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THE LINDELÖF PROPERTY AND FRAGMENTABILITY

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ABSTRACT. Let K be a compact Hausdorff space, $C(K)$ the space of continuous real functions on K. In this paper we prove that any $t_p(K)$-Lindelöf subset of $C(K)$ which is compact for the topology $t_p(D)$ of pointwise convergence on a dense subset $D \subset K$ is norm fragmented, i.e., each non-empty subset of it contains a non-empty $t_p(D)$-relatively open subset of small supremum norm diameter. Several applications are given.

1. Introduction

In what follows K will be a compact Hausdorff space, $C(K)$ will stand for the space of continuous real functions on K, and for a given subset $F \subset K$, $t_p(F)$ will be the topology in $C(K)$ of pointwise convergence on F.

The notion of fragmentability as stated below was introduced by Jayne and Rogers [8].

Definition 1.1. Let (X, τ) a topological space and ρ a metric on X. We say that (X, τ) is *fragmented by* ρ (or ρ*-fragmented*) if for each non-empty subset A of X and for each $\varepsilon > 0$ there exists a non-empty τ-open subset U of X such that $U \cap A \neq \emptyset$ and ρ-diam$(U \cap A) \leq \varepsilon$.

A result by one of us in [10] implies that every $t_p(K)$-compact subset of $C(K)$ is fragmented by the supremum norm. On the other hand, a Bourgin's result in [3, p.98], proved by using a construction by Stegall, states that the Radon-Nikodým property holds for weakly Lindelöf and weak* compact convex subsets of dual Banach spaces; in other words, weakly Lindelöf and weak* compact convex subsets of dual Banach spaces are fragmented by the dual norm. The aim of the present paper is to solve affirmatively the problem below:

PROBLEM 1. *Let D a dense subset of K and let H be a $t_p(D)$-compact subset of $C(K)$. If H is $t_p(K)$-Lindelöf, is H fragmented by the norm of $C(K)$?*

that appears in [4]. Thus our main result, Theorem B, states that, if H satisfies the hypotheses of Problem 1, then the compact space $(H, t_p(D))$ is fragmented by the norm of $C(K)$. Our Theorem B is a common generalization of the results in [10] and [3] cited above. It also extends [5, Proposition 1.1] and the main result in [4]. Our results here are very much related to the problem of knowing if $\ell^\infty = C(\beta\mathbb{N})$ contains a $t_p(\beta\mathbb{N})$-Lindelöf subset Y separating the points of $\beta\mathbb{N}$. We prove that this is impossible if Y is assumed to be $t_p(\mathbb{N})$-Čech-analytic.

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Compact sets fragmented by a lower-semicontinuous metric are called Radon-Nikodým compact and they are homeomorphic to a weak*-compact subset of a dual Banach space with the Radon-Nikodým property [11]. One of the consequences of the positive solution of Problem 1 is that the space $(H, t_p(D))$ appearing there is Radon-Nikodým compact and so, for instance, it is sequentially compact as well [11, Lemma 5.3].

2. Preliminary results on $B_1(H)$

For a topological space H, $C_b(H)$ stands for the space of bounded continuous real functions on H and $B_1(H)$ stands for the space of pointwise limits of sequences of continuous functions on H. $2^{\mathbb{N}}$ denotes the compact space of sequences of 0's and 1's endowed with its product topology and $2^{(\mathbb{N})}$ is the set of finite sequences of 0's and 1's. For a $t \in 2^{(\mathbb{N})}$, we let $|t|$ denote the length of t. Given $\sigma \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $\sigma|n = (\sigma(1), \dots \sigma(n)) \in 2^{(\mathbb{N})}$.

Since the following theorem, which appears in Pol [12, p. 34] with only a sketch of a proof, is essential in our paper, we give a full proof.

Theorem 2.1. *Let (H, d) be a complete metric space, D a subset of $C_b(H)$ which is uniformly bounded by 1 and $K = \overline{D}$ the closure of D in $[-1,1]^H$. Then the following are equivalent*

(a) $K \not\subset B_1(H)$,
(b) *There is a homeomorphism $\varphi : 2^{\mathbb{N}} \to \varphi(2^{\mathbb{N}}) \subset H$, a sequence $(f_n)_{n\in\mathbb{N}}$ in D and numbers $-1 < s < t < 1$ such that*

$$f_n(\varphi(\sigma)) \in G_{\sigma(n)} \text{ for every } \sigma \in 2^{\mathbb{N}} \text{ and } n \in \mathbb{N}$$

where $G_0 := [-1, s)$ and $G_1 := (t, 1]$.

