

**WEIGHT OF PRECOMPACT SUBSETS AND TIGHTNESS
(J. MATH. ANAL. APPL.)**

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ABSTRACT. Pfister (1976) and Cascales and Orihuela (1986) proved that precompact sets in (DF) - and (LM) -spaces have countable weight; i.e., are metrizable. Improvements by Valdivia (1982), Cascales and Orihuela (1987) and Kałkol and Saxon (preprint) have varying methods of proof. For these and other improvements a refined method of upper semi-continuous compact-valued maps applied to uniform spaces will suffice. At the same time, this method allows us to dramatically improve Kaplansky's theorem, that the weak topology of metrizable spaces has countable tightness, extending it to include all (LM) -spaces and all quasiseparated (DF) -spaces, both in the weak and original topologies. One key is showing that for a large class \mathfrak{G} including all (DF) - and (LM) -spaces, countable tightness of the weak topology of E in \mathfrak{G} is equivalent to realcompactness of the weak* topology of the dual of E .

1. INTRODUCTION

Recall that the *weight* $w(X)$ of a topological space X is the minimal cardinality of a basis for the topology of X . For a set B we denote by $|B|$ its cardinal. The *tightness* $t(X)$ of X is the smallest infinite cardinal number \mathfrak{m} such that for any set A of X and any point $x \in \overline{A}$ (the closure in X) there is a set $B \subset A$ for which $|B| \leq \mathfrak{m}$ and $x \in \overline{B}$. The notion of countable tightness arises as a natural generalization of the *Fréchet-Urysohn* notion. X is said to be Fréchet-Urysohn if for every set $A \subset X$ and every $x \in \overline{A}$ there is a sequence in A which converges to x .

In [3] Cascales and Orihuela showed (answering an (LF) -space question of Floret [11]) that $w(K) \leq \aleph_0$ for any precompact set K in an (LM) -space, i.e., precompact subsets are metrizable in inductive limits of increasing sequences of metrizable locally convex spaces (LCS). They continued this line of research in [4] and introduced a large class \mathfrak{G} of LCS with good stability properties containing (LM) -spaces and dual metric spaces which, themselves, respectively generalize the intensely studied (LF) - and (DF) -spaces. Pfister and Valdivia, respectively, had earlier demonstrated countable weight for precompact sets in (DF) - and dual metric spaces. Cascales and Orihuela [4] unified and extended these result by showing that $w(K) \leq \aleph_0$ for every precompact set $K \subset E$ whenever $E \in \mathfrak{G}$. In particular, E is angelic, and they proved that E with its weak topology is also angelic.

We study first the weight of precompact subsets in uniform spaces with decreasing bases for their uniformity, indexed in the product of a countable

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family of infinite directed sets, see Theorem 3.1. The proof of this result uses arguments similar to those in Theorem 1 of [4], which is extended here, and technically relies heavily on pure topological results included in Proposition 2.1 and Theorem 2.3 below (proved in section 2 for the sake of completeness). Theorem 3.1 applies to many LCS and extends, among others, the main result of [3] by showing [Theorem 4.2(ii)] that if E is the inductive limit of a sequence (E_n) of LCS in \mathfrak{G}_m , where m is an infinite cardinal number and \mathfrak{G}_m denotes the class of LCS E with character $\chi(E) \leq m$, then $w(K) \leq m$ for any precompact set $K \subset E$. In particular $w(K) \leq \aleph_0$ for any precompact set K in an (LM) -space.

More topological than Kąkol and Saxon's proof [13], the approach to Theorem 4.2(ii) via Theorem 3.1 leads to entirely new results on countable tightness of LCS. Although among (LM) -spaces only the metrizable ones are Fréchet-Urysohn, cf. [12], it turns out that $t(E) \leq \aleph_0$ for every (LM) -space E . This is a consequence of Theorem 4.2(i): If E is the inductive limit of at most m locally convex spaces in \mathfrak{G}_m , then $t(E) \leq m$ and $t(E, \sigma(E, E')) \leq m$. For a space $E \in \mathfrak{G}$ we prove that $[(E, \sigma(E, E')) \text{ has countable tightness}] \Leftrightarrow [(E', \sigma(E', E)) \text{ is realcompact}] \Leftrightarrow [E \text{ has countable tightness}]$, see Theorem 4.6 and Proposition 4.7. Consequently, there exist (DF) -spaces which do not have countable tightness. In fact, we provide examples of (DF) -spaces which are very nearly barrelled (are Mackey and \aleph_0 -barrelled) and yet have uncountable tightness. However, all quasibarrelled spaces in \mathfrak{G} have countable tightness, see Theorem 4.8, proving yet again countable tightness for (LM) -spaces E under both the original and weak topologies. When E is metrizable we have Kaplansky's theorem [14, §24.1.6] as a corollary. At the end of the paper we pose problems for future study.

Our notation and terminology are standard and we take [10, 14, 15] as our basic reference texts for topology and topological vector spaces. E' and E^* denote the topological and algebraic duals of a LCS E , respectively. All topological spaces X are assumed to be Tychonoff, i.e. T_1 completely regular spaces. The *character* of a point x in X is defined (and denoted by $\chi(x, X)$) as the smallest cardinal number of a basis of neighborhoods of x . Then $\chi(X) = \sup\{\chi(x, X) : x \in X\}$ denotes the character of X . By the *density* $d(X)$ we mean the minimal cardinality of a dense subset of X . The *Lindelöf number* $l(X)$ of X is the smallest infinite cardinal number m such that every open cover of X has a subcover of cardinality $\leq m$. By $C(X)$ we denote the space of continuous real functions on the topological space X ; $C_p(X)$ denotes the space $C(X)$ endowed with the topology of pointwise convergence on X . It is known that $\sup_n l(X^n) = t(C_p(X))$, see [1, Theorem II.1.1]. For a compact and Hausdorff space K , the sup-norm $\|\cdot\|_\infty$ of $C(K)$ is defined by $\|f\|_\infty := \sup\{|f(x)| : x \in K\}$, for every $f \in C(K)$. Then, see [1],

$$w(K) = d(C(K), \|\cdot\|_\infty) = d(C_p(K)).$$

If A is a subset in a vector space E (real or complex), $\Gamma(A)$ denotes its absolutely convex hull.

2. RESULTS ON SET-VALUED UPPER SEMI-CONTINUOUS MAPS

The next two results, Proposition 2.1 and Theorem 2.3, will play a crucial role when proving our main results in this paper: Theorems 3.1, 4.2, 4.6 and 4.8. Proposition 2.1 is a useful observation and Theorem 2.3 is a more elaborated result based upon ideas in [2, 6]. Recall that a map ψ from a topological space X to the power set 2^Y of a topological space Y is upper semi-continuous if for each $x \in X$ and each open set G of Y containing $\psi(x)$, there is an open neighborhood U of x in X such that $\psi(U) \subset G$.

