendium about Stegall classes we refer the interested reader to [11]. We know that the result below, that directly follows from corollary 3.2, is known and easy. However it is difficult to find a published reference for it –see hand written lecture notes [22].

Corollary 3.3. *For a completely regular topological space Y the following statement are equivalent:*

(i) *Y belongs to $\mathcal{S}$;*
(ii) *for every Baire space X and every usco map $\psi : X \to 2^Y$ there is a selector $f : X \to Y$ which is continuous at each point of a dense $\mathcal{G}_\delta$ subset of X.*

4. A FEW REMARKS ABOUT K-ANALYTIC SPACES

We shall start this section by recalling some definitions from descriptive set theory. $\mathbb{N}^{\mathbb{N}}$ denotes the space of sequences of positive integers endowed with its product topology and $\mathbb{N}^{(\mathbb{N})}$ is the set of finite sequences of positive integers. Given $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, we write $\alpha|k = (\alpha_1, \dots \alpha_k) \in \mathbb{N}^{(\mathbb{N})}$. A subset A of Y is said to be analytic if there is a continuous onto map $g : \mathbb{N}^{\mathbb{N}} \to A$. The subset A of Y is called K-analytic (resp. K-countably determined) if there is an upper semi-continuous map ψ from $\mathbb{N}^{\mathbb{N}}$ (resp. a subset of $\mathbb{N}^{\mathbb{N}}$) to the family of compact subsets of Y such that $A = \bigcup_\alpha \psi(\alpha)$. A good reference for analytic and K-analytic spaces is [15]. A Banach space that is K-analytic for its weak topology will be referred as weakly K-analytic, see [28].

All the results below about K-analytic spaces remain true when the term K-analytic spaces is replaced by K-countably determined spaces and analytic spaces are then replaced by images of separable metrizable spaces. Nonetheless, we stick to the concept of K-analyticity throughout this section.

The following easy and useful consequence of proposition 2.3 is pointed out in [28, Theorem 2.1], with a different proof.

Corollary 4.1. *Let τ and δ be two topologies on Y such that δ coincides with τ on all τ-compact sets. If (Y, τ) is K-analytic, then (Y, δ) is K-analytic.*

Proof. Since (Y, τ) is K-analytic then there is a τ-usco map $\psi : \mathbb{N}^{\mathbb{N}} \to 2^Y$ such that $Y = \bigcup_\alpha \psi(\alpha)$. Given $\alpha \in \mathbb{N}^{\mathbb{N}}$ fix $\mathcal{N}_\alpha$ a countable base of neighbourhoods of α in $\mathbb{N}^{\mathbb{N}}$. The map ψ will be upper δ-semi-continuous if, and only if, $\psi(\mathcal{N}_\alpha) \rightsquigarrow \psi(\alpha)$ in (Y, δ) for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. Thus, (Y, δ) is K-analytic after proposition 2.3. □

Next corollary, for the particular case of (Y, δ) being metric, is the Lemma in [14] – read the comments therein– that is the key to easily prove the existence of Čech-analytic K-countably determined spaces which are not K-analytic.

Corollary 4.2. *Let A be a K-analytic space, B a subset of A, and let f be a map from B onto a regular topological space (Y, δ) in which relatively countably compact subsets are relatively compact. Assume f satisfies the following condition:*

if a sequence $(b_n)_n$ in B has a cluster point in A, then $(f(b_n))_n$ has a cluster point in (Y, δ). (13)

Then (Y, δ) is K-analytic.

Proof. Let us take $\Phi : \mathbb{N}^{\mathbb{N}} \to 2^A$ a usco map with $A = \bigcup_\alpha \Phi(\alpha)$. Consider the set $Z := \{\alpha \in \mathbb{N}^{\mathbb{N}} : \Phi(\alpha) \cap B \neq \emptyset\}$ and set $X := \overline{Z}$ in $\mathbb{N}^{\mathbb{N}}$. Define $\varphi : Z \to 2^Y$ by $\varphi(\alpha) := f(\Phi(\alpha) \cap B)$, $\alpha \in Z$. X is a first countable space, the relatively countably compact subsets of (Y, δ) are relatively compact, and φ satisfies (10) in theorem 3.1. Indeed, if $(\alpha_n)_n$ in Z converges to α in X then

$$\bigcup_n \varphi(\alpha_n) = \bigcup_n f(\Phi(\alpha_n) \cap B) = f(\bigcup_n \Phi(\alpha_n) \cap B),$$

because $\bigcup_n \Phi(\alpha_n) \cap B$ is relatively compact in A and f satisfies (13). Thus, theorem 3.1 applies to produce a δ-usco map

$$\psi : X \to 2^Y \text{ with } \varphi(\alpha) \subset \psi(\alpha) \text{ for every } \alpha \in Z. \tag{14}$$