Proof. We note that $f \in [-1,1]^H$ is not in $B_1(H)$ if, and only if, for some non-empty closed set $F \subset H$ and some pair $-1 < s < t < 1$ of real numbers, $\{x \in F : f(x) < s\}$ and $\{x \in F : f(x) > t\}$ are both dense in F, cf. [2, Proposition 1E].
$(a) \Rightarrow (b)$ Assume that there is $f \in K \setminus B_1(H)$. From the former remark we have a closed set $F \subset H$ and s, t as above. Let $G_0 := [-1, s)$, $G_1 := (t, 1]$. Then, $\{x \in F : f(x) \in G_0\}$ and $\{x \in F : f(x) \in G_1\}$ are both dense in F.

By induction on $n = |t|$, we choose a family $\{U_t : t \in 2^{(\mathbb{N})}\}$ of non-empty relatively open subsets of F and a sequence $(f_n)_{n\in\mathbb{N}}$ in D such that

(i) $U_\emptyset = F$,
(ii) for each $t \in 2^{(\mathbb{N})}$, $\overline{U_{t0}} \cup \overline{U_{t1}} \subset U_t$ and $\overline{U_{t0}} \cap \overline{U_{t1}} = \emptyset$,
(iii) d-diam$(U_t) < \frac{1}{|t|}$ for each $t \in 2^{(\mathbb{N})}$, and
(iv) $f_n(U_{tj}) \subset G_j$ for $j = 0$ or 1 and $|t| = n - 1$.

Construction. (i) begins the induction. For $n \geq 1$, suppose $\{U_t : |t| < n\}$, $\{f_i : i < n\}$ have been chosen so as to satisfy $(i) - (iv)$. For each $t \in 2^{(\mathbb{N})}$ with $|t| = n - 1$, choose $a_t, b_t \in U_t$ such that $f(a_t) \in G_0$ and $f(b_t) \in G_1$. Since $f \in \overline{D}$, there is $f_n \in D$ such that $f_n(a_t) \in G_0$ and $f_n(b_t) \in G_1$ for all t with $|t| = n-1$. Since f_n is continuous there are open neighbourhoods U_{t0}, U_{t1} of a_t, b_t respectively such that $(ii) - (iv)$ are satisfied. This completes the construction.

Now let

$$\Delta := \bigcap_n \bigcup_{|t|=n} \overline{U_t} = \bigcap_n \bigcup_{|t|=n} U_t.$$

Then $\Delta \subset H$ is compact and the map $\varphi : 2^{\mathbb{N}} \to \Delta$ given by $\varphi(\sigma) = \bigcap_{n=1}^{\infty} \overline{U_{\sigma|n}}$ is a homeomorphism that together with (f_n) fulfills condition (b).
$(b) \Rightarrow (a)$ Let $g : \overline{G_0} \cup \overline{G_1} \to \{0,1\}$ be the obvious map. Let f be a pointwise cluster point of (f_n) and let $\Delta := \varphi(2^{\mathbb{N}})$. Then $f(\Delta) \subset \overline{G_0} \cup \overline{G_1}$, and $g \circ f_n \circ \varphi$ is the n-th projection of $2^{\mathbb{N}}$ onto $\{0,1\}$. Since $g \circ f \circ \varphi$ is a cluster point of the sequence $(g \circ f_n \circ \varphi)$, $g \circ f \circ \varphi$ is not Borel measurable by Sierpinski's theorem; cf. [14]. Hence $f|_\Delta$ is not measurable and therefore $f \in K \setminus B_1(H)$. □

The concept of independent sequence of functions as appears below was introduced by Rosenthal in [13] (see also [2]),

Definition 2.2. A sequence of functions (f_n) in $\mathbb{R}^\Omega$ is called *independent on* $A \subset \Omega$ if there are numbers $s < t$ such that for each pair of finite disjoint subsets $P, Q \subset \mathbb{N}$ we have

$$\bigcap_{n\in P} \{\omega \in A : f_n(\omega) \le s\} \cap \bigcap_{n \in Q} \{\omega \in A : f_n(\omega) \ge t\} \neq \emptyset \tag{2.1}$$

If (f_n) is a sequence of *continuous* functions independent on a *compact* set A, then (2.1) holds for arbitrary disjoint subsets P and Q of $\mathbb{N}$. This fact is used in the proof of the next lemma, which links $\beta\mathbb{N}$ and independent sequences of continuous functions.