Proposition 2.1. *Let X and Y be topological spaces and let $\psi : X \rightarrow 2^Y$ be an upper semi-continuous compact-valued map such that the set $Y = \bigcup\{\psi(x) : x \in X\}$. Assume that $w(X)$ is infinite. Then,*

- (i) *the Lindelöf number $l(Y^n) \leq w(X)$, for every $n = 1, 2, \dots$;*
- (ii) *if Y is moreover assumed to be metric then $d(Y) \leq w(X)$.*

Proof. To prove (i) we observe first that for every $n = 1, 2, \dots$ the multi-valued map $\psi^n : X^n \rightarrow 2^{Y^n}$ given by

$$\psi^n(x_1, x_2, \dots, x_n) := \psi(x_1) \times \psi(x_2) \times \cdots \times \psi(x_n)$$

is compact-valued, upper semi-continuous and

$$Y^n = \bigcup\{\psi^n(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in X^n\}.$$

Since $w(X)$ is infinite we have that $w(X^n) = w(X)$ and therefore we only need to prove (i) for $n = 1$. Take $(G_i)_{i \in I}$ any open cover of Y . For each $x \in X$ the compact set $\psi(x)$ is covered by the family $(G_i)_{i \in I}$ and therefore we can choose a finite subset $I(x)$ of I such that

$$\psi(x) \subset \bigcup_{i \in I(x)} G_i.$$

By upper semi-continuity, for each x in X we can take an open set O_x of X such that $x \in O_x$ and

$$\psi(O_x) \subset \bigcup_{i \in I(x)} G_i.$$

The family $(O_x)_{x \in X}$ is an open cover of X and therefore there is a set $F \subset X$ such that $|F| \leq w(X)$ and $X = \bigcup_{x \in F} O_x$, see [10, Theorem 1.1.14]. Then

$$Y = \psi(X) = \bigcup_{x \in F} \psi(O_x) = \bigcup_{x \in F} \bigcup_{i \in I(x)} G_i.$$

Hence $(G_i)_{i \in I}$ has a subcover of at most $w(X)$ elements.

Now we consider (ii). Assume Y is a metric space, and for every $n \in \mathbb{N}$ choose $F_n \subset Y$ a maximal set of points the distance between any two of which is at least $1/n$. It is not difficult to check that F_n is closed, each $x \in X$ has a neighborhood U such that $\psi(U) \cap F_n$ is finite, and therefore $|F_n| \leq w(X)$. It is then quite easy to see that $F = \bigcup_{n=1}^{\infty} F_n$ is dense in Y and thus we obtain

$$d(Y) \leq w(X)$$

which finishes the proof (see also [10, Theorem 4.1.15]). ■

Corollary 2.2. *Let X and Y be topological spaces and let $\psi : X \rightarrow 2^Y$ be an upper semi-continuous compact-valued map such that $Y = \cup\{\psi(x) : x \in X\}$. Assume that $w(X)$ is infinite. If $Y_0 \subset Y$ is a closed subspace then the Lindelöf number $l(Y_0^n) \leq w(X)$, for every $n = 1, 2, \dots$.*

Proof. Since Y_0^n is closed in Y^n for every $n = 1, 2, \dots$ we have that $l(Y_0^n) \leq l(Y^n)$, and then we apply the last result. ■

A subset A of a topological space Y is said to be relatively countably compact if every sequence $(y_n)_n$ in A has a cluster point in Y ; if the cluster point can be taken in A then we say that A is countably compact.

Theorem 2.3. *Let X be a first-countable topological space, Y a topological space in which the relatively countably compact subsets are relatively compact and let $\phi : X \rightarrow 2^Y$ be a set-valued map satisfying the property*

$$(1) \quad \bigcup_{n \in \mathbb{N}} \phi(x_n) \text{ is relatively compact for each convergent sequence } (x_n)_n \text{ in } X.$$

If for each x in X we define

$$(2) \quad \psi(x) := \bigcap \{\overline{\phi(V)} : V \text{ neighborhood of } x \text{ in } X\},$$

then the map so defined $\psi : X \rightarrow 2^Y$ is upper semi-continuous, compact-valued and satisfies $\phi(x) \subset \psi(x)$ for every x in X .

Proof. Given x in X we define

$$C(x) := \{y \in Y : \text{there is } x_n \rightarrow x \text{ in } X, \text{ for every } n \in \mathbb{N} \text{ there is } y_n \in \phi(x_n) \text{ and } y \text{ is cluster point of } (y_n)_n\}.$$

Fix $V_1^x \supset V_2^x \supset \dots \supset V_n^x \supset \dots$ a basis of open neighborhoods of x in the space X . We will establish now several claims leading eventually to the proof.

CLAIM 1. $C(x)$ is countably compact and thus $\overline{C(x)}$ is compact.

We have to prove that every sequence in $C(x)$ has a cluster point in $C(x)$. Take $(y_j)_j$ in $C(x)$ and let $x_n^j \rightarrow x$ for every $j \in \mathbb{N}$ and let $y_n^j \in \phi(x_n^j)$ such that y_j is a cluster point of (y_n^j) . There are natural numbers $n_i^j, i, j \in \mathbb{N}$ such that

$$1 \leq n_1^j < n_2^j < \dots < n_i^j < \dots, \quad j = 1, 2, \dots$$

and

$$x_n^j \in V_k^x \text{ whenever } n_k^j \leq n < n_{k+1}^j, \quad k = 1, 2, \dots, \quad j = 1, 2, \dots$$

The sequence $(x_n)_n$ given by

$$(3) \quad \{x_1^1, x_2^1, \dots, x_{n_2^1-1}^1, x_{n_2^1}^1, \dots, x_{n_3^1-1}^1, x_{n_3^1}^1, \dots, x_{n_4^1-1}^1, x_{n_4^1}^1, \dots, x_{n_2^2}, \dots, x_{n_4^2-1}^2, x_{n_3^3}, \dots, x_{n_4^3-1}^3, x_{n_4^1}^1, \dots\}$$

clearly converges to x and

$$y_j \in \overline{\bigcup_{n \in \mathbb{N}} \phi(x_n)}, \text{ for every } j = 1, 2, \dots$$

Property (1) implies that the sequence $(y_j)_j$ has a cluster point y in Y . The point y actually belongs to $C(x)$ because if we consider the sequence $(z_n)_n$ corresponding to (3) but defined by

$$\{y_1^1, y_2^1, \dots, y_{n_2-1}^1, y_{n_2}^1, \dots, y_{n_3-1}^1, y_{n_3}^2, \dots, y_{n_3-1}^2, y_{n_3}^1, \dots, y_{n_4-1}^1, \\ , y_{n_3}^2, \dots, y_{n_4-1}^2, y_{n_4}^3, \dots, y_{n_4-1}^3, y_{n_4}^1, \dots\}$$

then $z_n \in \phi(x_n)$ and it is easy to see that y is a cluster point of $(z_n)_n$ and thus the claim is proved.

CLAIM 2. If G is a open set in Y such that $C(x) \subset G$, then there is an open neighborhood V of x such that $\phi(V) \subset G$.