Then, $Y = \bigcup_{\alpha \in X} \psi(\alpha)$ and since X is Polish, X is the continuous image of $\mathbb{N}^{\mathbb{N}}$ and consequently (Y, δ) is K-analytic. □

The corollary below gathers Choquet's Theorem saying that a metrizable K-analytic space is analytic, see [7] and Talagrand's improvement saying that a K-analytic space with a coarser metrizable topology is analytic, see [27].

Corollary 4.3. *Let (Y, τ) be a regular topological space. The following statements are equivalent:*

(i) *Y is analytic;*
(ii) *Y is K-analytic and there is a metric d on Y whose topology is coarser than τ;*
(iii) *Y is K-analytic and there is a metric d on Y metrizing all compact subsets of Y.*

Proof. The implication (i)⇒(ii) goes as follows: if Y is analytic there is a sequence $(f_n)_n$ of continuous functions on Y which separates the points (i.e. if $x \neq y$ in Y, then there is a $n \in \mathbb{N}$ such that $f_n(x) \neq f_n(y)$), [15, Theorem 5.5.1]. Thus the topology associated to the obvious metric

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(x) - f_n(y)|}{1 + |f_n(x) - f_n(y)|},\ x, y \in Y,$$

is coarser than τ and therefore (ii) holds. The implication (ii)⇒(iii) is clear. Let us prove (iii)⇒(i). If Y is K-analytic and the compact subsets of Y are d-metrizable then the k-space topology τ^k associated to τ by the rules in (6) is finer than the topology associated to d. Moreover, (Y, τ^k) is K-analytic after corollary 4.1. Let $\psi : \mathbb{N}^{\mathbb{N}} \to 2^{(Y,\tau^k)}$ be usco such that $Y = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \psi(\alpha)$. Given $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$ set

$$V_k^{\alpha} := \{\beta : \beta \text{ in } \mathbb{N}^{\mathbb{N}} \text{ such that } \beta|k = \alpha|k\}.$$

(Y, d) is separable, since it is Lindelöf. Let $\{z_n : n \in \mathbb{N}\}$ be a dense subset of (Y, d). Define

$$D_{n_1,n_2,\ldots,n_m} := B_d(z_{n_1}; 1) \cap \ldots \cap B_d(z_{n_m}; \frac{1}{m}),$$

where $B_d(z; r)$ stands for the d-closed ball in Y of centre z and radius $r > 0$. Define

$$T = \{(\alpha, \beta) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : \beta = (b_n)_n \text{ and } \psi(V_n^{\alpha}) \cap D_{b_1,\ldots,b_n} \neq \emptyset, n \in \mathbb{N}\}.$$

T is a closed, hence Polish, subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. Given $(\alpha, \beta) \in T$ observe that there is a unique point $x =: g(\alpha, \beta) \in Y$ such that

$$\psi(V_1^{\alpha}) \cap D_{b_1} \supset \cdots \supset \psi(V_n^{\alpha}) \cap D_{b_1,\ldots,b_n} \supset \cdots \rightsquigarrow \{x\}$$

in (Y, τ^k). This decreasing sequence is compactoid in (Y, τ^k) due to the fact that ψ is usco, and it has a unique cluster point because the d-diameter of the sets tends to zero. The map $g : T \to Y$ so defined is onto and τ^k-continuous. Therefore (Y, τ^k) is analytic. Being this so, the space (Y, τ) is also analytic and we are done. □

As said in the introduction the question whether a K-analytic space with metrizable compacta has to be analytic is known to be undecidable, see [13]. It is also shown in [25] that it is undecidable whether a closed linear subspace of an L^1 space with separable weakly compact subsets is itself separable. Observe: a) closed linear subspaces of L^1 are weakly K-analytic, [28]; b) separable weakly compact sets of weakly K-analytic spaces are weakly metrizable, [28]; c) separable Banach spaces are weakly analytic.