Lemma 2.3. *Let H be a compact Hausdorff space and $(f_n)_{n\in\mathbb{N}}$ a sequence in $C(H)$ uniformly bounded by 1. If $(f_n)_{n\in\mathbb{N}}$ is independent on H, then the mapping $n \mapsto f_n$ extends to a homeomorphism of $\beta\mathbb{N}$ onto the closure of $\{f_n : n \in \mathbb{N}\}$ in $[-1,1]^H$.*

Proof. By the definition of $\beta\mathbb{N}$, the map $n \mapsto f_n$ extends to a continuous map $\delta : \beta\mathbb{N} \to [-1,1]^H$. Clearly $\delta(\beta\mathbb{N}) = \overline{\{f_n : n \in \mathbb{N}\}}$. To show that δ is a homeomorphism, it is sufficient to prove that δ is one-to-one. Suppose $\alpha, \beta \in \beta\mathbb{N}$ and $\alpha \neq \beta$. Then there are disjoint clopen neighbourhoods of α and β respectively, *i.e.* for some $P, Q \subset \mathbb{N}$, $\alpha \in \overline{P}, \beta \in \overline{Q}$ and $\overline{P} \cap \overline{Q} = \emptyset$. Since H is compact and $\{f_n : n \in \mathbb{N}\}$ is independent on H, there exist real numbers $s < t$ and an $h \in H$ such that $\delta(n)(h) = f_n(h) \le s$ for all $n \in P$ and $\delta(n)(h) = f_n(h) \ge t$ for all $n \in Q$. Since $\alpha \in \overline{P}$, $\delta(\alpha)(h) \le s$ by the continuity of δ. Similarly $\delta(\beta)(h) \ge t$. Hence $\delta(\alpha) \neq \delta(\beta)$. □

Lemma 2.4. *Let H be a compact metrizable space, D a uniformly bounded subset of $C(H)$ that separates the points of H and let K be the pointwise closure of D in $\mathbb{R}^H$. If H is Lindelöf relative to the weak topology σ_K induced by K, then $K \subset B_1(H)$.*

Proof. We may and do assume that D is uniformly bounded by 1 and so $K \subset [-1,1]^H$. We show that $K \not\subset B_1(H)$ implies that (H, σ_K) is not Lindelöf.

Since H is a Polish space, according to Theorem 2.1, $K \not\subset B_1(H)$ implies the existence of a sequence $(f_n)_{n\in\mathbb{N}}$ in D, real numbers $-1 < s < t < 1$ and a homeomorphism of $2^{\mathbb{N}}$ into H, denoted as $\chi_M \mapsto h_M \in H$, such that for each $M \subset \mathbb{N}$

$$f_n(h_M) \quad < \quad s \quad \text{for each} \quad n \in \mathbb{N} \setminus M, \ \text{and} \tag{2.2}$$

$$f_n(h_M) \quad > \quad t \quad \text{for each} \quad n \in M \tag{2.3}$$

The image Δ of this homeomorphism is a compact subset of H and so σ_K-closed, because, by the assumption, D separates the points of H and so the topology of H is induced by D and is weaker than σ_K. The inequalities (2.2) and (2.3) imply that the sequence $(f_n)_{n\in\mathbb{N}}$ is independent on H, and hence by Lemma 2.3, there exists a homeomorphism δ of $\beta\mathbb{N}$ onto $C \stackrel{\text{def}}{=} \overline{\{f_n : n \in \mathbb{N}\}} \subset K$ extending the map $n \mapsto f_n$. From (2.2) and (2.3), it follows that, for each $M \subset \mathbb{N}$,

$$x(h_M) \leq s \text{ if } x \in C \setminus \delta(\overline{M}) \quad \text{and} \quad x(h_M) \geq t \text{ if } x \in \delta(\overline{M}). \tag{2.4}$$

For each $x \in C$, let

$$G_x = \{h_M \in \Delta : x(h_M) \geq t\} = \{h_M \in \Delta : x(h_M) > s\}. \tag{2.5}$$

Then G_x is a σ_K–closed and σ_K-open subset of Δ.