Assume that the claim is not true. Then for every $n \in \mathbb{N}$ there is $x_n \in V_n^x$ such that $\phi(x_n) \not\subset G$ and consequently we can choose $y_n \in \phi(x_n)$ such that $y_n \in Y \setminus G$. Observe now that $x_n \rightarrow x$ and therefore $(y_n)_n$ has a cluster point y in Y because of property (1) and the inclusion

$$\{y_n\}_n \subset \bigcup_{n \in \mathbb{N}} \phi(x_n).$$

But we have reached a contradiction: on the one hand by definition $y \in C(x)$, on the other hand $y \in Y \setminus G$ because $Y \setminus G$ is closed, violating the hypothesis that $C(x) \cap (Y \setminus G) = \emptyset$. This finishes the proof of the claim.

CLAIM 3. If G is a open set in Y such that $\overline{C(x)} \subset G$, then there is an open neighborhood V of x such that $\overline{\phi(V)} \subset G$.

Indeed, take $O \subset Y$ open such that $\overline{C(x)} \subset O \subset \overline{O} \subset G$ and apply the former claim to $C(x)$ and O ; we get an open neighborhood V of x such that $\phi(V) \subset O$. Now, $\overline{\phi(V)} \subset \overline{O} \subset G$ and we are done.

CLAIM 4. $\psi(x) = \overline{C(x)}$ and thus $\psi(x)$ is compact-valued.

The inclusion $C(x) \subset \psi(x)$ is a consequence of the definitions of the sets $C(x)$, $\psi(x)$ and the definition of cluster point of a sequence; the inclusion $\overline{C(x)} \subset \psi(x)$ follows now from the fact that $\psi(x)$ is closed. Conversely, take $z \in \psi(x)$. We prove that $z \in \overline{C(x)}$: for any closed neighborhood U of z in Y and for every $n \in \mathbb{N}$ there is some $y_n \in U \cap \phi(V_n^x)$. Choose a point $x_n \in V_n^x$ and a point $y_n \in \phi(x_n) \cap U$. Then $x_n \rightarrow x$ and therefore $(y_n)_n$ has a cluster point y in Y because of (1). By definition $y \in C(x)$ and because U is closed we have $y \in U$ which means that $z \in \overline{C(x)}$. This proves the equality $\psi(x) = \overline{C(x)}$.

CLAIM 5. $\psi : X \rightarrow 2^Y$ is upper semi-continuous.

We have to prove that for every open set $G \supset \psi(x)$ there is an open neighborhood V of x such that $\psi(V) \subset G$. Take G as above. Since $\psi(x) = \overline{C(x)}$, we apply claim 3 and find an open neighborhood V of x satisfying

$$(4) \quad \overline{\phi(V)} \subset G.$$

Now V is also an open neighborhood of any $y \in V$, and definition (2) implies

$$\psi(y) \subset \overline{\phi(V)} \subset G.$$

Hence $\psi(V) \subset G$ and the upper semi-continuity of ψ has been proved.

Finally observe that by definition $\phi(x) \subset \psi(x)$ for every x in X and the claims prove the theorem. \blacksquare

Note that the class of spaces Y with the above property includes many topological and topological vector spaces as for instance: the Lindelöf spaces, the realcompact spaces (i.e., spaces homeomorphic to closed subspaces of Cartesian product of copies of the real line, see [10], pages 271-277), angelic spaces (see [4, 5] and references therein), Banach spaces with their weak topology, dual Banach spaces with their weak* topology, etc.

3. PRECOMPACT SUBSETS IN UNIFORM SPACES

Given a uniform space (Z, \mathcal{U}) the *weight of the uniformity* $uw(Z)$ is the minimal cardinality of a basis for the uniformity \mathcal{U} . For every compact Hausdorff space K there is exactly one uniformity \mathcal{U} on the set K that induces the original topology of K ; all the sets containing the diagonal $\Delta \subset K \times K$ which are open in the cartesian product $K \times K$ form a basis for the uniformity, see [10, Theorem 8.3.13]. Thus for a compact space the equality $w(K) = uw(K)$ always holds. The aim of the section is to prove that in a uniform space (Z, \mathcal{U}) with a decreasing basis for the uniformity indexed in a countable product of directed sets the weight of the precompact subsets can be dramatically decreased, see Theorem 3.1 below, from $uw(Z)$ up to the point of sometimes being able to decide even metrizability, see Corollary 3.2.

In what follows if $(J_\ell, \leq_\ell)_{\ell \in L}$ is any family of directed sets we consider the cartesian product $\prod_{\ell \in L} J_\ell$ directed by \leq where

$$\alpha = (a_\ell)_{\ell \in L} \leq \beta = (b_\ell)_{\ell \in L} \text{ if (and only if) } a_\ell \leq_\ell b_\ell \text{ for every } \ell \in L.$$

We will consider each J_ℓ as a discrete space and $\prod_{\ell \in L} J_\ell$ as a topological space endowed with the product topology.

Theorem 3.1. *Let (Z, \mathcal{U}) be a uniform space and let us suppose that the uniformity \mathcal{U} has a basis $\mathfrak{B}_\mathcal{U} = \{N_\alpha : \alpha \in \prod_{s \in S} I_s\}$, where $(I_s, \leq_s)_{s \in S}$ is a finite or a countable family of infinite directed sets, satisfying*

$$(5) \quad N_\beta \subset N_\alpha \text{ whenever } \alpha \leq \beta \text{ in } \prod_{s \in S} I_s.$$

Then for every precompact subset K of (Z, \mathcal{U}) we have the inequality

$$w(K) \leq \sup_s |I_s|.$$

Proof. It will be enough to prove the result for the compact subsets of Z : certainly, the corresponding result for precompact subsets can be then obtained reasoning with the completion $(\tilde{Z}, \tilde{\mathcal{U}})$ of (Z, \mathcal{U}) and having in mind that the closure of the elements of \mathfrak{B} in $\tilde{Z} \times \tilde{Z}$ is a basis for the uniformity $\tilde{\mathcal{U}}$, see [14, §5.5.4], and that the precompact subsets of (Z, \mathcal{U}) are relatively compact in \tilde{Z} . Let us put $J_1 = \mathbb{N}$ endowed with its discrete topology and directed by its natural order \leq_1 and for $n = 2, 3, \dots$ let us define $J_n = \prod_{s \in S} I_s$ directed

by $\leq_n := \leq$ and endowed with the product of discrete topologies. Now, take the directed product

$$X := \prod_{n \in \mathbb{N}} J_n = \mathbb{N} \times \prod_{s \in S} I_s \times \prod_{s \in S} I_s \times \cdots \times \prod_{s \in S} I_s \times \cdots$$

also endowed with its product topology. The reader can either easily check or see [10, Theorem 2.3.13] to be convinced that

$$(6) \quad w(X) \leq \sup_s |I_s|.$$

Let K be a compact subset of Z . For $x = (m, \alpha_1, \alpha_2, \dots, \alpha_n, \dots)$ in X we define

$$\begin{aligned} \phi(x) := \{ & f \in C(K) : \|f\|_\infty \leq m \\ & \text{and } |f(s) - f(t)| \leq 1/n \text{ if } (s, t) \in (K \times K) \cap N_{\alpha_n}, n = 1, 2, \dots \}. \end{aligned}$$