The assumption required for the function f in (13) appears quite naturally. It has been used in [14], to be precise in the Lemma and Proposition 2 of this paper that led to one of the main results there. Condition (13) appears also in different, non trivial, situations in the papers [3, Proposition 2] and [6, Lema 5.5].

Let $(Y, \|\cdot\|)$ be a Banach space. A subset S of the unit sphere S_{Y^*} is called a boundary if for any $y \in Y$, there is $y^* \in S$ such that $y^*(y) = \| y \|$. A simple example of boundary is provided by the set $Ext(B_{Y^*})$ of extreme points of B_{Y^*}. Nonetheless, there are boundaries disjoint from the set of extreme points. If K is compact then the set of Dirac measures $\{\pm\delta_k : k \in K\}$ is a boundary for $C(K)$ when endowed with its supremum norm $\|\cdot\|_{\infty}$. If $S \subset S_{C(K)^*}$, denote by $\tau_p(S)$ the topology defined on $C(K)$ by the pointwise convergence on S. The lemma below has been proved in [3, Lemma 1] and [6, Lemma 5.7].

Lemma 2. *Let K be a compact space and $S \subset S_{C(K)^*}$ a boundary for $C(K)$. If $(f_n)_n$ is a sequence in $C(K)$ and $x \in K$, then there exists $\mu \in S$ such that $f_n(x) = f_n(\mu)$, for every $n \in \mathbb{N}$.*

Now, we can improve a result by Talagrand, see [28, Theorem 3.4].

Corollary 4.4. *Let K be a compact space. The following statement are equivalent:*

(i) *For every boundary $S \subset S_{C(K)^*}$ the space $(C(K), \tau_p(S))$ is K-analytic;*
(ii) *There is a boundary $S \subset S_{C(K)^*}$ for which $(C(K), \tau_p(S))$ is K-analytic;*
(iii) *$(C(K), \tau_p(K))$ is K-analytic;*
(iv) *$C(K)$ is weakly K-analytic.*

Proof. The implications (i)⇒(ii) and (iv)⇒(i) are obvious. The implication (ii)⇒(iii) follows from corollary 4.2: take $A = B = (C(K), \tau_p(S))$, $Y = (C(K), \tau_p(K))$ and f the identity map; in the angelic space Y the relatively countably compact sets are relatively compact, [12, Theorem 3.7], and f satisfies condition (13) after lemma 2. The implication (iii)⇒(iv) appears in [28, Theorem 3.4] but we reproduce it for the sake of completeness and because we have all the ingredients here. Assume (iii) holds. Statement (iv) will hold if $(B_{C(K)}, w)$ is K-analytic. By assumption $(B_{C(K)}, \tau_p(K))$ is K-analytic (as closed subspace of a K-analytic space) and $\tau_p(K)$ and w have the same compact sets in $B_{C(K)}$, see Grothendieck's theorem [12, Theorem 4.2]. Corollary 4.1 applies to say that $(B_{C(K)}, w)$ is K-analytic then. □

In the same vein we have the following (maybe known) consequence of corollary 4.2. For a completely regular topological space X we write νX to denote its Hewitt realcompactification (repletion), see [10, Section 3.11].

Corollary 4.5. *Let X be a completely regular topological space such that $(C(X), \tau_p(X))$ is angelic. The space $(C(X), \tau_p(X))$ is K-analytic if, and only if, $(C(\nu X), \tau_p(\nu X))$ is K-analytic.*

Proof. The restriction map R from $(C(\nu X), \tau_p(\nu X))$ onto $(C(X), \tau_p(X))$ is a continuous bijection. So if $(C(\nu X), \tau_p(\nu X))$ is K-analytic then $(C(X), \tau_p(X))$ is K-analytic. Conversely, assume that $(C(X), \tau_p(X))$ is K-analytic. Let $^\nu$ be the map sending each $f \in C(X)$ to its unique extension $f^\nu \in C(\nu X)$. Given a sequence $(f_n)_n$ in $C(X)$ and $x^\nu \in \nu X$ there is $x \in X$ such that $f_n(x) = f_n^\nu(x^\nu)$, for every $n \in \mathbb{N}$, [12, Theorem 4.6(1)]. On the other hand $(C(\nu X), \tau_p(\nu X))$ is an angelic space because the restriction map R is a continuous bijection and $(C(X), \tau_p(X))$ is angelic: use the angelic lemma [12, Lemma 3.1]. The K-analyticity of $(C(\nu X), \tau_p(\nu X))$ follows now from corollary 4.2: take $A = B = (C(X), \tau_p(X))$, $Y = (C(\nu X), \tau_p(\nu X))$ and $f = {}^\nu$. □