We show that for each $A \subset C$, $a \in \overline{A}$ if and only if $G_a \subset \bigcup\{G_x : x \in A\}$. Suppose $a \in \overline{A}$. Then $h_M \in G_a$ implies $a(h_M) > s$ and so $x(h_M) > s$ for some $x \in A$. Hence $h_M \in G_x$ for some $x \in A$ by (2.5). Conversely, if $a \notin \overline{A}$, then there is a closed and open neighbourhood of a that is disjoint from A, *i.e.* for some $M \subset \mathbb{N}$, $a \in \delta(\overline{M})$ and $\delta(\overline{M}) \cap A = \emptyset$. Then by (2.4), $a(h_M) \geq t$ and $x(h_M) \leq s$ for each $x \in A$, *i.e.* $h_M \in G_a \setminus \bigcup\{G_x : x \in A\}$ by (2.5).

Finally suppose (H, σ_K) were Lindelöf. Then the σ_K–closed subset Δ is also σ_K–Lindelöf. This implies that C is countably tight, i.e. for each subset A of C, each point in $\overline{A}$ is in the closure of a countable subset of A. For, suppose $A \subset C$ and $a \in \overline{A}$. Then from above $G_a \subset \bigcup\{G_x : x \in A\}$. Since G_a, G_x are all σ_K–closed and σ_K-open, there is a countable subset B of A such that $G_a \subset \bigcup\{G_x : x \in B\}$ which implies that $a \in \overline{B}$. This proves that C (and hence $\beta\mathbb{N}$) is countably tight. But if this were the case, then each compact separable space, being a continuous image of $\beta\mathbb{N}$, is countably tight. But, for instance, $[0,1]^{[0,1]}$ with the product topology is separable and compact without being countably tight. This contradiction shows that (H, σ_K) cannot be Lindelöf. □

3. Fragmentability

In this section, we prove the main theorem of this paper. The following lemma is its very special case, but, nevertheless, it contains the crux of the matter.

Lemma 3.1. *Let D be a uniformly bounded subset of $C(2^{\mathbb{N}})$ such that, for some $\varepsilon > 0$, whenever $x, x' \in 2^{\mathbb{N}}, x \neq x'$,then*

$$\rho(x, x') \stackrel{\text{def}}{=} \sup_{f \in D} |f(x) - f(x')| \geq \varepsilon.$$

If K is the closure of D in $\mathbb{R}^{2^{\mathbb{N}}}$, then $2^{\mathbb{N}}$ is not Lindelöf relative to σ_K, the weak topology on $2^{\mathbb{N}}$ induced by K.

Proof. Suppose that $(2^{\mathbb{N}}, \sigma_K)$ is Lindelöf, and we will reach a contradiction. First, by Lemma 2.4, $K \subset B_1(2^{\mathbb{N}})$. Let μ denote the normalized Haar measure on $2^{\mathbb{N}}$, and let

$$\mathcal{U} = \{U \subset 2^{\mathbb{N}} : U \text{ is Borel, } \sigma_K\text{-open and } \mu(U) = 0\}.$$

Also let $G = \bigcup \mathcal{U}$ and $C = 2^{\mathbb{N}} \setminus G$. Then $C \neq \emptyset$, for otherwise, using the Lindelöf property of $(2^{\mathbb{N}}, \sigma_K)$, $2^{\mathbb{N}}$ can be covered by countably many members of $\mathcal{U}$ and consequently $\mu(2^{\mathbb{N}}) = 0$.