Each $\phi(x)$ is bounded and closed for $\|\cdot\|_\infty$ and uniformly equicontinuous; by Ascoli's theorem, [10, Theorem 8.2.10], $\phi(x)$ is a compact subset of $(C(K), \|\cdot\|_\infty)$. On the other hand, by property (5) we have $\phi(x) \subset \phi(y)$ whenever $x \leq y$ in X . It is easily checked now that if $x_n \rightarrow x$ in X then there is $y \in X$ such that $\bigcup_{n \in \mathbb{N}} \phi(x_n) \subset \phi(y)$. Since X is a first-countable space we can apply Theorem 2.3 to this ϕ and $Y := (C(K), \|\cdot\|_\infty)$ to obtain an upper semi-continuous, compact-valued map $\psi : X \rightarrow 2^{C(K)}$ with the property

$$(7) \quad \phi(x) \subset \psi(x) \text{ for every } x \in X.$$

As every continuous function on K is $\|\cdot\|_\infty$ -bounded and uniformly continuous for $\mathcal{U}|_{K \times K}$ we obtain $C(K) = \bigcup \{\phi(x) : x \in X\}$; the inclusions (7) show that $C(K) = \bigcup \{\psi(x) : x \in X\}$ and then Proposition 2.1 applies to yield

$$d(C(K), \|\cdot\|_\infty) \leq w(X).$$

The inequality (6) and the equality $w(K) = d(C(K), \|\cdot\|_\infty)$ lead us to

$$w(K) \leq \sup_s |I_s|$$

and the proof is complete. ■

As a very special case of the former theorem we get the metrizable result that follows.

Corollary 3.2 (Cascales and Orihuela, [4]). *Let (Z, \mathcal{U}) be a uniform space and let us suppose that the uniformity \mathcal{U} has a basis $\mathfrak{B}_\mathcal{U} = \{N_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ satisfying*

$$N_\beta \subset N_\alpha \text{ whenever } \alpha \leq \beta \text{ in } \mathbb{N}^\mathbb{N}.$$

Then the precompact subsets of (Z, \mathcal{U}) are metrizable for the induced uniform topology.

4. APPLICATIONS TO LOCALLY CONVEX SPACES

If \mathfrak{B} is a basis of absolutely convex neighborhoods of the origin for the topology \mathfrak{T} of a locally convex space (E, \mathfrak{T}) , then the topology is associated to the uniformity \mathcal{U} for which a basis is given by $\mathfrak{B}_{\mathcal{U}} := \{N_U : U \in \mathfrak{B}\}$ where

$$N_U := \{(x, y) \in E \times E : x - y \in U\}.$$

It is not difficult to prove that $\chi(E, \mathfrak{T}) = uw(E, \mathcal{U})$ and $t(E, \mathfrak{T}) \leq \chi(E, \mathfrak{T})$ with the latter inequality strict at times. Given an infinite cardinal number \mathfrak{m} let us denote by $\mathfrak{G}_{\mathfrak{m}}$ the class of those locally convex spaces (E, \mathfrak{T}) for which $\chi(E, \mathfrak{T}) \leq \mathfrak{m}$.

We start this section with the following simple observation.

Proposition 4.1. *$\mathfrak{G}_{\mathfrak{m}}$ is stable by taking subspaces, quotients by closed subspaces, completions and products of no more than \mathfrak{m} spaces.*

Proof. The trace in a subspace F of a basis of neighborhoods of 0 in a LCS (E, \mathfrak{T}) form a basis of neighborhoods of 0 in $(F, \mathfrak{T}|_F)$. The image under the canonical projection $\pi : E \rightarrow E/F$ (F closed subspace) of a basis of neighborhoods of 0 in a LCS (E, \mathfrak{T}) form a basis of neighborhoods of 0 for the quotient topology on E/F . The closures in the completion $(\tilde{E}, \tilde{\mathfrak{T}})$ of a basis of neighborhoods of 0 in a LCS (E, \mathfrak{T}) form a basis of neighborhoods of 0 in $(\tilde{E}, \tilde{\mathfrak{T}})$, [14, §15.3.1]. If $(X_s)_{s \in S}$ is a family of topological spaces with $\chi(X_s) \leq \mathfrak{m}$, for every $s \in S$, and $|S| \leq \mathfrak{m}$, then $\chi(\prod_{s \in S} X_s) \leq \mathfrak{m}$, [10, Theorem 2.3.13]. ■

Note that $\mathfrak{G}_{\mathfrak{m}}$ is not stable by inductive operations. Indeed, take $\mathfrak{m} = \aleph_0$, that is, take the class of the metrizable locally convex spaces \mathfrak{G}_{\aleph_0} : any non-metrizable inductive limit of a sequence of Fréchet spaces does not belong to \mathfrak{G}_{\aleph_0} (for instance the test-space for distributions $\mathfrak{D}(\Omega)$). Do observe also that if (E, \mathfrak{T}) belongs to a certain $\mathfrak{G}_{\mathfrak{m}}$ then the space with its weak topology $(E, \sigma(E, E'))$ need not belong to $\mathfrak{G}_{\mathfrak{m}}$ in general: for example, any infinite dimensional Banach space E belongs to \mathfrak{G}_{\aleph_0} but $(E, \sigma(E, E'))$ does not. However, if we talk about tightness of spaces or weight of precompact sets we can complete the properties in proposition 4.1 with the following properties about inductive operations involving spaces of the class $\mathfrak{G}_{\mathfrak{m}}$

Theorem 4.2. *Let $(E_s, \mathfrak{T}_s)_{s \in S}$ be a family of LCS in the class $\mathfrak{G}_{\mathfrak{m}}$, and let $\{f_s : E_s \rightarrow E\}_{s \in S}$ be linear maps and let $(E, \mathfrak{T}) = \sum_{s \in S} f_s(E_s, \mathfrak{T}_s)$ be the locally convex hull of $f_s(E_s, \mathfrak{T}_s)$. Then we have:*

- (i) $t(E, \mathfrak{T}) \leq \mathfrak{m}$ and $t(E, \sigma(E, E')) \leq \mathfrak{m}$, when $|S| \leq \mathfrak{m}$;
- (ii) the weight of the precompact subsets of (E, \mathfrak{T}) is at most \mathfrak{m} when $|S| \leq \aleph_0$.