We stress again that the results in this section, in particular corollary 4.5, remain true when K-analytic spaces are replaced by K-countably determined. Recall that for a K-countably determined space X, the space $(C(X), \tau_p(X))$ is also angelic, [23]. We refer the reader [1, Chapter IV. Section §9] for useful links between Hewitt real-compactification and K-countably determined function spaces.

5. USCO MAPS DEFINED ON PRODUCT OF DIRECTED SETS

In what follows $(J_n, \leq_n)_{n\in\mathbb{N}}$ is a sequence of directed sets. We consider the cartesian product $J := \prod_{n\in\mathbb{N}} J_n$ directed by $\leq$, where

$$\alpha = (a_n)_n \leq \beta = (b_n)_n \ \text{ if, and only if, } \ a_n \leq_n b_n \ \text{ for every } \ n \in \mathbb{N}.$$

Each J_n is also consider as a topological space with its discrete topology and then the product $J = \prod_{n\in\mathbb{N}} J_n$ is their topological product.

Corollary 5.1. *Let $(J_n, \leq_n)_{n\in\mathbb{N}}$ be a sequence of directed sets and $J := \prod_{n\in\mathbb{N}} J_n$. Let Y be a regular topological space in which relatively countably compact subsets are relatively compact and let $\varphi : J \to 2^Y$ be a multi-valued map with the properties:*

(i) *for every $\alpha \in J$ the set $\varphi(\alpha)$ is relatively compact;*
(ii) *$\varphi(\alpha) \subset \varphi(\beta)$ whenever $\alpha \leq \beta$ in J.*

Then, there exists a usco map $\psi : J \to 2^Y$ satisfying $\varphi(\alpha) \subset \psi(\alpha)$ for every α in J.

Proof. We apply theorem 3.1 for $Z = X = J$. Observe first that J is a metric space, hence first-countable. To apply theorem 3.1 we simply need to verify that condition (10) is satisfied. We prove that for any compact subset K of J the set $\varphi(K)$ is relatively compact in Y. Indeed, for each $n \in \mathbb{N}$ we consider the n-th projection $\pi_n : J \to J_n$. The set $\pi_n(K) \subset J_n$ is finite. Then there is $b_n \in J_n$ such that $j_n \leq_n b_n$, for every $j_n \in \pi_n(K)$. If we write $\beta := (b_n)_n$, we have $\alpha \leq \beta$ for every $\alpha \in K$. Consequently, condition (ii) applies to get $\varphi(K) \subset \varphi(\beta)$ and then condition (i) says that $\varphi(K)$ is relatively compact. As said before the rest of the proof can be now entrusted to theorem 3.1. ☐

Corollary 5.1 extends Corollary 1.1 in [2] for $J = \mathbb{N}^{\mathbb{N}}$. We finish the section, and the paper, with two more results: the first application that follows, corollary 5.2, offers an alternative proof of a classical result by Dieudonné, see [20, §27.1.(5)]; our final application, corollary 5.3, gives a simpler proof of one of the main results in [4] and [18].

Corollary 5.2 (Dieudonné)**.** *Every Fréchet-Montel locally convex space is separable.*

Proof. Let Y be a Fréchet-Montel space. Let $(V_n)_n$ be a base of absolutely convex neighbourhoods of the origin in Y. For every $\alpha = (a_n)_n$ in $\mathbb{N}^{\mathbb{N}}$ we define the bounded set $\varphi(\alpha) := \bigcap_n a_n V_n$. Since Y is Montel each $\varphi(\alpha)$ is relatively compact. The equality $Y = \bigcup_\alpha \varphi(\alpha)$ is clear. For $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$ we have $\varphi(\alpha) \subset \varphi(\beta)$. The hypothesis in corollary 5.1 are fulfilled and then there is a usco map $\psi : \mathbb{N}^{\mathbb{N}} \to 2^Y$ with $\varphi(\alpha) \subset \psi(\alpha)$, $\alpha \in \mathbb{N}^{\mathbb{N}}$. Thus Y is K-analytic, then Lindelöf and metrizable, consequently separable. ☐

We write $|A|$ to denote the cardinal of the set A. Recall that the *weight* $\mathrm{w}(Z)$ of a topological space Z is the minimal cardinality of a base for the topology of Z. Given a locally convex space E and $A \subset E$, $A^o \subset E'$ stands for the absolute polar of A with respect to the dual pair $\langle E, E' \rangle$. The absolutely convex hull of A is denoted by $\Gamma(A)$.