Let $\varphi : (K, \tau_p) \to (L^1(2^{\mathbb{N}}, \mu),$ norm) be the map that assigns to each $f \in K$ the class $[f]$. Here τ_p is the topology of pointwise convergence in $\mathbb{R}^{2^{\mathbb{N}}}$. Since (K, τ_p) is angelic by [2, Theorem 3F], one can show that φ is continuous. In fact, if $A \subset K$ and $f \in A^{-\tau_p}$, then there is a sequence in A converging pointwise to f. Hence by the bounded convergence theorem, $\varphi(f) \in \varphi(A)^-$. It follows that $\varphi(K)$ is compact and metrizable and $\varphi : (K, \tau_p) \to \varphi(K)$ is a quotient map. We claim that if $f, g \in K$ and $\varphi(f) = \varphi(g)$ then $f(x) = g(x)$ for each $x \in C$. For, let $V = \{t \in 2^{\mathbb{N}} : f(t) \neq g(t)\}$, then $V \in \mathcal{U}$ and so V and C are disjoint. It follows that each member x of C defines a continuous function $\hat{x}$ on $\varphi(K)$ satisfying $\hat{x}(\varphi(f)) = f(x)$. Hence $\hat{C}$ is a subset of $C(\varphi(K))$ and the latter is norm separable. Note that if $x, x' \in C$, then

$$\rho(x, x') = \sup_{f \in D} |f(x) - f(x')| = \sup_{f \in K} |f(x) - f(x')| = \| \hat{x} - \hat{x'} \| .$$

Here $\| \cdot \|$ is the supremum norm of $C(\varphi(K))$. Hence $(\hat{C},$ norm) and (C, ρ) are isometric. Since the former is separable and the latter discrete, we see that C is countable. It follows that C and $G = 2^{\mathbb{N}} \setminus C$ are Borel sets and $\mu(C) = 0$. We reach our contradiction by observing that $\mu(G) = 0$. To see this, let L be a compact subset of G. Then L is σ_K closed in $2^{\mathbb{N}}$, because the assumption that D separates points of $2^{\mathbb{N}}$ implies that σ_K is finer than the topology of $2^{\mathbb{N}}$. It follows that L is σ_K-Lindelöf and hence it is covered by countably many members from $\mathcal{U}$. Therefore $\mu(L) = 0$. By the regularity of μ, $\mu(G) = 0$. □

From now onwards, except in the last corollary in this section, D will be a dense subset of K. Given a $t_p(D)$-compact subset H of $C(K)$ we will look at the elements of K as functions on H: for each point k in K we will denote by $\hat{k}$ the restriction to H of the "point mass" at k, that is $\hat{k}(f) := f(k)$. It is clear that $\hat{D} = \{\hat{d} : d \in D\}$ is a pointwise bounded set of continuous functions on the compact space $(H, t_p(D))$, and $\hat{K} = \{\hat{k} : k \in K\}$ is a pointwise compact set of continuous functions on $(H, t_p(K))$. Obviously, the closure of $\hat{D}$ in $\mathbb{R}^H$ is $\hat{K}$.

In the proof of the following theorem, we use the simple fact that, for (X, τ) in Definition 1 to be ρ-fragmented, it is sufficient that each τ-closed non-empty subset of X has non-empty relatively τ-open subsets of arbitrarily small ρ-diameter. Also in the proof $\| \cdot \|$ will denote the supremum norm of $C(K)$.

Theorem 3.2. *Let K be a compact Hausdorff space and let D be a dense subset of K. Then, every $t_p(D)$-compact subset of $C(K)$ which is $t_p(K)$-Lindelöf is fragmented by the supremum norm, and so, it is a Radon-Nikodým compact space.*

Proof. Let H be a $t_p(D)$ compact subset of $C(K)$ and let B denote the unit ball of $C(K)$. Then B is $t_p(D)$-closed. If A is a non-empty $t_p(D)$-closed subset of H, then by the Baire category theorem, there exists an $n \in \mathbb{N}$ such that $(nB) \cap A$ has non-empty relative $t_p(D)$-interior. Hence in order to prove that $(H, t_p(D))$ is fragmented by the norm it is sufficient to prove each $(nB) \cap A$ is fragmented by the norm. So we may and do assume that H is uniformly bounded.

Suppose that $(H, t_p(D))$ is not fragmented by the norm. Then, for some non-empty $t_p(D)$-compact subset C of H and $\varepsilon > 0$, each non-empty $t_p(D)$-open subset of C has norm diameter greater that ε. By induction on $n = |t|$, $t \in 2^{(\mathbb{N})}$, we construct a family $\{U_t : t \in 2^{(\mathbb{N})}\}$ of non-empty relatively $t_p(D)$-open subsets of C and a family $\{x_t : t \in 2^{(\mathbb{N})}\}$ of points of D, satisfying the following conditions, where the closures are relative to $t_p(D)$:

(*i*) $U_\emptyset = C$,
(*ii*) for each t, $\overline{U_{t0}} \cup \overline{U_{t1}} \subset U_t$,
(*iii*) $|(f-g)(x_t)| > \varepsilon$ for each $f \in U_{t0}$ and $g \in U_{t1}$, and
(*iv*) whenever $s, t \in 2^{(\mathbb{N})}$ and $|s| < |t|$, $\operatorname{diam} \hat{x_s}(\overline{U_{tj}}) < |t|^{-1}$ for $j = 0, 1$.