Proof. We shall start by fixing for every $s \in S$ a basis \mathfrak{B}_s of absolutely convex neighborhoods of 0 in (E_s, \mathfrak{T}_s) such that $|\mathfrak{B}_s| \leq \mathfrak{m}$. We will prove first that $t(E, \mathfrak{T}) \leq \mathfrak{m}$. In order to prove that $t(E, \mathfrak{T}) \leq \mathfrak{m}$ it is enough to show that if $A \subset E$ and $0 \in \overline{A}^{\mathfrak{T}}$ then there is a set $B \subset A$ with $|B| \leq \mathfrak{m}$ and such that $0 \in \overline{B}^{\mathfrak{T}}$. The family

$$(8) \quad \mathfrak{B} := \{\Gamma(\cup_{s \in S} f_s(U_s)) : U_s \in \mathfrak{B}_s, s \in S\}$$

is a basis of 0 in (E, \mathfrak{T}) and the family

$$\mathfrak{B}_0 := \{\Gamma(\cup_{s \in S'} f_s(U_s)) : U_s \in \mathfrak{B}_s, S' \text{ finite subset of } S\}$$

has at most \mathfrak{m} elements. Given $A \subset E$, $0 \in \overline{A}^{\mathfrak{T}}$ we define

$$B := \{x_{U_0} : x_{U_0} \text{ is a chosen point in } U_0 \cap A \text{ if } U_0 \cap A \neq \emptyset, U_0 \in \mathfrak{B}_0\}.$$

It is clear that $B \subset A$, $|B| \leq \mathfrak{m}$ and, moreover, $0 \in \overline{B}^{\mathfrak{T}}$. Indeed, if $U \in \mathfrak{B}$ then $A \cap U \neq \emptyset$. Hence, there is $U_0 \in \mathfrak{B}_0$ with $U_0 \subset U$ and $U_0 \cap A \neq \emptyset$; this means that the corresponding $x_{U_0} \in B \cap U$ and therefore $0 \in \overline{B}^{\mathfrak{T}}$. Now we prove that $t(E, \sigma(E, E')) \leq \mathfrak{m}$. Since there is a homeomorphic embedding from $(E, \sigma(E, E'))$ into $C_p(E', \sigma(E', E))$, it suffices to show that for every $n = 1, 2, \dots$ we have $l(E', \sigma(E', E))^n \leq \mathfrak{m}$ because in this case $t(C_p(E', \sigma(E', E))) \leq \mathfrak{m}$, [1, Theorem II.1.1], and consequently $t(E, \sigma(E, E')) \leq \mathfrak{m}$. According to [14, §19.1.3] the space (E, \mathfrak{T}) is topologically isomorphic to a quotient $\widehat{E} = (\bigoplus_{s \in S} E_s)/H$ of the locally convex sum of (E_s, \mathfrak{T}_s) by a closed subspace H ; again according to [14, §22.2.2, and page 287] the weak* dual $(E', \sigma(E', E))$ is isomorphically homeomorphic to a closed subspace of $\prod_{s \in S} (E'_s, \sigma(E'_s, E_s))$. Bearing in mind now Corollary 2.2 we only have to prove that $\prod_{s \in S} (E'_s, \sigma(E'_s, E_s))$ is an upper semi-continuous compact-valued image of a space of weight at most \mathfrak{m} : indeed, for $s \in S$ we consider \mathfrak{B}_s as a discrete space. Then the map $\psi_s : \mathfrak{B}_s \rightarrow 2^{(E'_s, \sigma(E'_s, E_s))}$ defined by

$$\psi_s(U) := U^o, \text{ for every } U \in \mathfrak{B}_s$$

is upper semi-continuous, compact-valued and $E'_s = \cup\{\psi_s(U) : U \in \mathfrak{B}_s\}$; now the map $\psi : \prod_s \mathfrak{B}_s \rightarrow 2^{\prod_s (E'_s, \sigma(E'_s, E_s))}$ given by

$$\psi((U_s)_s) := \prod_s \psi_s(U_s)$$

for $(U_s)_s \in \prod_s \mathfrak{B}_s$ is compact-valued (Tychonoff theorem) and upper semi-continuous, see [9, Proposition 3.6], and satisfies

$$\prod_s E_s = \cup\{\psi((U_s)_s) : (U_s)_s \in \prod_s \mathfrak{B}_s\}.$$

By [10, Theorem 2.3.13] we have that $w(\prod_s \mathfrak{B}_s) \leq \mathfrak{m}$ and we deduce that $t(E, \sigma(E, E')) \leq \mathfrak{m}$.

Now let us prove (ii). Assume $|S| \leq \aleph_0$ and let \mathcal{U} be the uniformity in E associated to \mathfrak{T} . A basis for \mathcal{U} is given by $\mathfrak{B}_{\mathcal{U}} := \{N_U : U \in \mathfrak{B}\}$, where \mathfrak{B} is the basis of neighborhoods of 0 described in (8). Consequently

$$\mathfrak{B}_{\mathcal{U}} = \{N_U : U = \Gamma(\cup_s U_s), (U_s)_s \in \prod_s \mathfrak{B}_s\}.$$

When directing each \mathfrak{B}_s downwards by inclusion then $\mathfrak{B}_{\mathcal{U}}$ does satisfy condition (5) in Theorem 3.1. Thus we get that the weight of \mathfrak{T} -precompact subsets of E is less or equal $\sup_s |\mathfrak{B}_s|$, so at most \mathfrak{m} , and the proof is complete. \blacksquare

Let us mention that statement (ii) in the previous theorem appears in [13] with a very different proof.

Corollary 4.3. *Let $(E, \mathfrak{T}) = \varinjlim (E_n, \mathfrak{T}_n)$ be an inductive limit of metrizable LCS. Then,*

- (i) (E, \mathfrak{T}) and $(E, \sigma(E, E'))$ have countable tightness;
- (ii) the precompact subsets of (E, \mathfrak{T}) are metrizable.

Proof. It is just the former theorem for $\mathfrak{m} = \aleph_0$. ■

Note that every (LM) -space has a basis of neighborhoods of 0 with at most the cardinality of the real numbers. By the last corollary if (E, \mathfrak{T}) is a non-metrizable (LM) -space then the strict inequality $t(E, \sigma(E, E')) < \chi(E, \sigma(E, E'))$ holds. It has to be stressed that although (LM) -spaces always have countable tightness, the non-metrizable (LM) -spaces are never Fréchet-Urysohn, see [12]. Nevertheless, the metrizability of precompact subsets ensures us that every point in the closure of a precompact set A of (E, \mathfrak{T}) is actually the limit of a sequence in A .

Theorem 3.1 has been used to get the upper bound of the weight of precompact subsets in locally convex hulls of spaces in \mathfrak{G}_m . But Theorem 3.1 has a bit more potential yet.

Theorem 4.4. *Let (E, \mathfrak{T}) be a LCS with a family $\{A_\alpha : \alpha \in \prod_{s \in S} I_s\}$ of subsets of E' , where $(I_s, \leq_s)_{s \in S}$ is a finite or a countable family of directed sets satisfying*

- (i) $\cup\{A_\alpha : \alpha \in \prod_{s \in S} I_s\} = E'$;
- (ii) $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$ in $\prod_{s \in S} I_s$;
- (iii) for any $\alpha \in \prod_{s \in S} I_s$ the countable subsets of A_α are \mathfrak{T} -equicontinuous.

