NOTES

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POINTWISE LIMITS OF ANALYTIC FUNCTIONS

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What are the possible pointwise limits of analytic functions? Since analytic functions are well behaved, you might expect such limits to be "nice" also. Yet it turns out that some bizarre things can occur. The results in this paper are accessible to a student of complex analysis, but they do not appear to have been collected together before.

Classical Background. It is easy to construct a sequence of continuous functions which converges pointwise to a function with discontinuities. However, not every function is the pointwise limit of continuous functions. The limit of a sequence of continuous functions which converges uniformly is necessarily continuous. We will be considering continuous functions on an open subset Ω of the complex plane. In this context, it is more natural to consider a topology of convergence in between uniform and pointwise convergence—*uniform convergence on compact subsets* (u.c.c.). A sequence of functions f_n on Ω converges u.c.c. if for every compact subset K of Ω, the restrictions $f_n|K$ converge uniformly. If every f_n is continuous, then the limit function is continuous on each K and thus is continuous on Ω. If every f_n is analytic, the u.c.c. limit is analytic also.

In both real and complex analysis, mathematicians have looked for and found conditions on a family of functions $\mathfrak{F}$ on Ω which guarantee that sequences taken from $\mathfrak{F}$ have convergent subsequences. These results are closely related to our problem. To explain the classical theorems, we need some additional terminology.

Recall that a function is continuous at z_0 if for each $\varepsilon > 0$, there is a $\delta > 0$ so that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \varepsilon$. A family of functions $\mathfrak{F}$ on Ω is *equicontinuous* if for each z_0 in Ω and each $\varepsilon > 0$ the $\delta > 0$ given above can be chosen independent of f in $\mathfrak{F}$. That is, $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \varepsilon$ for all f in $\mathfrak{F}$.

A family of functions $\mathfrak{F}$ is *pointwise bounded* if for each z_0 in Ω, the set $\{f(z_0): f \in \mathfrak{F}\}$ is bounded. The family $\mathfrak{F}$ is *locally bounded* if each z_0 in Ω is contained in an open ball U so that the set $\{f(z)\colon z \in U, f \in \mathfrak{F}\}$ is bounded. Since a convergent sequence is bounded, a sequence of continuous functions converging pointwise is pointwise bounded. However, it need not satisfy the stronger condition of local boundedness. A sequence converging u.c.c., though, must be locally bounded because every z_0 is contained in a ball U on which the sequence converges uniformly.

The following classical theorem from real analysis is proved in many introductory texts.

THEOREM A (*Arzela-Ascoli*). *Let $\mathfrak{F}$ be a family of continuous functions on Ω. Every sequence in $\mathfrak{F}$ has a subsequence converging u.c.c. if and only if $\mathfrak{F}$ is equicontinuous and pointwise bounded.*

The analogous theorem for analytic functions is:

THEOREM M (*Montel*). *Let $\mathfrak{F}$ be a family of analytic functions on Ω. Every sequence in $\mathfrak{F}$ has a subsequence converging u.c.c. if and only if $\mathfrak{F}$ is locally bounded.*

Notice that the equicontinuity condition is eliminated at the price of a stronger boundedness condition. A natural proof of Montel's Theorem is obtained by deducing equicontinuity and applying Theorem A.

Sketch. Fix w_0 in Ω and let U be a ball of radius r about w_0 so that $|f(w)| \leq M$ for all w in U and f in $\mathcal{F}$. Let $\mathcal{C}$ be the circle of radius $r/2$ centered at w_0. For every w within $r/4$ of w_0, Cauchy's Theorem gives

$$f(w) - f(w_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{z-w} - \frac{f(z)}{z-w_0}\,dz = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)\,dz}{(z-w)(z-w_0)}(w-w_0).$$

A simple estimate gives $|f(w) - f(w_0)| \leq 4Mr^{-1}|w - w_0|$ for all f in $\mathcal{F}$. From this, equicontinuity is immediate. □

What happens when a family of analytic functions is only pointwise bounded? As we noted earlier, this question applies in particular to a sequence of analytic functions which converges pointwise on Ω. The answers are dramatically different than for Montel's Theorem. There are some positive results, but first we will examine some of the pathology. Another classical result, Runge's Theorem, is needed to provide a way of obtaining analytic functions with prescribed behavior.