Construction. (*i*) starts the induction from $n = 0$. Next, for some $n > 0$, assume that $\{U_t : |t| < n\}$ and $\{x_s : |s| < n-1\}$ have been constructed. Fix a $t \in 2^{(\mathbb{N})}$ with $|t| = n-1$. By assumption, for some $f_0, f_1 \in U_t$, $\| f_0 - f_1 \| > \varepsilon$, which means that $|(f_0 - f_1)(x_t)| = |\hat{x_t}(f_0) - \hat{x_t}(f_1)| > \varepsilon$ for some $x_t \in D$. Since $\hat{x_t}$ and $\hat{x_s}$, $|s| < |t|$, are all continuous on $(H, t_p(D))$, one can select $t_p(D)$-open neighbourhoods U_{t0} and U_{t1} of f_0 and f_1, respectively, so that $(ii), (iii)$ and (iv) are satisfied. This completes the construction. Note that (iii) implies that $\overline{U}_{t0} \cap \overline{U}_{t1} = \emptyset$ for each $t \in 2^{(\mathbb{N})}$.

Let $F := \bigcap_n \bigcup_{|t|=n} \overline{U_t}$. Then F is a compact subset of $(H, t_p(D))$ and it is partitioned as $F = \bigcup_{\sigma \in 2^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} \overline{U_{\sigma|n}}$. Define $\varphi : F \to 2^{\mathbb{N}}$ by $\varphi^{-1}(\sigma) = \bigcap_{n \in \mathbb{N}} \overline{U_{\sigma|n}}$. Then clearly φ is a continuous and onto map. For each $t \in 2^{(\mathbb{N})}$ and $\sigma \in 2^{\mathbb{N}}$, $\hat{x_t}(\varphi^{-1}(\sigma))$ is a singleton by (iv). Hence $\hat{x_t}$ 'lifts' to a continuous function x_t^* on $2^{\mathbb{N}}$ such that

$$f(x_t) = \hat{x_t}(f) = x_t^*(\varphi(f)) \tag{3.1}$$

for every $f \in F$. If $\sigma, \sigma' \in 2^{\mathbb{N}}$ and $\sigma \neq \sigma'$, then for some $n \in \{0\} \cup \mathbb{N}$, $\sigma|n = \sigma'|n$ and $\sigma(n+1) \neq \sigma'(n+1)$. If we let $t = \sigma|n$, then by (iii), $|x_t^*(\sigma) - x_t^*(\sigma')| > \varepsilon$. This means that the hypothesis for D in Lemma 3.1 is satisfied by $\{x_t^* : t \in 2^{(\mathbb{N})}\}$. Let L be the closure of $\{x_t^* : t \in 2^{(\mathbb{N})}\}$ in $\mathbb{R}^{2^{\mathbb{N}}}$. Then by Lemma 3.1, $2^{\mathbb{N}}$ is not Lindelöf for the weak topology σ_L. However, we show below that our assumptions imply that $(2^{\mathbb{N}}, \sigma_L)$ is Lindelöf and this contradiction proves the theorem. To see that $(2^{\mathbb{N}}, \sigma_L)$ is Lindelöf, it is sufficient to prove that φ is $(t_p(K) - \sigma_L)$-continuous, since $(F, t_p(K))$ is Lindelöf by hypothesis. For the continuity of φ, we must prove that the map $f \mapsto \xi(\varphi(f))$ is $t_p(K)$-continuous on F for each $\xi \in L$. Fix $\xi \in L$. then there is a net $\{x_{t_\alpha}^*\}$ that converges to ξ pointwise. By the compactness of K, we may assume that x_{t_α} converges to $x \in K$. Then by (3.1) we have that

$$\xi(\varphi(f)) = \lim_\alpha x_{t_\alpha}^*(\varphi(f)) = \lim_\alpha f(x_{t_\alpha}) = f(x)$$