Then for every precompact subset K of (E, \mathfrak{T}) we have the inequality

$$\text{weight}(K) \leq \sup_s |I_s|.$$

Proof. Let us first observe that a set in a LCS is precompact for the given topology if and only if every sequence in the set is precompact [14, §5.6.3]. Let \mathfrak{T}' be the topology in E of uniform convergence on the family of sets $\{A_\alpha : \alpha \in \prod_{s \in S} I_s\}$ and let \mathfrak{T}^{seq} be the topology in E of uniform convergence on all the sequences contained in some A_α , $\alpha \in \prod_{s \in S} I_s$. It is clear that $\sigma(E, E') \leq \mathfrak{T}^{seq} \leq \mathfrak{T}$ and $\sigma(E, E') \leq \mathfrak{T}^{seq} \leq \mathfrak{T}'$. By Theorem 3.1 the weight of \mathfrak{T}' -precompact subsets is at most $\sup_s |I_s|$. On the one hand \mathfrak{T}' and \mathfrak{T}^{seq} coincide on sequences, and therefore they have the same precompact sets; on the other hand every \mathfrak{T} -precompact subset in E is \mathfrak{T}^{seq} -precompact. Now we use [14, §28.5.2] to get that the three topologies \mathfrak{T} , \mathfrak{T}' and \mathfrak{T}^{seq} coincide on \mathfrak{T} -precompact subsets and so the proof is finished. ■

Recall that Cascales and Orihuela [4] defined the class \mathfrak{G} as those LCS satisfying conditions (i), (ii) and (iii) in the former theorem with $S = \mathbb{N}$ and $I_n := \mathbb{N}$, for every $n = 1, 2, \dots$. Theorem 4.4 says, in particular, that for a space (E, \mathfrak{T}) in \mathfrak{G} , the \mathfrak{T} -precompact subsets are metrizable, see [4, Theorem 2]. The many results in [4] about \mathfrak{G} , see Introduction, provided impetus to the study of compactness and weak compactness in locally convex spaces, answering questions open at the time and extending results by [3], [16] and [18], among others.

Now we give a characterization of when spaces in class \mathfrak{G} have countable tightness for the weak topology. To do so we will use the following characterization of weakly real-compact LCS that can be found in [19, Page 137]

Theorem 4.5. *Let $\langle E, E' \rangle$ be a dual pair and let $\{E_i : i \in I\}$ be the family of all separable closed subspaces of $(E', \sigma(E', E))$. Then the following statements are equivalent:*

- (i) $(E, \sigma(E, E'))$ is real-compact;
- (ii) $E = \{x^* \in (E')^* : x^*|_{E_i} \text{ is } \sigma(E', E)\text{-continuous for each } i \in I\}$.

A topological space Y is said to be K -analytic, see [7], if there is an upper semi-continuous set-valued map with compact values $\psi : \mathbb{N}^{\mathbb{N}} \rightarrow 2^Y$ such that $Y = \{\psi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since $\mathbb{N}^{\mathbb{N}}$ is metric and separable, we have $w(\mathbb{N}^{\mathbb{N}}) \leq \aleph_0$ and consequently for any K -analytic space we have $l(Y^n) \leq \aleph_0$, Proposition 2.1. This simple fact is one of the keys used to prove the next result.

Theorem 4.6. *Let (E, \mathfrak{T}) be a LCS in the class \mathfrak{G} . The following statements are equivalent:*

- (i) $(E, \sigma(E, E'))$ has countable tightness;
- (ii) For every topological space (Y, \mathcal{G}) , any function from E into Y that is $\sigma(E, E')$ -continuous restricted to $\sigma(E, E')$ -closed and separable subsets of E is $\sigma(E, E')$ -continuous on E ;
- (iii) Every linear form on E that is $\sigma(E, E')$ -continuous restricted to $\sigma(E, E')$ -closed and separable subspaces of E is $\sigma(E, E')$ -continuous on E ;
- (iv) $(E', \sigma(E', E))$ is realcompact;
- (v) $(E', \sigma(E', E))$ is K -analytic;
- (vi) $(E', \sigma(E', E))^n$ is Lindelöf for every $n = 1, 2, \dots$;
- (vii) $(E', \sigma(E', E))$ is Lindelöf.

Proof. (i) \Rightarrow (ii) Let $f : E \rightarrow Y$ be $\sigma(E, E')$ -continuous restricted to $\sigma(E, E')$ -closed and separable subsets of E . To prove that f is continuous it is enough to prove that for any set $A \subset E$ if $x \in \overline{A}^{\sigma(E, E')}$ then $f(x) \in \overline{f(A)}^{\mathcal{G}}$; but this is so because by hypothesis in this situation there is $D \subset A$ countable such that $x \in \overline{D}^{\sigma(E, E')}$, thus $f|_{\overline{D}^{\sigma(E, E')}}$ is continuous and so $f(x) \in \overline{f(D)}^{\mathcal{G}} \subset \overline{f(A)}^{\mathcal{G}}$. (ii) \Rightarrow (iii) Given a countable subset $D \subset E$ it is easy to check that $\overline{\text{span}_{\mathbb{Q}} D}^{\sigma(E, E')}$ is a $\sigma(E, E')$ -closed and separable vector subspace of E . Then for any topological space (Y, \mathcal{G}) and any function $f : E \rightarrow Y$, the $\sigma(E, E')$ -continuity of f restricted to $\sigma(E, E')$ -closed separable subsets of E is equivalent to the $\sigma(E, E')$ -continuity of f restricted to $\sigma(E, E')$ -closed separable subspaces. Clearly then (iii) is a consequence of (ii). (iii) \Rightarrow (iv) It is a consequence of Theorem 4.5. (iv) \Rightarrow (v) Take (E, \mathfrak{T}) in \mathfrak{G} . Then there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets in E' satisfying:

- (a) $E' = \cup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$;
- (b) sequences in every A_α are \mathfrak{T} -equicontinuous;
- (c) $A_\alpha \subset A_\beta$ when $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$.

Condition (b) implies that each A_α is relatively $\sigma(E', E)$ -countably compact by Alaoglu-Bourbaki's theorem, [14, §20.9.4], and so A_α is relatively $\sigma(E', E)$ -compact because $(E', \sigma(E', E))$ is real-compact. If we define $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow 2^{E'}$ by $\phi(\alpha) := A_\alpha$, $\alpha \in \mathbb{N}^{\mathbb{N}}$, then condition (c) on A_α 's implies that ϕ satisfies condition (1) in Theorem 2.3. This last mentioned theorem ensures us of the existence of an upper semi-continuous $\sigma(E', E)$ -compact-valued map $\psi : \mathbb{N}^{\mathbb{N}} \rightarrow 2^{E'}$ such that $\phi(\alpha) \subset \psi(\alpha)$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. Condition (a) on A_α 's gives us $E' = \cup\{\psi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and so $(E', \sigma(E', E))$ is K -analytic. (v) \Rightarrow (vi) This is a consequence of Proposition 2.1 already noted. (vi) \Rightarrow (i) Since $l(E', \sigma(E', E))^n \leq \aleph_0$, for $n = 1, 2, \dots$, then by [1, Theorem II.1.1], the space of continuous functions $C_p(E', \sigma(E', E))$ has countable tightness. Subspaces of spaces of countable tightness have countable tightness and thus $(E, \sigma(E, E'))$ has countable tightness and (i) is proved. To finish the equivalences we observe that obviously (vi) \Rightarrow (vii) and that Lindelöf spaces are real-compact, [10, Theorem 3.11.12], and thus (vii) \Rightarrow (iv). ■