Corollary 5.3. *Let (E_n, τ_n) be a sequence of locally convex spaces. Let P be a precompact subset in the inductive limit $(E, \tau) = \lim\limits_{\to}(E_n, \tau_n)$ and let $\mathcal{B}_n$ be a base of absolutely convex neighbourhoods of the origin in (E_n, τ_n) for $n = 1, 2, \ldots$ Then*

$$\mathrm{w}(P) \leq \sup_n |\mathcal{B}_n|.$$

Proof. Assume first P is compact. Write $\mathfrak{m} = \sup_n |\mathcal{B}_n|$. We will prove that there is a set C with $|C| \leq \mathfrak{m}$ and a continuous injection $\Phi : P \to \mathbb{R}^C$. Suppose for the moment that Φ has been constructed. Then since $\mathrm{w}(\mathbb{R}^B) \leq \mathfrak{m}$, see [10, Theorem 2.3.13], and Φ is a homeomorphism from P onto $\Phi(P)$, we have $\mathrm{w}(P) \leq \mathfrak{m}$ and the proof is finished.

The construction of Φ uses corollary 5.1. Let us define the seminorm q on E' given by

$$q(x') := \sup\{|x'(p)| : p \in P|\}, \text{ for } x' \in E'.$$

We consider the quotient space $Y := E'/q^{-1}(0)$, endowed with its canonical norm $\mathbf{q}$ defined by $\mathbf{q}(x' + q^{-1}(0)) := q(x')$, $x' \in E'$. We write $\pi : E' \to E'/q^{-1}(0)$ for the canonical quotient map. For each $n \in \mathbb{N}$ take $J_n := \mathcal{B}_n$ directed by inclusion downwards and $J = \prod_{n \in \mathbb{N}} J_n$ directed by the product order. For $\alpha = (V_n)_n$ in J we set $V_\alpha := \Gamma(\bigcup_n V_n)$. The family $\{V_\alpha\}_{\alpha \in J}$ is a base of neighbourhoods of the origin in (E, τ), [20, §19.1]. Thus the family of absolute polars $\{V_\alpha^o\}_{\alpha \in J}$ is a fundamental family of equicontinuous sets in E'. In particular, $E' = \bigcup_{\alpha \in J} V_\alpha^o$ and each V_α^o is compact for the topology $\mathfrak{T}_c$ on E' of uniform convergence on the precompact subsets of (E, τ), see [20, §21.6.(3)]. Observe also that if $\alpha \leq \beta$ in J then $V_\alpha^o \subset V_\beta^o$. Now, we define the multi-valued map $\varphi : J \to 2^Y$ by $\varphi(\alpha) := \pi(V_\alpha^o)$, $\alpha \in J$. The map φ satisfies conditions (i) and (ii) in corollary 5.1 and then we can produce $\psi : J \to 2^Y$ usco with $\varphi(\alpha) \subset \psi(\alpha)$, $\alpha \in J$. In particular, $Y = \bigcup_{\alpha \in J} \psi(\alpha)$. On the other hand, we know that $\mathrm{w}(J) \leq \mathfrak{m}$, see [10, Theorem 2.3.13], what implies —together with the fact that ψ is onto and usco— that every open cover of Y has a sub-cover of cardinality at most $\mathfrak{m}$, see [4, Proposition 2.1]. This implies that the normed space Y contains a dense subset $\mathbf{C}$ of at most $\mathfrak{m}$ elements, see [10, Theorem

4.1.15] and also [4, Proposition 2.1]. Fix C a subset of E' with $|C| \leq \mathfrak{m}$ and such that $\mathbf{C} = \{x' + q^{-1}(0) : x' \in C\}$. It is routine to prove that the map $\Phi : P \to \mathbb{R}^C$ given by $\Phi(p) = (x'(p))_{x' \in C}$ is a continuous injection.