for each $f \in F$, and $f \mapsto f(x)$ is clearly $t_p(K)$-continuous. □

Properties of $t_p(D)$-compact norm bounded sets H which are fragmented by the supremum norm can be found in [5]. For instance, it is proved there that for such an H the closed convex hull of it, $\overline{\mathrm{co}(H)}^{t_p(D)}$, is again $t_p(D)$-compact, satisfies $\overline{\mathrm{co}(H)}^{t_p(D)} = \overline{\mathrm{co}(H)}^{\|\ \|}$ and this closed convex hull has the usual Radon-Nikodým property.

If a topological space (X, τ) has a countable base (or more generally, is hereditarily Lindelöf) and if it is fragmented by a metric ρ, then a simple argument with points of condensation shows that (X, ρ) is separable. Combining this with Theorem B, we obtain the following.

Corollary 3.3. *Let K be a compact and D a dense and countable subset of K. Then, every $t_p(D)$-compact subset of $C(K)$ which is $t_p(K)$-Lindelöf is separable for the supremum norm.*

In the next corollary, the rôles of K, D and H will be as in Lemma 2.

Corollary 3.4. *Let H be a compact Hausdorff space, D a pointwise bounded subset of $C(H)$ that separates points of H and let K be the pointwise closure of D in $\mathbb{R}^H$. If H is Lindelöf relative to the weak topology σ_K induced by K, then, for each closed subset F of H, there is a dense G_δ- subset Z_F of F such that $\{f|_F : f \in K\}$ is equicontinuous at each point of Z_F.*

Proof. With the pointwise topology K is a compact Hausdorff space with the dense subset D. As usual, each $h \in H$ gives rise to $\hat{h} \in C(K)$ such that $\hat{h}(f) = f(h)$ for each $f \in K$. Then $h \mapsto \hat{h}$ is a homeomorphism $H \to (\hat{H}, t_p(D))$ and by hypothesis $\hat{H}$ is $t_p(K)$-Lindelöf. Hence by Theorem B, $(\hat{H}, t_p(D))$ is fragmented by the norm, or equivalently H is ρ-fragmented, where ρ is a metric on H given by

$$\rho(h, h') = \sup_{f \in K} |f(h) - f(h')| \quad \text{for} \quad h, h' \in H.$$

Consequently, using the category argument, cf. [11], one sees that, given a closed subset F of H, there is a dense G_δ-subset Z_F of F such that the identity map $H \to (H, \rho)$ is continuous at each point of Z_F, which is equivalent to the conclusion of the corollary. □

Remark 3.5. The corollary above is a generalization of Théorèm 4.1 in [17] where the space (H, σ_K) is assumed to be K-analytic. However, Talagrand remarks that one can prove the theorem with only assuming that (H, σ_K) be Lindelöf and that "*la démonstration serait alors beaucoup plus longue*". The proof of the stronger theorem has never been published.

4. Applications

Given a Banach space $(X, \|\cdot\|)$, a subset B of the dual unit ball B_{X^*} is said to be norming if $\| x \| = \sup\{|x^*(x)| : x^* \in B\}$ for every $x \in X$. As a consequence of Hahn-Banach separation theorem, if B is norming then its absolutely convex hull D is weak* dense in B_{X^*}. In this way, X endowed with the topology $\sigma(X, B)$ of pointwise convergence on B (or equivalently, on D) appears as a subspace of $(C(B_{X^*}), t_p(D))$ and Theorem 3.2 can be used to state the corollary below.

Corollary 4.1. *Let $(X, \|\cdot\|)$ a Banach space and $B \subset B_{X^*}$ a norming subset. Then, every weakly Lindelöf $\sigma(X, B)$-compact subset of X is fragmented by the norm and so it is a Radon-Nikodým compact space under $\sigma(X, B)$.*

This corollary implies the result in [3] mentioned in the Introduction without the convexity assumption: a weakly Lindelöf weak*-compact subset of a dual Banach space is fragmented by the dual norm. In [15], Srivatsa proves that, if f is a continuous map from a metric space T into a Banach space X with its weak topology, then f is the pointwise (norm)-limit of a sequence of continuous functions $T \to (X, \|\cdot\|)$, *i.e.* $f \in B_1(T, X)$. The following corollary shows that, if the weak topology in the above is replaced by $\sigma(X, B)$ for some norming subset B of B_{X^*}, one still reaches the same conclusion provided that the image of f is contained in a weakly Lindelöf subset of X and T is complete.