THEOREM R (*Runge*). *Let K be a compact subset of the complex plane with connected complement. Let f be a function analytic in a neighborhood of K. Then there is a sequence of polynomials which converges to f uniformly on K.*

Sketch. Inside the neighborhood U of K on which f is analytic, it is possible to draw a finite collection $\mathcal{C}$ of piecewise smooth curves which surround K in such a way that Cauchy's Theorem is valid for f. That is,

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(w)}{w-z}\,dw \quad \text{for all } z \text{ in } K.$$

This integral can be approximated by Riemann sums $\Sigma_{j=1}^{n} f(w_j)(w_{j+1} - w_j)/(w_j - z)$. Since $\mathcal{C}$ is bounded away from K, simple estimates show that these sums converge uniformly on K. This approximates f by rational functions. In order to get polynomials, it suffices to approximate each $g(z) = (w - z)^{-1}$ by polynomials uniformly on K.

When $|w|$ is strictly larger than $|z|$ for all z in K, the geometric series $\Sigma_{n=0}^{\infty} z^n/w^{n+1}$ converges uniformly to $(w - z)^{-1}$ on K. A similar argument can be used to express $(w' - z)^{-1}$ in terms of powers of $(w - z)^{-1}$ for w' near w. Thus if $(w - z)^{-1}$ can be approximated by polynomials in K, so can $(w' - z)^{-1}$. Because $\mathbb{C} \setminus K$ is a connected open set, a finite number of repetitions will get us from the "big" values of w to any point in $\mathbb{C} \setminus K$. Thus a polynomial approximation is obtained. □

Some Examples.

EXAMPLE 1. Let K_n be the union of the point $\{0\}$, the line segment $[(1/n), n]$ and the compact set $S_n = \{z \in \mathbb{C}: |z| \leq n \text{ and } \operatorname{dist}(z, \mathbb{R}_+) \geq 1/n\}$. (See Diagram 1 on p. 393.) Let g_n be an analytic function which vanishes in a neighborhood of S_n and $[(1/n), n]$ and is constantly one in a ball about $\{0\}$. Let p_n be a polynomial obtained by using Runge's Theorem such that $|p_n(z) - g_n(z)| < 1/n$ for all z in K_n.

Every point z in the complex plane belongs to K_n for large enough n. Thus $\lim_{n\to\infty} p_n(z)$ exists pointwise and equals zero everywhere except at $\{0\}$ where the limit is one! This limit is uniform on each set S_n by construction. Hence the convergence is u.c.c. on $\mathbb{C} \setminus [0, \infty]$. The sequence $\{p_n\}$ cannot converge uniformly on any neighborhood of $\{0\}$ because the limit function is not continuous at $\{0\}$. In fact, it cannot converge uniformly near any point on the positive real axis.

To see this, notice that if $\{p_n\}$ converges uniformly on a ball U of radius r centered at some $x > 0$, then for some $n \geq 2$, $r > 1/n$ and $|p_n(z)| < 1/2$ on U. In addition, $|p_n(z)| < 1/n \leq 1/2$ on S_n. But the union of S_n and U contains the circle of radius x about zero. So the maximum modulus principle implies that $|p_n(0)| < 1/2$ which is a contradiction. □