Since (LM) -spaces E belong to \mathfrak{G} , Corollary 4.3 and Theorem 4.6 apply to show that $(E', \sigma(E', E))$ is K -analytic. This can be proved also directly with techniques similar to those used in the proof of Theorem 4.2 which essentially was done in the proof of [3, Theorem 2]. But not every space in class \mathfrak{G} has countable tightness for its weak topology. Indeed, there is a Fréchet space (E, \mathfrak{T}) such that $(E'', \sigma(E'', E'))$ is not K -analytic [19, Page 67, Proposition (24) and section 4 in §5, chapter II]. The strong dual $(E', \beta(E', E))$ is a (DF) -space which, when endowed with its weak topology, has uncountable tightness via Theorem 4.6. Therefore, in contrast to part (ii) of Corollary 4.3, part (i) does not extend to all spaces in \mathfrak{G} . How far in \mathfrak{G} countable tightness does extend motivates the remainder of the paper. Even as it is, Corollary 4.3(i) substantially extends Kaplansky's theorem, as do, indeed, both of the remaining results.

Proposition 4.7. *Let (E, \mathfrak{T}) be a space in the class \mathfrak{G} . If (E, \mathfrak{T}) has countable tightness then $(E, \sigma(E, E'))$ has countable tightness.*

Proof. Assume that (E, \mathfrak{T}) has countable tightness. Reasoning as we did in the proof of (i) \Rightarrow (ii) \Rightarrow (iii) in theorem 4.6, we obtain that every linear form in E that is \mathfrak{T} -continuous on \mathfrak{T} -separable and closed subspaces of E is \mathfrak{T} -continuous. The \mathfrak{T} continuous linear forms are exactly the $\sigma(E, E')$ continuous linear forms; on the other hand the family of \mathfrak{T} -closed and separable subspaces of E is exactly the family of $\sigma(E, E')$ -closed and separable subspaces of E . With all this, we must conclude that the countable tightness of (E, \mathfrak{T}) implies condition (iii) in Theorem 4.6 and so $(E, \sigma(E, E'))$ has countable tightness. ■

Proposition 4.7 yields examples of (DF) -spaces with uncountable tightness: Take, as above, any (DF) -space whose weak topology has uncountable tightness. Moreover, there exist (DF) -spaces with uncountable tightness whose weak topology has countable tightness, denying the converse to Proposition 4.7: Take E as in the first example of Section 5 of [13]; i.e., fix a positive (finite) number p , let Λ be an uncountable indexing set, for each

$S \subset \Lambda$ define

$$E_S = \{u \in \ell^p(\Lambda) : u(x) = 0 \text{ for } x \notin S\},$$

and let E be the Banach space $\ell^p(\Lambda)$ endowed with the coarsest topology such that the projection of E onto the Banach space E_S along $E_{\Lambda \setminus S}$ is continuous for every countable $S \subset \Lambda$. A base of 0-neighborhoods for E consists of the sets U of the form $U = V + E_{\Lambda \setminus S}$, where V is a positive multiple of the unit ball in the Banach space $\ell^p(\Lambda)$ and S is a countable subset of Λ . We observed in [13] that E is a sequentially complete \aleph_0 -barrelled (DF) -space and has the same dual as the Banach space $\ell^p(\Lambda)$. Therefore according to Corollary 4.3, $t(E, \sigma(E, E')) = \aleph_0$. But E itself has uncountable tightness: the set B of characteristic functions of singleton subsets of Λ has 0 in its closure but not in the closure of any countable subset of B . We see from this example that a (DF) -space may fail to have countable tightness even when both the weak and Mackey topologies do have countable tightness.

Saxon and Tweddle [17] showed that if the Banach space $\ell^\infty(\Lambda)$ is given the coarsest topology making continuous the projections onto the Banach subspaces

$$E_S = \{u \in \ell^\infty(\Lambda) : u(x) = 0 \text{ for } x \notin S\},$$

where S runs through the countable subsets of Λ , then the resulting space E is a Mackey \aleph_0 -barrelled space which is not barrelled. As observed in Section 5 of [13], E is also a sequentially complete (DF) -space. Again, the set B of characteristic functions of singleton subsets of Λ shows that E has uncountable tightness.

Recall that E is [quasi]barrelled if and only if every $\sigma(E', E)$ -bounded [every $\beta(E', E)$ -bounded] set in E' is equicontinuous. Every (LM) -space is quasibarrelled, but Kōmura produced a barrelled (DF) -space which is not an (LM) -space, see [14]. The next step in the progression from \aleph_0 -barrelled to Mackey \aleph_0 -barrelled is to ask if barrelled (DF) -spaces have countable tightness. Indeed, they do. Even quasibarrelled (DF) -spaces do. In fact, we have the following generalization of Corollary 4.3(i).

Theorem 4.8. *Every quasibarrelled space (E, \mathfrak{T}) in \mathfrak{G} has countable tightness, and therefore the same also holds true for $(E, \sigma(E, E'))$.*

Proof. Proposition 4.7 permits us to prove only the first part. By definition there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets in E' satisfying:

- (i) $E' = \cup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$;
- (9) (ii) $A_\alpha \subset A_\beta$ when $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$;
- (iii) in each A_α , sequences are \mathfrak{T} -equicontinuous.

Since E is quasibarrelled and (iii) holds, we have each A_α is equicontinuous. Replacing each A_α by its $\sigma(E', E)$ -closed absolutely convex hull we may and do assume that each A_α is a $\beta(E', E)$ -Banach disc (strong duals of quasibarrelled spaces must be quasicomplete). In the terminology of [20], $(E', \beta(E', E))$ is then a quasi-LB space and therefore [20, Proposition 2.2] ensures the existence of a family of $\beta(E', E)$ -Banach discs of E' that we

again label as $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that:

- (a) $E' = \cup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$;
- (b) $A_\alpha \subset A_\beta$ when $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$;
- (c) for every $\beta(E', E)$ – Banach disc $B \subset E'$ there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $B \subset A_\alpha$.