Corollary 4.2. *Let K be a compact Hausdorff space, $D \subset K$ a dense subset, T is a complete metric space and $f : T \to C(K)$ a $t_p(D)$-continuous function. If there is a $t_p(K)$-Lindelöf subset $Y \subset C(K)$ such that $f(T) \subset Y$ then $f \in B_1(T, C(K))$.*

Proof. To prove that $f \in B_1(T, C(K))$ it is enough to prove that for every compact subset W of T the restriction $f|_W$ has a point of norm continuity, [16]. Given a compact $W \subset T$, the image $f(W)$ is $t_p(D)$-compact and so norm fragmented after Theorem 3.2. According to [11, Lemma 1.1] the identity map

$$id : (f(W), t_p(D)) \to (f(W), \|\cdot\|)$$

has a point of continuity and thus we get that $f|_W$ has a point of norm continuity and the proof is done. □

It is a usual exercise in elementary measure theory that, if f is a real-function on $\mathbb{R}\times\mathbb{R}$ such that $f_t \overset{\text{def}}{=} f(t,\cdot)$ is continuous for each $t \in \mathbb{R}$ and such that $f^s \overset{\text{def}}{=} f(\cdot, s)$ is continuous for each s belonging to a dense subset of $\mathbb{R}$, then $f \in B_1(\mathbb{R}\times\mathbb{R})$. The following corollary, which is a straightforward consequence of the previous one, is a far reaching generalization of this. For related results concerning measurability of separately continuous functions, see [9], [18] and references cited therein.

Corollary 4.3. *Let K be a compact Hausdorff space and T a complete metric space. Let $f : T \times K \to [-1, 1]$ be a function verifying*

(i) *There is a $t_p(K)$-Lindelöf subset $Y \subset C(K)$ such that $\{f_t : t \in T\} \subset Y$.*
(ii) *The set $\{x \in K : f^x \in C(T)\}$ is dense in K.*

Then, $f \in B_1(T \times K)$.

In [1, p. 610], Arkhangelskii raises the following question: Suppose K is a compact Hausdorff space. If there exists a Lindelöf subset Y of $(C(K), t_p(K))$ that separates points of K, must K be countably tight? As far as we know, this question, in the usual set theory, is still open. When $K = \beta\mathbb{N}$, which is not countably tight as seen in Section 2, the question above is the same as Problem 2 in [4]. The following corollary is a partial answer to this problem.

Corollary 4.4. *$\ell^\infty = C(\beta\mathbb{N})$ can not contain a $t_p(\mathbb{N})$-Čech-analytic and $t_p(\beta\mathbb{N})$-Lindelöf subset Y separating the points of $\beta\mathbb{N}$.*

Proof. Assume that there is such a Y. Since $(C(\beta\mathbb{N}), t_p(\mathbb{N})) = (\ell^\infty, t_p(\mathbb{N}))$ is a metric analytic space, if Y is $t_p(\mathbb{N})$-Čech analytic then Y is metric analytic too. The last implies the existence of a continuous map from a Polish space P onto $(Y, t_p(N))$, say,

$$\varphi : P \to (Y, t_p(N)) \hookrightarrow (C(\beta\mathbb{N}), t_p(\mathbb{N})).$$

Now, Corollary 4.2 can be applied to deduce that $\varphi \in B_1(P, C(\beta\mathbb{N}))$. This implies that its range, which contains Y, is norm separable and so $\beta\mathbb{N}$ must be metrizable which is impossible. □

Next result is more general than [4, Corollary F] and it is in the same vein as results [7, Theorem 1] and [6, Corollary 8]. The ideas of [4, Corollary F] can be used to provide a proof of this corollary.

Corollary 4.5. *Let K be a compact Hausdorff space, $D \subset K$ a dense subset, T a topological space that contains a dense Čech-complete subspace and $f : T \to C(K)$ is a $t_p(D)$-continuous function. If there is a $t_p(K)$-Lindelöf subset $Y \subset C(K)$ such that $f(T) \subset Y$, then f is norm-continuous at each point of a dense G_δ subset of T.*

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