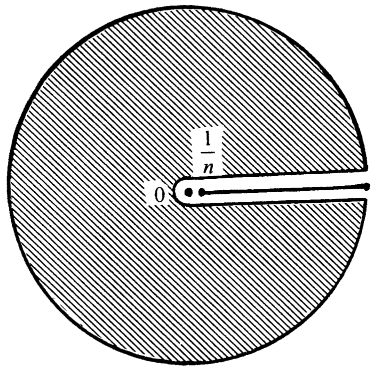

DIAGRAM 1

EXAMPLE 2. Let L_n be the union of K_n and the point $\{1/2n\}$. Let h_n be an analytic function on a neighborhood of L_n which is zero near K_n and one at $\{1/2n\}$. Again, Runge's Theorem provides polynomials q_n such that $|q_n(z) - h_n(z)| < 1/n$ on L_n. Clearly, the sequence $\{q_n\}$ converges pointwise to zero everywhere. However, convergence is not uniform near $\{0\}$ because $q_n(1/2n)$ is approximately equal to one. As in Example 1, convergence is uniform on each S_n but cannot converge uniformly near any point on the positive real axis. □

Some Positive Results.

PROPOSITION 1. *Let $\mathfrak{F}$ be a pointwise bounded family of analytic functions on Ω. There is a (maximal) dense open set $\Omega_0 \subseteq \Omega$ such that $\mathfrak{F}$ restricted to Ω_0 is locally bounded.*

Proof. For each z in Ω, let $\phi(z)$ denote the least upper bound for $\{|f(z)|: f \in \mathfrak{F}\}$. Because each f in $\mathfrak{F}$ is continuous, it is easy to see that $K_n = \{z: \phi(z) \leq n\}$ is (relatively) closed in Ω. Since $\phi(z)$ is finite for each z, Ω is the union of the K_n. The Baire Category Theorem shows that for every ball U in Ω, $K_n \cap U$ has interior for large n. Thus, the union Ω_0 of the interiors int K_n is a dense open subset of Ω. Obviously, $\mathfrak{F}$ is bounded on each int K_n, and hence $\mathfrak{F}$ is locally bounded on Ω_0. Conversely, if $\mathfrak{F}$ is bounded by n on an open set U, then U is contained in int K_n and thus is in Ω_0. □

COROLLARY 2. *Let f_n be a sequence of analytic functions converging pointwise on Ω. Then the limit function f is analytic on a dense open subset of Ω.*

Proof. A sequence of functions converging pointwise is pointwise bounded. By Proposition 1, $\{f_n\}$ is locally bounded on Ω_0. By Montel's Theorem, a subsequence converges u.c.c. to f on Ω_0 and hence f is analytic on Ω_0. □

PROPOSITION 3. *If f_n is a sequence of bounded analytic functions on Ω converging pointwise to f, then f is analytic and the convergence is u.c.c..*

Proof. The family $\mathfrak{F} = \{f_n\}$ is bounded and thus locally bounded. By Montel's Theorem, every subsequence of f_n has a sub-subsequence which converges u.c.c. This limit is f perforce, so f is analytic. If the whole sequence does not converge to f u.c.c., there would be a compact set K in Ω, an $\varepsilon > 0$, a subsequence f_{n_k}, and points z_k in K such that $|f_{n_k}(z_k) - f(z_k)| \geq \varepsilon$ for $k \geq 1$. Thus no subsequence of f_{n_k} could converge uniformly to f on K, which is a contradiction. □

What Limits Are Possible? How can the set of pointwise limits of analytic functions be described? If f is such a limit, then by Proposition 1 there is a dense open set Ω_0 on which the convergence is u.c.c. and f is analytic. On the relatively closed, nowhere dense set $C = \Omega \setminus \Omega_0$, f is merely the pointwise limit of continuous functions. That is all that can be said, for a modification of Example 1 can be arranged to converge to any function with these properties.

The idea is to choose increasing sequences K_n and C_n of subsets of Ω_0 and C respectively so that the complement of $K_n \cup C_n$ is connected and the union of all K_n's and C_n's is Ω.