The general case P precompact is reduced to the case already proved, P compact, dealing with the completion of (E, τ). $\square$

References

[1] A.V. Arkhangel'skii, *Topological function spaces*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1992.

[2] B. Cascales, *On K-analytic locally convex spaces*, Arch. Math. (Basel) **49** (1987), 232–244.

[3] B. Cascales and G. Godefroy, *Angelicity and the boundary problem*, Mathematika **45** (1998), no. 1, 105–112.

[4] B. Cascales, J. Kąkol, and S. A. Saxon, *Weight of precompact subsets and tightness*, J. Math. Anal. Appl. **269** (2002), 500–518.

[5] B. Cascales and J. Orihuela, *A sequential property of set-valued maps*, J. Math. Anal. Appl. **156** (1991), no. 1, 86–100.

[6] B. Cascales and R. Shvidkoy, *Riemann-Lebesgue integral sums and Krein-Šmulian-type theorem*, Preprint, 2002.

[7] G. Choquet, *Ensembles K-analytiques et K-sousliniens. Cas général et cas métrique*, Ann. Inst. Fourier. Grenoble **9** (1959), 75–81.

[8] S. Dolecki, G. Greco, and A. Lechicki, *Compactoid and compact filters*, Pac. J. Math. **117 (1)** (1985), 69–98.

[9] L. Drewnowski and I. Labuda, *On minimal upper semicontinuous compact-valued maps*, Rocky Mountain J. Math. **20** (1990), no. 3, 737–752.

[10] R. Engelking, *General topology*, PWN- Polish Scientific Publishers., 1977.

[11] M. Fabian, *Gâteaux differentiability of convex functions and topology*, John Wiley & Sons Inc., New York, 1997.

[12] K. Floret, *Weakly compact sets*, vol. 801, LNM. Springer-Verlag, 1980.

[13] D. H. Fremlin, *K-analytic spaces with metrizable compacta*, Mathematika **24** (1977), no. 2, 257–261.

[14] D. H. Fremlin, P. Holick´y, and J. Pelant, *Lindelöf modifications and K-analytic spaces*, Mathematika **40** (1993), no. 1, 1–6.

[15] J. E. Jayne and C. A. Rogers, *Analytic sets*, ch. K-analytic sets, pp. 1–181, Academic Press, 1980.

[16] V. Kadets, B. Shumyatskiy, R. Shvidkoy, L. Tseytlin, and K. Zheltukhin, *Some remarks on vector-valued integration*, Mat. Fiz. Anal. Geom. **9** (2002), no. 1, 48–65.

[17] V. M. Kadets and L. M. Tseytlin, *On "integration" of non-integrable vector-valued functions*, Mat. Fiz. Anal. Geom. **7** (2000), no. 1, 49–65.

[18] J. Kąkol and S. Saxon, *Montel (DF)-spaces, sequential $(LM)-$-spaces and the strongest locally convex topology*, J. London Math. Soc. (to appear), 2000.

[19] J. L. Kelley, *General topology*, Springer-Verlag, New York, 1975.

[20] G. K¨othe, *Topological vector spaces I*, Springer-Verlag, 1969.

[21] A. Lechicki and S. Levi, *Extensions of semicontinuous multifunctions*, Forum Math. **2** (1990), no. 4, 341–360.

[22] I. Namioka, *On class $\mathcal{S}$ of stegall*, Rainwater Seminar talks, 1983.

[23] J. Orihuela, *Pointwise compactness in spaces of continuous functions*, J. London Math. Soc. (2) **36** (1987), 143–152.

[24] J. P. Penot, *Compact nets, filters, and relations*, J. Math. Anal. Appl. **93** (1983), no. 2, 400–417.

[25] H. P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math. **28** (1974), 83–111.

[26] R. E. Smithson, *Subcontinuity for multifunctions*, Pacific J. Math. **61** (1975), no. 1, 283–288.

[27] M. Talagrand, *Sur la structure borélienne des espaces analytiques*, Bull. Sci. Math. (2) **101** (1977), no. 4, 415–422.

[28] ______, *Espaces de Banach faiblement K-analytiques*, Ann. of Math. **110** (1979), 407–438.

Departamento de Matemáticas, Facultad de Matemáticas, Universidad de Murcia, 30.100 Espinardo, Murcia, Spain

E-mail address: beca@um.es, luis@um.es