We define a web $\mathcal{W} = \{C_{n_1, n_2, \dots, n_k}\}$ as follows:

Given $\alpha = (n_k)_k$ in $\mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$ let us write $\alpha|k := (n_1, n_2, \dots, n_k)$. Now, for $k, n_1, n_2, \dots, n_k \in \mathbb{N}$ we define

$$(11) \quad C_{n_1, n_2, \dots, n_k} := \bigcup\{A_\beta : \beta \in \mathbb{N}^{\mathbb{N}}, \beta|k = (n_1, n_2, \dots, n_k)\}.$$

The family $\mathcal{W} = \{C_{n_1, n_2, \dots, n_k}\}$ is a web in the sense of De Wilde, [8]. The web \mathcal{W} enjoys the following properties:

$$(12) \quad C_{n_1, n_2, \dots, n_k} \subset C_{m_1, m_2, \dots, m_k}, \text{ for } n_j \leq m_j, k \in \mathbb{N}, j = 1, 2, \dots, k;$$

- (13) For every $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ and every $\beta(E', E)$ – neighborhood of 0 $U \subset E'$ there is $n_U \in \mathbb{N}$ and $p_U \geq 0$ such that $C_{n_1, n_2, \dots, n_U} \subset p_U U$.

The order condition in (12) immediately follows from the definitions. Condition (13) is proved as follows: every A_α is $\beta(E', E)$ -bounded because every sequence in it is \mathfrak{T} -equicontinuous; if we assume that (13) does not hold we would find $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ and a $\beta(E', E)$ -neighborhood U of 0 in E' such that $C_{n_1, n_2, \dots, n_k} \not\subset kU$, $k = 1, 2, \dots$. For every positive integer k there is $\alpha_k = (a_n^k)_n \in \mathbb{N}^{\mathbb{N}}$ with $\alpha_k|k = (n_1, n_2, \dots, n_k)$, such that $A_{\alpha_k} \not\subset kU$. We define now $a_n = \max\{a_n^k : k = 1, 2, \dots\}$, $n = 1, 2, \dots$, and $\gamma = (a_n)_n$. It is clear that $\gamma \geq \alpha_k$ and $A_\gamma \not\subset kU$, $k = 1, 2, \dots$, which contradicts the boundedness of A_γ and validates (13).

Given positive integers k, n_1, n_2, \dots, n_k we define

$$D_{n_1, n_2, \dots, n_k} := \overline{C_{n_1, n_2, \dots, n_k}}^{\sigma(E', E)}.$$

Since the topology $\beta(E', E)$ has a base of neighborhoods of 0 consisting of $\sigma(E', E)$ -closed sets and as the web \mathcal{W} satisfies condition (13) we obtain:

- (14) For every $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ and every $\beta(E', E)$ – neighborhood of 0 $U \subset E'$ there is $n_U \in \mathbb{N}$ and $p_U \geq 0$ such that $D_{n_1, n_2, \dots, n_U} \subset p_U U$.

If we re-label $A_\alpha := \bigcap_{k=1}^{\infty} D_{n_1, n_2, \dots, n_k}$, then the new family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ still satisfies the properties in (10). Since $(E', \beta(E', E))$ is quasi-complete, every $\beta(E', E)$ -bounded set is contained in a $\beta(E', E)$ -Banach disc, which means that our reconstituted family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental family of \mathfrak{T} -equicontinuous subsets of E' . Taking polars in $\langle E, E' \rangle$, the family $\{A_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighborhoods of the origin in (E, \mathfrak{T}) . On the other hand, if we read (14) after taking polars in E , we see that for every $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$ the increasing sequence

$$D_{n_1}^\circ \subset D_{n_1, n_2}^\circ \subset \dots \subset D_{n_1, n_2, \dots, n_k}^\circ \subset \dots$$

is bornivorous in the sense of [15, Definition 8.1.15]; by [14, §20.9.7] and [15, Propostion 8.2.27] we have that for every $\varepsilon > 0$

$$(15) \quad A_\alpha^\circ = \overline{\bigcup_{k=1}^{\infty} D_{n_1, n_2, \dots, n_k}^\circ}^{\sigma(E, E')} \subset (1 + \varepsilon) \bigcup_{k=1}^{\infty} D_{n_1, n_2, \dots, n_k}^\circ.$$

Collecting all the information we have produced we know now that if we define for $\alpha = (n_k)_k$ in $\mathbb{N}^{\mathbb{N}}$

$$U_\alpha := \bigcup_{k=1}^{\infty} D_{n_1, n_2, \dots, n_k}^\circ$$

then $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of \mathfrak{T} -neighborhoods of the origin in E . Now, we finally prove that $t(E, \mathfrak{T}) \leq \aleph_0$, that is, we prove if $A \subset E$ and $0 \in \overline{A}^{\mathfrak{T}}$ then there is a countable subset $B \subset A$ such that $0 \in \overline{B}^{\mathfrak{T}}$; given such an A the set

$$B := \{x_{n_1, n_2, \dots, n_k} : x_{n_1, n_2, \dots, n_k} \text{ is a chosen point in } D_{n_1, n_2, \dots, n_k}^\circ \cap A, \\ \text{if } D_{n_1, n_2, \dots, n_k}^\circ \cap A \neq \emptyset, k, n_1, n_2, \dots, n_k \in \mathbb{N}\}$$

is countable and satisfies $0 \in \overline{B}^{\mathfrak{T}}$. \blacksquare

In the terminology of [2] the web \mathcal{W} satisfying (13) is a $\beta(E', E)$ -bounded web; we refer the reader to [2] for a more detailed account of bounded webs and their applications.

In light of [20, Theorem 3] we observe that the class of quasibarrelled LCS in \mathfrak{G} coincides with those quasibarrelled LCS whose strong duals are \mathcal{C} -webbed in the sense of De Wilde.

Trivially, countable tightness is enjoyed by the increasingly wider classes of normable, metrizable and Fréchet-Urysohn LCS. Since the latter are always bornological, see [22, 12], there is no distinction among (DF) -spaces: If E is a Fréchet-Urysohn (DF) -space then E is normable, either by the Theorem of [12] or by Webb's Corollary 5.4 in [21]. However, it is apparent, see Proposition 5.5(2),(3) of [21], that Webb did not know this result, being unaware that Fréchet-Urysohn implies bornological.

More generally, a topological space X is *sequential* if every sequentially closed set is closed in X . Webb proved (Proposition 5.5(1) of [21]) that every sequential (DF) -space is quasibarrelled; indeed, as we show in [13], every sequential (DF) -space is either normable or Montel, providing the converse to Proposition 5.7 of [21]. As an immediate consequence, every sequential (DF) -space has countable tightness. Is the same true of every sequential space in \mathfrak{G} ?

We conclude with questions related to Theorem 4.2, Proposition 4.7 and Theorem 4.8.

Problem 1. *Let $(E_s, \mathfrak{T}_s)_{s \in S}$ be a family of LCS in \mathfrak{G}_m , let $\{f_s : E_s \rightarrow E\}_{s \in S}$ be linear maps and let $(E, \mathfrak{T}) = \sum_{s \in S} f_s(E_s, \mathfrak{T}_s)$ be the locally convex hull of $f_s(E_s, \mathfrak{T}_s)$. If $|S| \leq m$, is it true that the weight of precompact subsets of (E, \mathfrak{T}) is at most m ?*

Problem 2. *Are there nice classes other than \mathfrak{G} for which Proposition 4.7 holds?*

Problem 3. *Must a (DF) -space with countable tightness be quasibarrelled?*

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