Use Runge's theorem to approximate f within $1/n$ on K_n by polynomials. On C, there is a sequence g_n of continuous functions converging pointwise to f. A theorem of Lavrentiev can be used to approximate g_n within $1/n$ on C_n by a polynomial. Finally, use Runge's Theorem again to approximate both these polynomials simultaneously by a polynomial f_n. It is clear that $\{f_n\}$ converges to f.

A careful choice of the sequence g_n, as in Example 2, allows one to ensure that the convergence is not uniform near any point of C. Thus one may achieve an analytic limit f with failure to converge u.c.c. on any given closed nowhere dense subset.

Sequences of univalent functions (one to one and analytic) are much more nicely behaved. A sequence of univalent functions converging pointwise to a nonconstant function must in fact converge u.c.c.. In particular, the limit is always analytic. These facts will not be proved here. We just mention that the proof relies on a deep theorem of Montel which states that a family of analytic functions which omits two values is locally bounded.

References

The Arzela-Ascoli Theorem can be found in *Elementary Classical Analysis* by Marsden, and in greater generality in *General Topology* by Kelley. Montel's Theorem and Runge's Theorem can be found in most texts on complex analysis, for example, *Functions of One Complex Variables* by Conway. Another proof of Proposition 3 is contained in *A Second Course in Complex Variables* by Veech. Examples related to ours are contained in the problems section of *Real and Complex Analysis*, 2nd ed., by Rudin (pages 294 and 359). Lavrentiev's Theorem is a special case of Mergelyan's Theorem, which can be found in the last chapter of Rudin's book. The theorem of Montel referred to in the last paragraph is in Chapter 4 of Veech.

A NET OF EXPONENTIALS CONVERGING TO A NONMEASURABLE FUNCTION

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It is a basic fact that the pointwise limit of a *sequence* of (Lebesgue) measurable functions must be measurable. It is not hard to see that when "sequence" is replaced by "net," the result fails. Nevertheless, it is surprising that from the family $T = \{f(x-t) : t \in \mathbb{R}\}$ of translates of $f(x) = \exp(ix^2)$, a pointwise convergent net may be drawn whose limit is nonmeasurable. This is just another way of saying that in the space of all complex-valued functions on $\mathbb{R}$, in the topology of pointwise convergence, the closure of T contains a nonmeasurable function. (For an extended discussion of nets and filters in topology, see [**BAR**]. Essentially, the same fact that we prove here is proved in [**LUS**, Example 8.4.45, p. 218] by other means.)

To prove that there exists such a net of translates of $\exp(ix^2)$, let n_k be a sequence of positive integers approaching ∞. There are several ways to see that there exists a subnet (n_{k_γ}) of (n_k) such that for any bounded continuous function g on $\mathbb{R}$, $\phi(g) = \lim_\gamma g(n_{k_\gamma})$ exists. In one kind of language, ϕ is a point of the Stone-Čech compactification of $\mathbb{R}$—see [**GIJ**, Chapter 6] for more details. From another viewpoint, ϕ arises as a weak-star limit point of the functionals of evaluation (on the Banach space of continuous bounded functions on $\mathbb{R}$) at the points n_k. In any case, it is clear that ϕ is a complex homomorphism—that is, ϕ takes complex values and $\phi(g+h) = \phi(g) + \phi(h)$ and $\phi(gh) = \phi(g)\cdot\phi(h)$. To stress the dependence on x as the running variable, we shall write $\phi_x(f(x))$ sometimes. We will see that if the sequence (n_k) satisfies property P below, then $\phi_x(f(x+t)) = \lim_\gamma f(t + n_{k_\gamma})$ is not a measurable function of t.

With $f(x) = \exp(ix^2)$ we have

$$\phi_x(f(x+t)) = \phi_x(e^{i(x+t)^2}) = \phi_x(e^{ix^2})\phi_x(e^{it^2})\phi_x(e^{2ixt}).$$